Almost Automorphism Groups of Trees

and Completions of Thompson's V

Waltraud Lederle

Ben Gurion University of the Negev

June 26, 2018

Let $\mathcal{T}$ be a tree without leaves and without isolated points in the boundary.

Let $\mathcal{T}$ be a tree without leaves and without isolated points in the boundary.

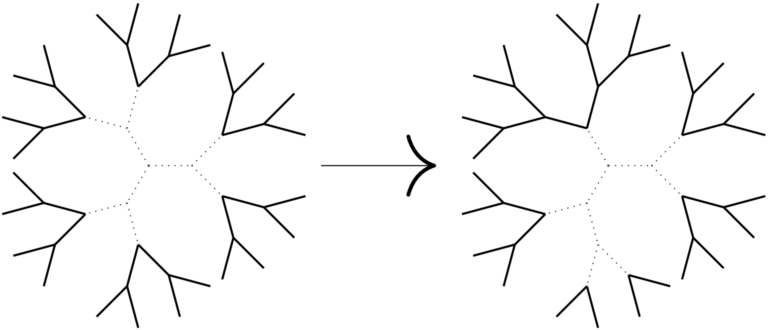
Tree almost automorphism:

Let $\mathcal{T}$ be a tree without leaves and without isolated points in the boundary.

Tree almost automorphism: Equivalence class of forest isomorphisms $\mathcal{T} \setminus T_1 \to \mathcal{T} \setminus T_2$ for finite subtrees T_1, T_2 of $\mathcal{T}$

Let $\mathcal{T}$ be a tree without leaves and without isolated points in the boundary.

Tree almost automorphism: Equivalence class of forest isomorphisms $\mathcal{T} \setminus T_1 \to \mathcal{T} \setminus T_2$ for finite subtrees T_1, T_2 of $\mathcal{T}$



Equivalence relation:

Equivalence relation:
$(\varphi\colon \mathcal{T} \setminus T_1 \to \mathcal{T} \setminus T_2) \sim (\varphi'\colon \mathcal{T} \setminus T_1' \to \mathcal{T} \setminus T_2')$ iff
there exists $T \supset T_1 \cup T_1'$ such that $\varphi|_{\mathcal{T}\setminus T} = \varphi'|_{\mathcal{T}\setminus T}$.

Equivalence relation:
$(\varphi\colon \mathcal{T} \setminus T_1 \to \mathcal{T} \setminus T_2) \sim (\varphi'\colon \mathcal{T} \setminus T_1' \to \mathcal{T} \setminus T_2')$ iff
there exists $T \supset T_1 \cup T_1'$ such that $\varphi|_{\mathcal{T} \setminus T} = \varphi'|_{\mathcal{T} \setminus T}$.
Then they induce the same homeomorphism of $\partial \mathcal{T}$.

Equivalence relation:
$(\varphi\colon \mathcal{T} \setminus T_1 \to \mathcal{T} \setminus T_2) \sim (\varphi'\colon \mathcal{T} \setminus T_1' \to \mathcal{T} \setminus T_2')$ iff
there exists $T \supset T_1 \cup T_1'$ such that $\varphi|_{\mathcal{T} \setminus T} = \varphi'|_{\mathcal{T} \setminus T}$.
Then they induce the same homeomorphism of $\partial \mathcal{T}$.

AAut($\mathcal{T}$): the group of almost automorphisms of the tree $\mathcal{T}$

Equivalence relation:
$(\varphi\colon \mathcal{T} \setminus T_1 \to \mathcal{T} \setminus T_2) \sim (\varphi'\colon \mathcal{T} \setminus T_1' \to \mathcal{T} \setminus T_2')$ iff
there exists $T \supset T_1 \cup T_1'$ such that $\varphi|_{\mathcal{T} \setminus T} = \varphi'|_{\mathcal{T} \setminus T}$.
Then they induce the same homeomorphism of $\partial\mathcal{T}$.

