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I Some general facts about unitary representations.

G group (discrete)

$$\rho : G \xrightarrow{\text{homomorphism}} U(\mathcal{H})$$

ρ — unitary representation

$\mathcal{H}$ — Hilbert space

$U(\mathcal{H})$ — group of unitary operators

Subrepresentation,
irreducible representation

Theorem (Gelfand-Raikov) For any $x, y \in G$, $x \neq y$, there exists an irreducible unitary representation ρ of G with $\rho(x) \neq \rho(y)$.

(analogy with residual finiteness).

Knowledge of (irreducible) representations helps to understand group.

The problem of classification of irreducible representations is wild (if G is not virtually abelian).

Two representations ρ and η are equivalent ($\rho \simeq \eta$) if $\exists \varphi$ s.t. diagram

$$\begin{tikzcd} \mathcal{H}_\rho \arrow[r, "\rho(g)"] \arrow[d, "\varphi"'] & \mathcal{H}_\rho \arrow[d, "\varphi"] \\ \mathcal{H}_\eta \arrow[r, "\eta(g)"] & \mathcal{H}_\eta \end{tikzcd}$$

is commutative $\forall g \in G$.

Weak equivalence (Weak containment)

$\rho \prec \eta$ if $\forall \varepsilon > 0$ $\forall S \subset G$,

weak containment

$|S| < \infty$ and $\forall v \in \mathcal{H}_\rho$ $\exists$ $w_1, \dots, w_k \in \mathcal{H}_\eta$

such that

$$\left| (\rho(g)v, v) - \sum_{i=1}^{k} (\eta(g)w_i, w_i) \right| < \varepsilon$$

$(\rho(g)v, v)$ – matrix coefficient.

$$\rho \approx \eta \iff \rho \prec \eta, \ \eta \prec \rho$$

weak equivalence

Weak equivalence is much weaker equivalence relation than equivalence.

Theorem (Hulanicki – Reiter) TFAE

(i) G is amenable

(ii) $\mathbb{1}_G \prec \lambda_G$ regular representation

trivial repres.

(iii) $\rho \prec \lambda_G$ for every unitary representation ρ of G.

λ_G repres. in $\ell^2(G)$, for $f \in \ell^2(G)$

$$(\lambda_G(g) f)(x) = f(x^{-1} g)$$

Theorem – Definition. (Kazhdan)

A group G has property (T) if whenever a unitary representation $(\rho, \mathcal{H})$ of G weakly contains $\mathbb{1}_G$, it contains $\mathbb{1}_G$.

trivial representation

$$\mathbb{1}_G \prec \rho \Rightarrow \mathbb{1}_G \subset \rho$$

subrepresentation.

Spectra of representations.

$$m = \sum_{g \in G} c_g \, g \in \mathbb{C}[G] \qquad c_g \in \mathbb{C}$$

$$\rho(m) = \sum_{g} c_g \, \rho(g) \in B(\mathcal{H}_\rho)$$

↑ Hecke type operator

$$sp(\rho(m)) = \,?$$

Example: $G = \langle S \rangle$, $\quad S = S^{-1}$

$$m = \frac{1}{|S|} \sum_{g \in S} g$$

$\rho = \lambda_G$ – regular representation

$sp(\lambda_G(m))$ = spectrum of Cayley graph of G.

Proposition. Let ρ, η be two unitary representations of G. TFAE

1) $\rho \prec \eta$

2) $sp(\rho(m)) \subset sp(\eta(m))$ for all $m \in \mathbb{C}[G]$.

3) There exists a surjective homomorphism $\varphi : C^*_\eta \twoheadrightarrow C^*_\rho$ such that $\varphi(\eta(g)) = \rho(g)$ for all $g \in G$.

