Grigorchuk-Gupta-Sidki groups as a source for Beauville surfaces

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Beauville surfaces and groups

Definition

A Beauville surface of unmixed type is a compact complex surface which is the quotient of a product $C_1 \times C_2$ of two algebraic curves C_1 and C_2 of genera at least 2 by the action of a finite group G acting freely by holomorphic transformations, in such a way that:

(i) $C_i/G \cong \mathbb{P}_1(\mathbb{C})$ for $i = 1, 2$.

(ii) The covering map $C_i \to C_i/G$ is ramified over three points for $i = 1, 2$.

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Group-theoretical reformulation

Let G be a group. For every $x, y \in G$, we define

$$\Sigma(x,y) = \bigcup_{g \in G} \left(\langle x \rangle^g \cup \langle y \rangle^g \cup \langle xy \rangle^g \right).$$

Theorem (Bauer, Catanese, Grunewald)

A finite group G is a Beauville group if and only if

(i) *G is a 2-generator group.*

(ii) *G has two sets of generators, $\{x_1, y_1\}$ and $\{x_2, y_2\}$, such that*

$$\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = 1.$$

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We say that $\{x_1, y_1\}$ and $\{x_2, y_2\}$ form a Beauville structure for G.

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Known results

In the sequel, all groups will be finite and p will always be a prime.

Theorem (Catanese, 2000)

An abelian group is a Beauville group if and only if it is isomorphic to $C_n \times C_n$, where $n > 1$ and $\gcd(n, 6) = 1$.

Corollary

There are no abelian Beauville 2-groups or 3-groups.

Theorem (Bauer, Catanese, Grunewald, 2005; Fuertes, González-Diez, 2009)

The alternating groups A_n, for $n \geq 6$, and the symmetric groups S_n, for $n \geq 5$, are Beauville groups.

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The groups $SL(2, q)$, $PSL(2, q)$, for all prime powers $q > 5$, are Beauville groups.

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Known results: p-groups

- After abelian groups, the most natural class of finite groups to consider are nilpotent groups.
- The study of nilpotent Beauville groups is reduced to that of Beauville p-groups [Barker, Boston, Fairbairn, 2012].

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- The smallest non-abelian Beauville p-groups for $p = 2$ and $p = 3$ are of order 2^7 and 3^5.
- For $p \geq 5$, the smallest non-abelian Beauville p-group is of order p^3.

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Known results: infinite families of Beauville p-groups

Theorem (Barker, Boston, Fairbairn, 2012)

There are non-abelian Beauville p-groups of order p^n for every $p \geq 5$ and $n \geq 3$.

Theorem (Barker, Boston, Peyerimhoff, Vdovina, 2015)

There are infinitely many Beauville 2-groups.

Theorem (Stix, Vdovina, 2015)

For every prime p, there are infinitely many quotients of the ordinary triangle group

$$\Delta_{m,n,r} = \langle x, y \mid x^{p^m} = y^{p^n} = (xy)^{p^r} = 1 \rangle$$

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Known results: Beauville structures in p-central quotients

For any group G, the normal series

$$G = \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) \geq \ldots$$

given by $\lambda_n(G) = [\lambda_{n-1}(G), G]\lambda_{n-1}(G)^p$ for $n > 1$ is called the p-central series of G.

Conjecture (Boston):

If $p \geq 5$ and F is either

- the free group $\langle x, y \rangle$ or
- the free product $\langle x, y \mid x^p, y^p \rangle$,

then the p-central quotients $F/\lambda_n(F)$ are Beauville groups.

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Theorem (ŞG, 2016)

Let $F = \langle x, y \rangle$ be the free group. Then

$$F/\lambda_n(F) \text{ is a Beauville group} \iff p \geq 5 \text{ and } n \geq 2.$$

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Let $F = \langle x, y \mid x^p, y^p \rangle$ be the free product of two cyclic groups order p. Then

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Known results: quotients of the Nottingham group

The Nottingham group $\mathcal{N}$ is the group of normalized automorphisms of the ring $\mathbb{F}_p[[t]]$ of formal power series for a prime p. Namely,

$$\mathcal{N} = \{f \in \operatorname{Aut}\mathbb{F}_p[[t]] \mid f(t) \equiv t \pmod{t^2}\}.$$

If $\mathcal{N}_i = \{f \in \mathcal{N} \mid f(t) \equiv t \pmod{t^{i+1}}\}$, then $\mathcal{N}_i \trianglelefteq \mathcal{N}$ for all $i \geq 1$.

Notation: $z_m = p^m + p^{m-1} + \cdots + 2$ for $m \geq 1$, $z_0 = 2$.

Theorem (Fernández-Alcober, ŞG, 2016)

- *If $p \geq 5$ then all quotients $\mathcal{N}/\mathcal{N}_i$ with $i \geq 3$ are Beauville groups, except for those of the form $i = z_m$.*
- *If $p = 3$ then all quotients $\mathcal{N}/\mathcal{N}_i$ with $i \geq 6$ are Beauville groups, except for those of the form $i = z_m$.*

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2. Grigorchuk-Gupta-Sidki groups

Definitions

$\mathcal{T}$= p-adic tree: a tree with root $\emptyset$ such that every vertex has p 'children' for a prime p.

L_n = vertices at distance n from the root.