$\mathrm{AAut}(\mathcal{T})$: the group of almost automorphisms of the tree $\mathcal{T}$

Topology: $\mathrm{Aut}(\mathcal{T}) \hookrightarrow \mathrm{AAut}(\mathcal{T})$ continuous and open

Equivalence relation:
$(\varphi\colon \mathcal{T} \setminus T_1 \to \mathcal{T} \setminus T_2) \sim (\varphi'\colon \mathcal{T} \setminus T_1' \to \mathcal{T} \setminus T_2')$ iff
there exists $T \supset T_1 \cup T_1'$ such that $\varphi|_{\mathcal{T}\setminus T} = \varphi'|_{\mathcal{T}\setminus T}$.
Then they induce the same homeomorphism of $\partial\mathcal{T}$.

$\mathrm{AAut}(\mathcal{T})$: the group of almost automorphisms of the tree $\mathcal{T}$

Topology: $\mathrm{Aut}(\mathcal{T}) \hookrightarrow \mathrm{AAut}(\mathcal{T})$ continuous and open

With this topology $\mathrm{AAut}(\mathcal{T})$ is

Equivalence relation:
$(\varphi\colon \mathcal{T} \setminus T_1 \to \mathcal{T} \setminus T_2) \sim (\varphi'\colon \mathcal{T} \setminus T_1' \to \mathcal{T} \setminus T_2')$ iff
there exists $T \supset T_1 \cup T_1'$ such that $\varphi|_{\mathcal{T} \setminus T} = \varphi'|_{\mathcal{T} \setminus T}$.
Then they induce the same homeomorphism of $\partial\mathcal{T}$.

$\mathrm{AAut}(\mathcal{T})$: the group of almost automorphisms of the tree $\mathcal{T}$

Topology: $\mathrm{Aut}(\mathcal{T}) \hookrightarrow \mathrm{AAut}(\mathcal{T})$ continuous and open

With this topology $\mathrm{AAut}(\mathcal{T})$ is

- locally compact,

Equivalence relation:
$(\varphi\colon \mathcal{T} \setminus T_1 \to \mathcal{T} \setminus T_2) \sim (\varphi'\colon \mathcal{T} \setminus T_1' \to \mathcal{T} \setminus T_2')$ iff
there exists $T \supset T_1 \cup T_1'$ such that $\varphi|_{\mathcal{T} \setminus T} = \varphi'|_{\mathcal{T} \setminus T}$.
Then they induce the same homeomorphism of $\partial\mathcal{T}$.

$\mathrm{AAut}(\mathcal{T})$: the group of almost automorphisms of the tree $\mathcal{T}$

Topology: $\mathrm{Aut}(\mathcal{T}) \hookrightarrow \mathrm{AAut}(\mathcal{T})$ continuous and open

With this topology $\mathrm{AAut}(\mathcal{T})$ is

- locally compact,
- totally disconnected.

Equivalence relation:
$(\varphi\colon \mathcal{T} \setminus T_1 \to \mathcal{T} \setminus T_2) \sim (\varphi'\colon \mathcal{T} \setminus T_1' \to \mathcal{T} \setminus T_2')$ iff
there exists $T \supset T_1 \cup T_1'$ such that $\varphi|_{\mathcal{T}\setminus T} = \varphi'|_{\mathcal{T}\setminus T}$.
Then they induce the same homeomorphism of $\partial\mathcal{T}$.

AAut($\mathcal{T}$): the group of almost automorphisms of the tree $\mathcal{T}$

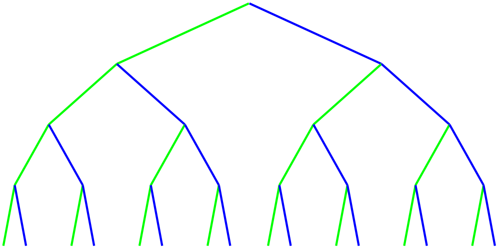
Topology: Aut($\mathcal{T}$) $\hookrightarrow$ AAut($\mathcal{T}$) continuous and open

With this topology AAut($\mathcal{T}$) is

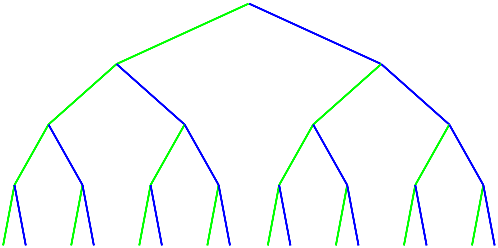
- locally compact,
- totally disconnected.