Here C^*_ρ is a C^*-algebra generated by ρ (i.e. the norm closure of the operators $\rho(m)$, $m \in \mathbb{C}[G]$.

$$C^*_\rho = \overline{\rho(\mathbb{C}[G])} \quad \text{— } C^*\text{-algebra}$$

$$\mathcal{M}_\rho = \overline{\rho(\mathbb{C}[G])}^{\,weak} \quad \text{— von Neumann algebra.}$$

II Examples of unitary representations.

a) Regular λ_G in $\mathcal{H} = \ell^2(G)$

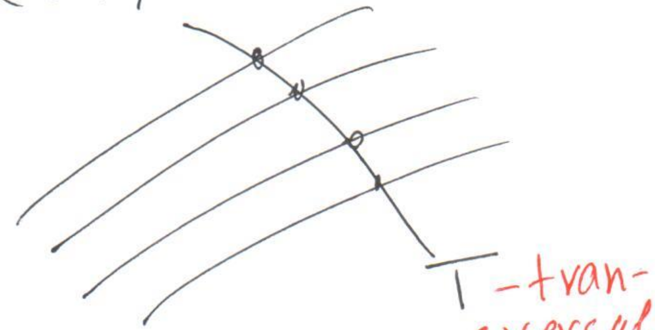
b) Quasiregular (permutational)

$H < G$, $\lambda_{G/H}$ in $\ell^2(G/H)$ for $f \in \ell^2(G/H)$

$$\left(\lambda_{G/H}(g)\right)(f)(hH) = f(g^{-1}hH)$$

c) Actional

$G \curvearrowright X$ in $\ell^2(X)$



$$\simeq \bigoplus_{x \in T} \lambda_{G/G_x}$$

$G_x = \{g \in G \mid gx = g\}$ - stabilizer

d) Koopman $\qquad (X, \mu)$ - probability space

$\qquad\qquad\qquad\qquad \uparrow$ invariant measure.

(G, X, μ)

$\underline{\underline{K}}$ in $L^2(X, \mu)$

$$\big(K(g)f\big)(x) = f(g^{-1}x)$$

if μ is quasi-invariant, then

$$\big(K(g)f\big)(x) = \sqrt{\frac{d\mu(g^{-1}x)}{d\mu(x)}}\; f(g^{-1}x)$$

$\uparrow$ Radon-Nikodim derivative.

e) in the above situation

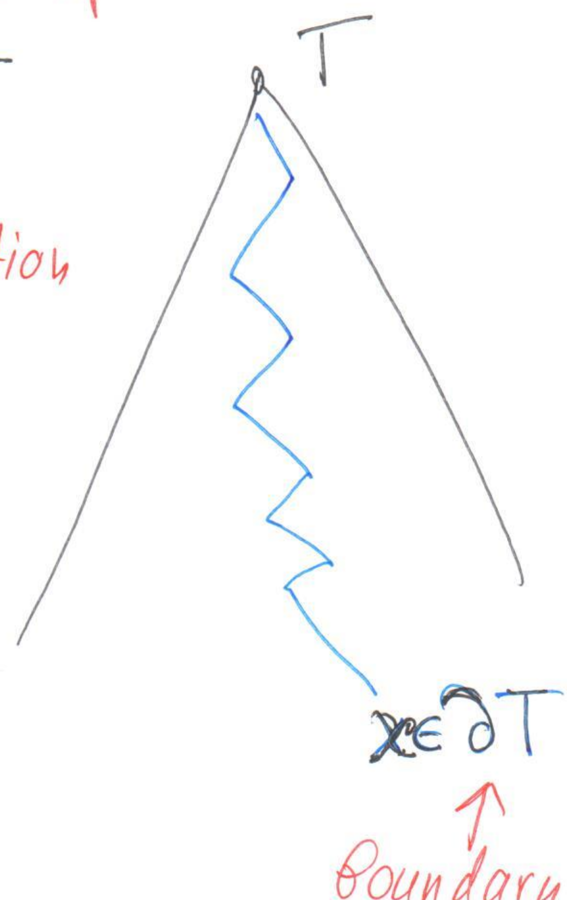
$\{\rho_x\}_{x \in X}$ - a family of permutational representations on orbits Gx (in $\ell^2(Gx)$).