$\operatorname{Aut}\mathcal{T}$ = the group of automorphisms of $\mathcal{T}$.

(an automorphism of $\mathcal{T}$ is a bijection of the vertices that preserves incidence.)

$\operatorname{st}(n) = \{f \in \operatorname{Aut}\mathcal{T} \mid f(u) = u \ \ \forall u \in L_n\}$ is the nth level stabilizer.

If $G \leq \operatorname{Aut}\mathcal{T}$ then $\operatorname{st}_G(n) = \operatorname{st}(n) \cap G$, and $\operatorname{st}_G(n) \trianglelefteq G$.

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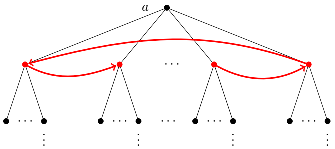
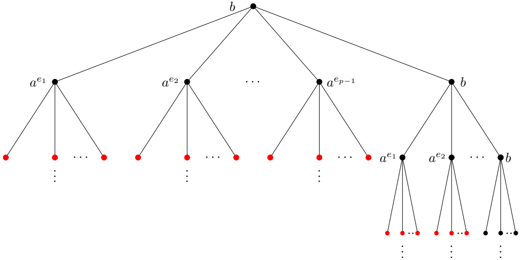
GGS-groups

$a=$ a rooted automorphism permuting the vertices hanging from the root according to the permutation $(1\,2\,\ldots\,p)$.



$b=$ an automorphism in st(1) acting on p main subtrees as $a^{e_1}, a^{e_2}, \ldots, a^{e_{p-1}}, b$ according to a given non-zero vector $\mathbf{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$.



Definition

$G = \langle a, b\rangle \leq \operatorname{Aut}\mathcal{T}$ is called the *Grigorchuk-Gupta-Sidki group* (GGS-group for short) corresponding to the *defining vector* $\mathbf{e}$.

- If G is a GGS-group, then the quotients $G/\operatorname{st}_G(n)$ are finite p-groups generated by two elements order p.
- If $p = 2$ then there is only one GGS-group, which is isomorphic to the infinite dihedral group D_∞. No finite quotient of D_∞ is a Beauville group [Bauer, Catanese, Grunewald, 2005].
- In the remainder, we assume that p is an odd prime.
- G is a periodic group if and only if $\sum_{i=1}^{p-1} e_i = 0$ [Vovkivsky, 2000].
- The property of being Beauville for the quotients $G/\operatorname{st}_G(n)$ depends on whether G is periodic or not.

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Non-periodic case

Let G be non-periodic GGS-group.

For every $n \geq 2$, the quotient $G_n = G/\operatorname{st}_G(n)$ have p+1 maximal subgroups:

$\langle a, G_n' \rangle$, $\langle b, G_n' \rangle$, and $M_{n,i} = \langle ab^i, G_n' \rangle$ for every $1 \leq i \leq p-1$.

Proposition [ŞG, Uria-Albizuri, 2018]

1. All elements in $M_{n,i} \smallsetminus G_n'$ are of order p^n for every $1 \leq i \leq p-1$.
2. For any $g, h \in \bigcup_{i=1}^{p-1} M_{n,i} \smallsetminus G_n'$,

$$\langle g^{p^{n-1}} \rangle = \langle h^{p^{n-1}} \rangle.$$

Theorem [ŞG, Uria-Albizuri, 2018]

Let G be a non-periodic GGS-group over the p-adic tree. Then the quotient $G/\operatorname{st}_G(n)$ is not a Beauville group for any $n \geq 1$.

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Theorem

Let G be a periodic GGS-group. Then

$$G/\operatorname{st}_G(2) \text{ is a Beauville group} \iff p \geq 5.$$

Theorem

Let G be a periodic GGS-group. Then $G/\operatorname{st}_G(3)$ is a Beauville group. Furthermore,

- *if $p \geq 5$ then $\{a^{-2}, ab\}$ and $\{ab^2, b\}$,*
- *if $p = 3$ then $\{a, b\}$ and $\{av, b^2u\}$, where u and v are elements of order 3 with some property,*

form a Beauville structure for $G/\operatorname{st}_G(3)$.

Question: What can we say about the quotients $G/\operatorname{st}_G(n)$ if $n > 3$?

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Periodic case – lifting Beauville structures

Proposition (Fuertes, Jones, 2011)

Let G be a finite group and let $\{x_1, y_1\}$ and $\{x_2, y_2\}$ be two sets of generators of G. Assume that, for a given $N \trianglelefteq G$, the following hold:

1. $\{x_1N, y_1N\}$ *and* $\{x_2N, y_2N\}$ *form a Beauville structure for* G/N.
2. $o(g) = o(gN)$ *for every* $g \in \{x_1, y_1, x_1y_1\}$.

Then $\{x_1, y_1\}$ *and* $\{x_2, y_2\}$ *form a Beauville structure for* G.

Theorem [ŞG, Uria-Albizuri, 2018]

Let G be a periodic GGS-group over the p-adic tree. Then the quotient $G/\mathrm{st}_G(n)$ is a Beauville group if $p \geq 5$ and $n \geq 2$, or $p = 3$ and $n \geq 3$.

Thus, a periodic GGS-group is a source for the construction of an infinite series of Beauville p-groups.

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