Motivation: For $\mathcal{T}$ regular, gave the first example of simple group without lattices (Kapoudjian; Bader–Caprace–Gelander–Mozes).

Thompson's group V = the group of all **colour preserving** almost automorphisms of

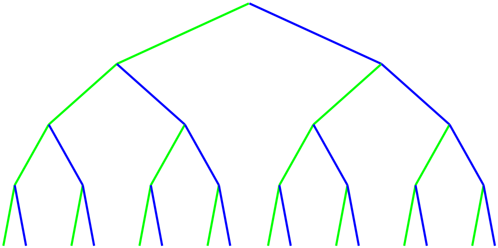


Thompson's group V = the group of all **colour preserving** almost automorphisms of



One of the first examples of a fininitely prestented, simple, infinite group (Thompson 1965)

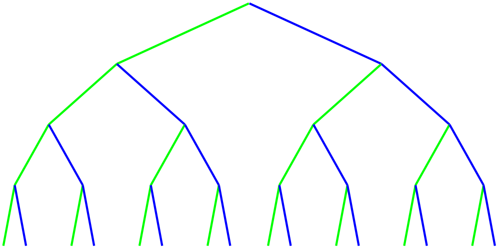
Thompson's group V = the group of all **colour preserving** almost automorphisms of



One of the first examples of a fininitely prestented, simple, infinite group (Thompson 1965)

Completion of a group Γ: locally compact group containing a dense copy of Γ, e.g. $\mathbb{Q} \leq \mathbb{R}$ or $\mathbb{Q} \leq \mathbb{Q}_p$

Thompson's group V = the group of all **colour preserving** almost automorphisms of



One of the first examples of a fininitely prestented, simple, infinite group (Thompson 1965)

Completion of a group Γ: locally compact group containing a dense copy of Γ, e.g. $\mathbb{Q} \leq \mathbb{R}$ or $\mathbb{Q} \leq \mathbb{Q}_p$

$V \leq \mathrm{AAut}(\mathcal{T})$ is dense, this gives a completion of V

Groupoid: Small category where every morphism is an isomorphism

Groupoid: Small category where every morphism is an isomorphism
étale: $Obj(\mathcal{G})$ and $Mor(\mathcal{G})$ Hausdorff topological spaces, $Obj(\mathcal{G})$ Cantor space, all structure maps continuous, source and range open and local homeomorphisms

Groupoid: Small category where every morphism is an isomorphism
étale: $Obj(\mathcal{G})$ and $Mor(\mathcal{G})$ Hausdorff topological spaces, $Obj(\mathcal{G})$ Cantor space, all structure maps continuous, source and range open and local homeomorphisms
Bisection: Clopen subset $\mathcal{U} \subset Mor(\mathcal{G})$ s.t. $s, r \colon \mathcal{U} \to Obj(\mathcal{G})$ homeomorphisms

Groupoid: Small category where every morphism is an isomorphism
étale: $Obj(\mathcal{G})$ and $Mor(\mathcal{G})$ Hausdorff topological spaces, $Obj(\mathcal{G})$ Cantor space, all structure maps continuous, source and range open and local homeomorphisms
Bisection: Clopen subset $\mathcal{U} \subset Mor(\mathcal{G})$ s.t. $s, r : \mathcal{U} \to Obj(\mathcal{G})$ homeomorphisms
Topological full group: (defined by Matui)
$TFG(\mathcal{G}) := \{r \circ (s|_{\mathcal{U}})^{-1} \in \mathrm{Homeo}(Obj(\mathcal{G})) \mid \mathcal{U} \text{ bisection}\}$