Can view as a random family with

distrubution μ.

Example. A very different representations can be weakly equivalent.

G weakly branch group acting on rooted tree T



κ - Koopman representation in $L^2(\partial T, \mu)$

uniform Bernoulli measure

$\rho_x \simeq \lambda_{G/G_x}$

orbital repres.

~~classical~~ quasi-regular repr.

$T = T_2$

$\partial T = \{0,1\}^{\mathbb{N}}$

$\mu = \{\frac{1}{2}, \frac{1}{2}\}^{\otimes \mathbb{N}}$

Bartholdi - Grigorchuk 2000-2001.

Th. 1) For each $x \in \partial T$ ρ_x is irreducible.

2) For $x, y \in \partial T$, x and y not in the same orbit $\rho_x \not\sim \rho_y$.

(different orbits correspond not unitary equivalent representations).

3) $\forall x \in \partial T$

$$K \approx \rho_x$$

weakly equivalent

under assumption that G is amenable.

4) K is a direct sum of finite dimensional representations. $K = \bigoplus_{n=1}^{\infty} \rho_n$

$\dim \rho_n < \infty$.

f) Groupoid representation

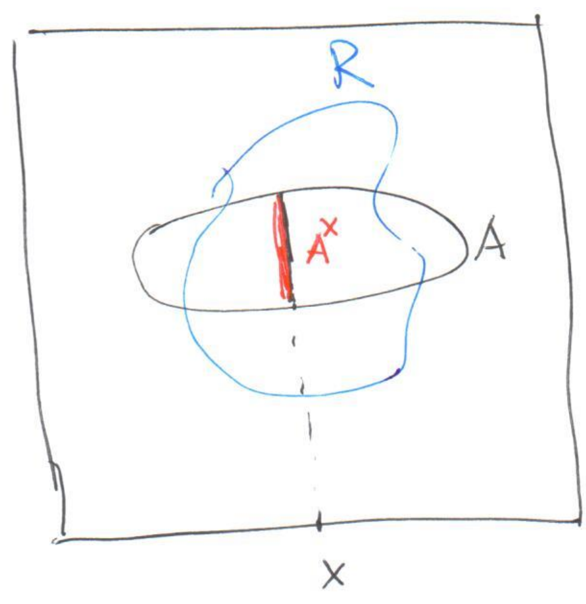
(G, X, μ), μ - quasi-invariant

R - orbit equivalence relation

$R \subset X \times X$

$R = \{(x,y) \mid x, y \in X,\ y = gx \text{ for some } g \in G\}$.

For $A \subset X \times X$



$A^x = A \cap (\{x\} \times X)$

ν_e - measure on R

$$\nu_e(A) = \int_X |A^x| \, d\mu(x)$$

$\boxed{\pi \text{ groupoid representation in } L^2(R, \nu_e)}$

$$(\pi(g) f)(x,y) = f(g^{-1}x, y).$$

π is unitary equivalent to

$$\int_X \rho_x \, d\mu(x) \quad \text{in } \mathcal{H} = \int_X \ell^2(Gx)\, d\mu(x).$$

Theorem (Artem Dudko & Gr 2017)

1) For an ergodic measure class preserving action of a countable group G on a standard probability space (X, μ) one has

$$\boxed{K \succ \pi}$$

and

$$\boxed{\pi \approx \rho_x}, \quad x \in X$$

μ-almost sure.

2) if additionally (G, X, μ) is hyperfinite, then $K \approx \pi$.

Corollary. Spectra of Schreier graphs Γ_x coincide μ-almost surely and coinside with the spectrum of groupoid representation. They are contained in the spectrum of Koopman representation and coinside with it if G is amenable.