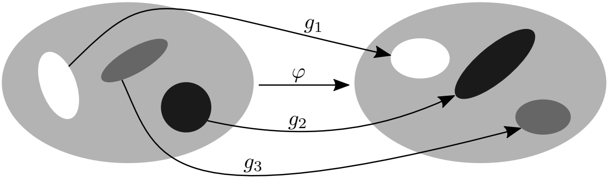
Groupoid: Small category where every morphism is an isomorphism

étale: $Obj(\mathcal{G})$ and $Mor(\mathcal{G})$ Hausdorff topological spaces, $Obj(\mathcal{G})$ Cantor space, all structure maps continuous, source and range open and local homeomorphisms

Bisection: Clopen subset $\mathcal{U} \subset Mor(\mathcal{G})$ s.t. $s, r \colon \mathcal{U} \to Obj(\mathcal{G})$ homeomorphisms

Topological full group: (defined by Matui)

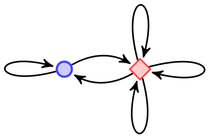
$TFG(\mathcal{G}) := \{r \circ (s|_{\mathcal{U}})^{-1} \in \text{Homeo}(Obj(\mathcal{G})) \mid \mathcal{U} \text{ bisection}\}$



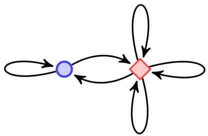
One-sided shift:

One-sided shift: start with directed graph

One-sided shift: start with directed graph

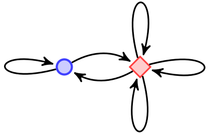


One-sided shift: start with directed graph



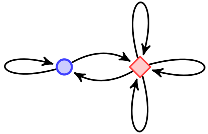
$X = \{\text{set of all half-infinite paths}\}$

One-sided shift: start with directed graph



$X = \{\text{set of all half-infinite paths}\}$
$\sigma : X \to X$, $(e_0, e_1, e_2, \dots) \mapsto (e_1, e_2, \dots)$

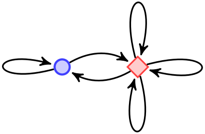
One-sided shift: start with directed graph



$X = \{\text{set of all half-infinite paths}\}$
$\sigma \colon X \to X$, $(e_0, e_1, e_2, \dots) \mapsto (e_1, e_2, \dots)$

Get an étale groupoid:

One-sided shift: start with directed graph

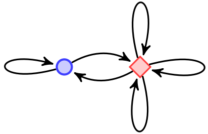


$X = \{$set of all half-infinite paths$\}$
$\sigma: X \to X, (e_0, e_1, e_2, \dots) \mapsto (e_1, e_2, \dots)$

Get an étale groupoid:

$Obj(\mathcal{G}_{\mathfrak{g}}) = X$

One-sided shift: start with directed graph

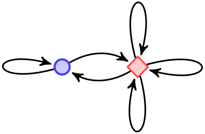


$X = \{\text{set of all half-infinite paths}\}$
$\sigma\colon X \to X,\ (e_0, e_1, e_2, \dots) \mapsto (e_1, e_2, \dots)$

Get an étale groupoid:

$Obj(\mathcal{G}_{\mathfrak{g}}) = X$
$Mor(\mathcal{G}_{\mathfrak{g}}) = \{(x, n-m, y) \in X \times \mathbb{Z} \times X \mid \sigma^n(x) = \sigma^m(y)\}$

One-sided shift: start with directed graph

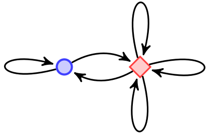


$X = \{\text{set of all half-infinite paths}\}$
$\sigma\colon X \to X,\ (e_0, e_1, e_2, \dots) \mapsto (e_1, e_2, \dots)$

Get an étale groupoid:

$Obj(\mathcal{G}_{\mathfrak{g}}) = X$
$Mor(\mathcal{G}_{\mathfrak{g}}) = \{(x, n-m, y) \in X \times \mathbb{Z} \times X \mid \sigma^n(x) = \sigma^m(y)\}$
$s(x, n-m, y) = y,\ r(x, n-m, y) = x,$
$(x, n-m, y) \cdot (y, m-k, z) = (x, n-k, z)$

One-sided shift: start with directed graph



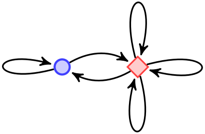
$X = \{\text{set of all half-infinite paths}\}$
$\sigma : X \to X,\ (e_0, e_1, e_2, \dots) \mapsto (e_1, e_2, \dots)$

Get an étale groupoid:

$Obj(\mathcal{G}_\mathfrak{g}) = X$
$Mor(\mathcal{G}_\mathfrak{g}) = \{(x, n-m, y) \in X \times \mathbb{Z} \times X \mid \sigma^n(x) = \sigma^m(y)\}$
$s(x, n-m, y) = y,\ r(x, n-m, y) = x,$
$(x, n-m, y) \cdot (y, m-k, z) = (x, n-k, z)$

Matui: $TFG(\mathcal{G}_\mathfrak{g})$ is finitely presented and has simple commutator subgroup.

One-sided shift: start with directed graph



$X = \{\text{set of all half-infinite paths}\}$
$\sigma\colon X \to X,\ (e_0, e_1, e_2, \dots) \mapsto (e_1, e_2, \dots)$

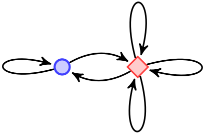
Get an étale groupoid:

$Obj(\mathcal{G}_{\mathfrak{g}}) = X$
$Mor(\mathcal{G}_{\mathfrak{g}}) = \{(x, n-m, y) \in X \times \mathbb{Z} \times X \mid \sigma^n(x) = \sigma^m(y)\}$
$s(x, n-m, y) = y,\ r(x, n-m, y) = x,$
$(x, n-m, y) \cdot (y, m-k, z) = (x, n-k, z)$

Matui: $TFG(\mathcal{G}_{\mathfrak{g}})$ is finitely presented and has simple commutator subgroup.

Thompson's V: topological full group coming from

One-sided shift: start with directed graph



$X = \{\text{set of all half-infinite paths}\}$
$\sigma \colon X \to X,\ (e_0, e_1, e_2, \dots) \mapsto (e_1, e_2, \dots)$

Get an étale groupoid:

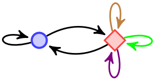
$Obj(\mathcal{G}_{\mathfrak{g}}) = X$
$Mor(\mathcal{G}_{\mathfrak{g}}) = \{(x, n-m, y) \in X \times \mathbb{Z} \times X \mid \sigma^n(x) = \sigma^m(y)\}$
$s(x, n-m, y) = y,\ r(x, n-m, y) = x,$
$(x, n-m, y) \cdot (y, m-k, z) = (x, n-k, z)$

Matui: $TFG(\mathcal{G}_{\mathfrak{g}})$ is finitely presented and has simple commutator subgroup.

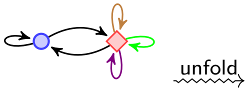
Thompson's V: topological full group coming from 

Other way to see the topological full group:

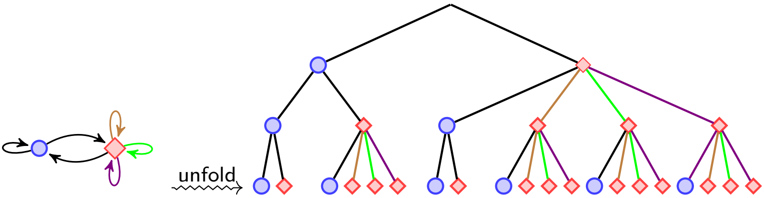
Other way to see the topological full group:



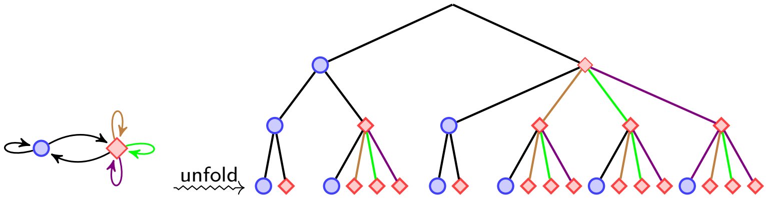
Other way to see the topological full group:



Other way to see the topological full group:

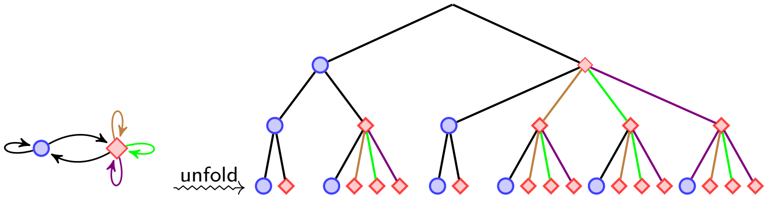


Other way to see the topological full group:



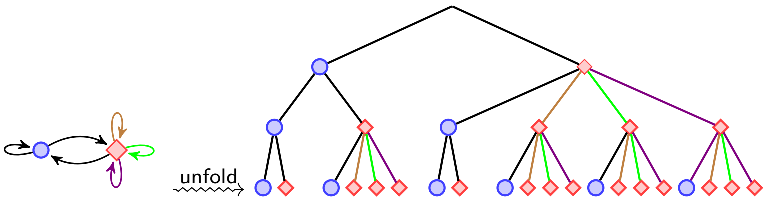
Top. full group

Other way to see the topological full group:



Top. full group = Group of all colour-preserving alm. aut.

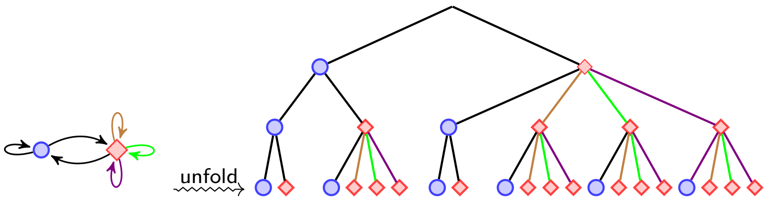
Other way to see the topological full group:



Top. full group = Group of all colour-preserving alm. aut.

Matsumoto: If graph is such that $\det(id - \text{adj. matrix}) = -1$, then top. full group is V.

Other way to see the topological full group:

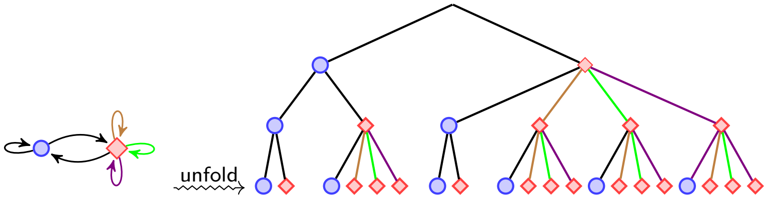


Top. full group = Group of all colour-preserving alm. aut.

Matsumoto: If graph is such that $\det(id - \text{adj. matrix}) = -1$, then top. full group is V.

$\mathrm{AAut}(\mathcal{T})$ has a locally compact topology

Other way to see the topological full group:



Top. full group = Group of all colour-preserving alm. aut.

Matsumoto: If graph is such that $\det(id - \text{adj. matrix}) = -1$, then top. full group is V.