Theorem (Dudko & Gr 17). The spectrum of the Cayley graph of the group $\mathcal{G} = \langle a, b, c, d \rangle$ of intermediate growth is $[-2, 0] \cup [2, 4]$ and coincides with the spectrum of the Schreier graph Γ_x of the action of $\mathcal{G}$ on the boundary ∂T_2 of binary rooted tree T_2 for any $x \in \partial T_2$ (i.e. with the spectrum of $\rho_x(a+b+c+d)$).

III Actions on rooted trees

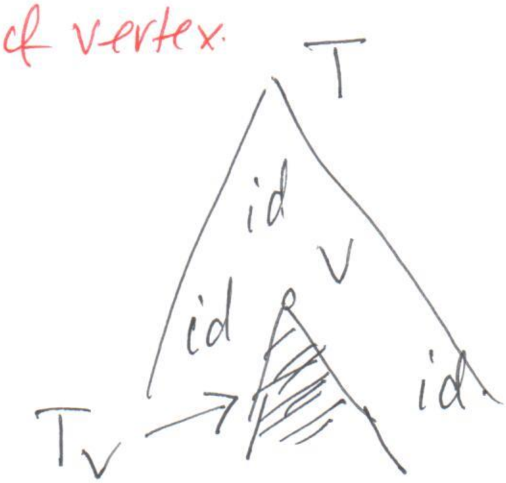
$G \leq \mathrm{Aut}\, T$, T - regular rooted tree.

$st_G(v)$ - stabilizer of vertex

$rist_G(v)$ - rigid stabilizer of vertex.

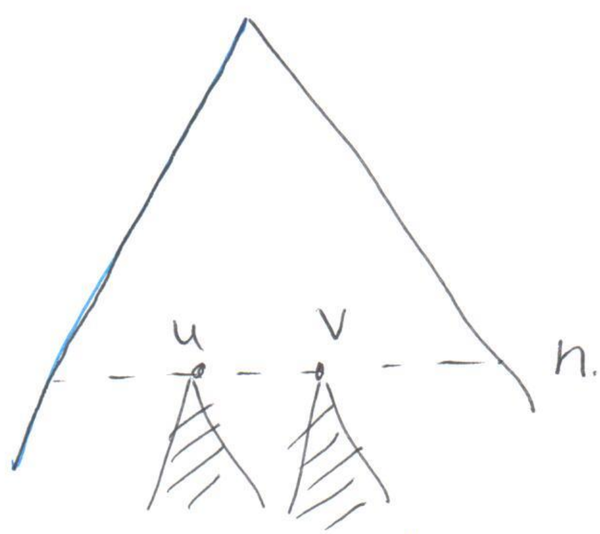
$$\parallel$$

$$\{g \in G \mid g = id \text{ on } T \setminus T_v\}$$



$$rist_G(n) = \langle rist_G(v) \mid |v| = n \rangle$$

$$= \prod_{|v|=n} rist_G(v)$$ - rigid stabilizer of level n.



Def. a) G is BRAHCH group if G act level transitive and $\forall n$ $[G : rist_G(n)] < \infty$.

b) G is weakly branch if it act level transitive on T and

$$\forall n, \quad \mathrm{rist}_G(n) \neq \{1\}.$$

Examples. $G = \langle a, b, c, d \rangle$ "first" group

$\tilde{G} = \langle a, \tilde{b}, \tilde{c}, \tilde{d} \rangle$ "first" overgroup (S. Samarakoon talk)

$IMG(z^2+i)$, are branch!

$\mathcal{H}^{(3)}$ - Hanoi tower group

Basilica $= IMG(z^2-1)$ is weakly branch

Branch $\subset$ weakly branch

Hanoi $\mathcal{H}^{k}$, $k \geq 4$ are weakly branch at least.

Q. Are $\mathcal{H}^{(k)}$, $k \geq 4$ branch?

Theorem (Dudko - Gr) For a countable weakly branch group acting on spherically homogeneous rooted tree T the centralizer

$$C_{Bij(\partial T)}(G) = \{1\} \quad (*)$$

is trivial.

$Bij(\partial T)$ - the group of all bijections of the boundary ∂T.

Recall: ∂T is homeomorphic to a Cantor set.

History 1) $C(G) = \{1\}$

2) $C_{Aut\,T}(G) = \{1\}$

and now $(*)$.

Question. Which groups can be embedded into $\mathrm{Bij}(X)$, $|X| = 2^{\aleph_0}$? so that the centralizer will be trivial?

(IV) Subexponentially bounded actions on trees.

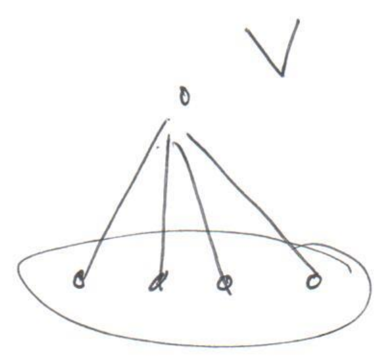
$$g \in \mathrm{Aut}\, T_d \longleftrightarrow \rho(g) \in \mathrm{Sym}(d)^V$$

portrait

$v \in V$

$\sigma_v \in \mathrm{Sym}(d)$

S. SIDKI



σ_v – permutation of descenders of v

$$\alpha_g(n) = \#\{v \in V_n : \sigma_v \neq 1\}$$

activity function of $g \in \operatorname{Aut} T$

g of polynomial activity if

$$\alpha_g(n) \leq n^k \quad \text{for some } k$$

subexponential activity if

$$\alpha_g(n) \leq \lambda^n \text{ for each } \lambda > 1$$

G has subexponential activity if each $g \in G$ has.

Let $\mathcal{P} = \{p = (p_1, p_2, \dots, p_d) : p_i > 0,\ \sum p_i = 1\}$

– set of probability distributions on the alphabet of the tree.

$$\mathcal{P}^* = \{p \in \mathcal{P} : p_i \neq p_j, \quad \forall i \neq j\}$$

μ_p – Bernoulli measure on $\partial T \simeq \{1, 2, \dots, d\}^{\mathbb{N}}$ determined by vector p. It is quasi-invariant if G has subexponential activity!

Theorem. (Dudko-Gr.)

1) The Koopman representation κ_p $\forall p \in \mathcal{P}^*$ associated to the action of G on $(\partial T, \mu_p)$ is irreducible.

2) This representation is not unitary equivalent to any of the quasi-regular representations ρ_x, $x \in \partial T$.

3) Koopman representations associated to different $p \in \mathcal{P}^*$ are pairwise disjoint.

4) All ρ_x and κ_p, $x \in X$, $p \in \mathcal{P}^*$ are pairwise weakly equivalent.

V Factor representations.

Theorem (D - Gr). Each countable weakly branch group has $2^{\aleph_0}$ pairwise distinct not quasi-equivalent factor representations of type II_1

ρ - unitary representation

$\mathcal{M}_\rho$ - ~~a~~ von Neuman algebra (generated by operators $\rho(g), g \in G$.

$\mathcal{M}_\rho$ is a factor if $\mathcal{C}(\mathcal{M}_\rho) = \{\lambda I\}$

center

scalar operators

ρ is a factor representation if $\mathcal{M}_\rho$ is a factor.

$\rho \sim \eta$ if there is an isomorphism

quasi-equivalence

$$\varphi : M_\rho \longrightarrow M_\eta$$

s.t.

$$\varphi(\rho(g)) = \eta(g) \quad \forall g \in G$$

Definition. A character on a group G is a function $\chi : G \to \mathbb{C}$ s.t.

1) $\chi(e) = 1$
2) χ is constant on conjugacy classes
3) $\forall n$, $\forall g_1, \dots, g_n \in G$ the matrix

$$\left(\chi(g_i g_j^{-1})\right)_{i,j=1}^{n}$$

is positively definite.