$\mathrm{AAut}(\mathcal{T})$ has a locally compact topology $\rightsquigarrow$ completions of V

Definition (Glöckner): The **local prime content** of a t.d.l.c. group G is $\{p \text{ prime } \mid G \text{ has a pro-}p \text{ subgroup}\}$

Definition (Glöckner): The **local prime content** of a t.d.l.c. group G is $\{p \text{ prime } \mid G \text{ has a pro-}p \text{ subgroup}\}$

Let $F \leq \mathrm{Aut}(\mathfrak{g})$ fix all vertices,

Definition (Glöckner): The **local prime content** of a t.d.l.c. group G is $\{p \text{ prime } \mid G \text{ has a pro-}p \text{ subgroup}\}$

Let $F \leq \mathrm{Aut}(\mathfrak{g})$ fix all vertices, then F defines local permutations at vertices of $\mathcal{T}$.

Definition (Glöckner): The **local prime content** of a t.d.l.c. group G is $\{p \text{ prime } \mid G \text{ has a pro-}p \text{ subgroup}\}$

Let $F \leq \mathrm{Aut}(\mathfrak{g})$ fix all vertices, then F defines local permutations at vertices of $\mathcal{T}$.

$\mathrm{AAut}_F(\mathcal{T}) = \{[\varphi] \mid \varphi\colon \mathcal{T} \setminus T_1 \to \mathcal{T} \setminus T_2, \forall u \in \mathcal{T} \setminus T_1\colon \varphi|_{(u)} \in F\}$

Definition (Glöckner): The **local prime content** of a t.d.l.c. group G is $\{p \text{ prime } \mid G \text{ has a pro-}p \text{ subgroup}\}$

Let $F \leq \mathrm{Aut}(\mathfrak{g})$ fix all vertices, then F defines local permutations at vertices of $\mathcal{T}$.

$\mathrm{AAut}_F(\mathcal{T}) = \{[\varphi] \mid \varphi\colon \mathcal{T} \setminus T_1 \to \mathcal{T} \setminus T_2, \forall u \in \mathcal{T} \setminus T_1 \colon \varphi|_{(u)} \in F\}$
local prime content = prime divisors of $|F|$

Definition (Glöckner): The **local prime content** of a t.d.l.c. group G is $\{p \text{ prime } \mid G \text{ has a pro-}p \text{ subgroup}\}$

Let $F \leq \mathrm{Aut}(\mathfrak{g})$ fix all vertices, then F defines local permutations at vertices of $\mathcal{T}$.

$\mathrm{AAut}_F(\mathcal{T}) = \{[\varphi] \mid \varphi\colon \mathcal{T} \setminus T_1 \to \mathcal{T} \setminus T_2, \forall u \in \mathcal{T} \setminus T_1\colon \varphi|_{(u)} \in F\}$
local prime content $=$ prime divisors of $|F|$

Theorem (L. 2018)

For every finite, nonempty set of primes P, Thompson's V has a topologically simple completion of local prime content P.

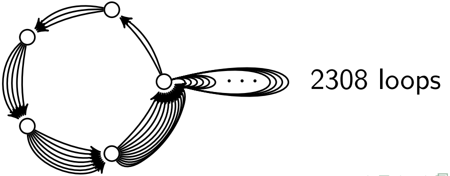
Definition (Glöckner): The **local prime content** of a t.d.l.c. group G is $\{p \text{ prime } \mid G \text{ has a pro-}p \text{ subgroup}\}$

Let $F \leq \mathrm{Aut}(\mathfrak{g})$ fix all vertices, then F defines local permutations at vertices of $\mathcal{T}$.

$\mathrm{AAut}_F(\mathcal{T}) = \{[\varphi] \mid \varphi\colon \mathcal{T} \setminus T_1 \to \mathcal{T} \setminus T_2, \forall u \in \mathcal{T} \setminus T_1\colon \varphi|_{(u)} \in F\}$
local prime content = prime divisors of $|F|$

Theorem (L. 2018)

For every finite, nonempty set of primes P, Thompson's V has a topologically simple completion of local prime content P.



Thank you!