$\mathcal{X}(G)$ – simplex of characters

$\mathcal{X}^{ex}(G)$ – extreme point (indecomposable characters)

Fact: indecomposable characters are in bijection with classes of quasi-equivalence of finite type factor representations.

$$|\mathcal{X}^{ex}(G)| = \begin{cases} 2 \\ \aleph_0 \\ 2^{\aleph_0} \end{cases}$$

(Higman-Thomson groups)
(Some inductive limits of finite groups)
S_∞
weakly branch groups.

VI Perfectly non-free actions.

(G, X, μ), μ - invariant

$Fix(g) = \{x \in X : g(x) = x\}$ set of fix points.

Action is (essentially) free if

$$\mu(Fix(g)) = 0 \quad \forall g \in G, g \neq 1.$$

We are interested in non free actions!

Fact: Function

$$\chi(g) = \mu(\mathrm{Fix}(g)), \quad g \in G.$$

is a character.

$$(G, X, \Sigma, \mu) \qquad (**)$$

Σ – sigma algebra of measurable subsets of X.

Definition. The action $(**)$ is:

i) totally non-free if the collection

A.M. VERSHIK

of sets $\mathrm{Fix}(g), g \in G$ and sets of zero measure generate the sigma algebra Σ.

ii) extremely non-free if

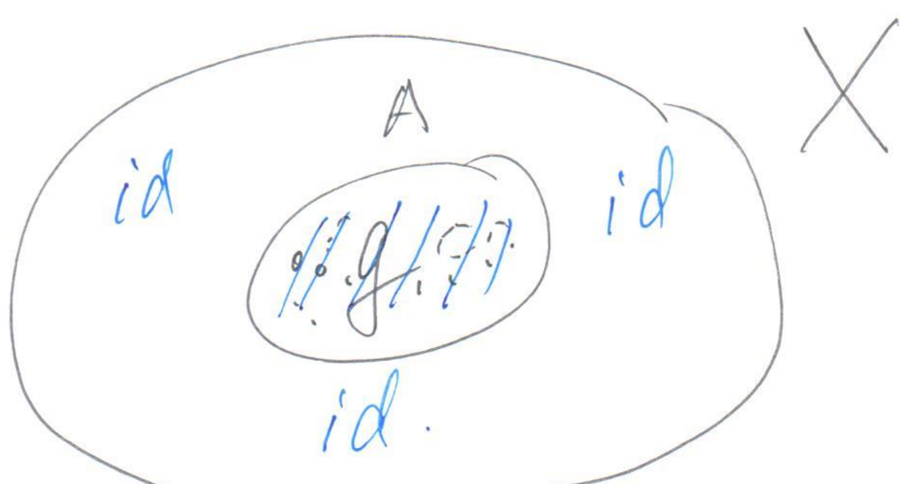
$\mathrm{St}_G(x) \neq \mathrm{St}_G(y)$ for $x \neq y$

μ- almost sure.

iii) *absolutely non-free* if $\forall A \subset X$ $\forall \varepsilon > 0$ $\exists g \in G$ s.t. $\mu(\mathrm{Fix}(g) \Delta A) < \varepsilon$.

(iv) *perfectly non-free* if there is a countable family $\{A_i\}_{i=1}^{\infty}$ such that it together with the sets of zero measure generates Σ and for each A_i the G_{A_i}-orbit $\{gx : g \in G_{A_i}\}$ is infinite for μ-almost all $x \in X$.

$$G_A = \{g \in G : \mathrm{supp}(g) \subseteq A\}$$



ANF $\Rightarrow$ PNF $\Rightarrow$ TNF $\Leftrightarrow$ ENF.

Th. ($\mathcal{D}$-Gr). Let (G, X, Σ, μ) be ergodic, measure-preserving and perfectly non-free. Let π be groupoid representation (in $L^2(\mathcal{R}, \nu)$). Then π is a factor representation and the corresponding character $\chi(g) = \mu(\mathrm{Fix}(g))$, $g \in G$ is indecomposable.

Th. ($\mathcal{D}$-Gr) For any weakly branch group G its action on $(\partial T, \mu)$ is absolutely non-free.