Highly-arc-transitive digraphs and connections with group theory

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Highly-arc-transitive digraphs

A k-arc in a digraph Γ is a sequence $v_0, v_1, \ldots, v_k$ of vertices such that (v_i, v_{i+1}) is an arc in Γ for $i = 0, 1, \ldots, k-1$.

The digraph Γ is said to be k-arc-transitive if the automorphism group acts transitively on the set of k-arcs.

The digraph Γ is said to be *highly-arc-transitive* (or a *hat-digraph*) if it is k-arc-transitive for all $k \geq 0$.

The concept of highly-arc-transitive digraphs was introduced in 1993 by Cameron, Praeger and Wormald.

Related concepts

A *k-arc* in an undirected graph X is a $k+1$ tuple $(v_0, v_1, \ldots, v_k)$ of vertices such that v_i and v_{i+1} are adjacent for $i = 0, 1, \ldots, k-1$. If the automorphism group acts transitively on the k-arcs for all k then T is a regular tree.

An undirected connected graph X is said to be *distance-transitive* if for all pairs u, v, u', v' such that $d(u, v) = d(u', v')$ there is an automorphism g such that $gu = u'$ and $gv = v'$. Locally finite distance-transitive graph were classified, independently, by Macpherson (1982) and Ivanov (1983). They have infinitely many ends. Extended to graphs that are not locally finite by Hamann and Potts (2012).

The class of highly-arc-transitive digraphs is far richer and defies a simple classification.

Outline

In this talk I want first to show you how highly-arc-transitive digraphs crop up in group theoretical investigations.

In the second part I want to discuss some work on the structure of highly-arc-transitive digraphs. In their paper Cameron, Praeger and Wormald put forward many questions and conjectures that are now getting resolved one by one.

Examples

Regular directed trees are the most basic examples of hat-digraphs. Many closely related examples based on trees.

Also easy to construct examples with two ends.

Thinking about analogous concepts for undirected graph then one can ask if every highly-arc-transitive digraph must have more than one end.

A subdigraph with more than one end

Let Γ be a locally finite highly-arc-transitive digraph and $L = \ldots, v_{-1}, v_0, v_1, v_2, \ldots$ a directed line in Γ.
The set of descendants of the line is the set of all vertices u such that the exists a directed path starting in some v_i and ending in u.

The subdigraph spanned by the set of descendants of the line L has more than one end.

One-ended example

Start with $T = \vec{T}_{1,2}$ the directed tree where each vertex has in-valency 1 and out-valency 2. (Corresponds to choosing a fixed end ω in the underlying 3-regular tree.) Pick a directed line $\ldots, v_{-1}, v_0, v_1, v_2, \ldots$. Let $\ldots, H_{-1}, H_0.H_1, H_2, \ldots$ be an enumeration of the horocycles in Γ (with respect to ω).

For $i \in \mathbb{Z}$ take copies T^i of the tree T andisomorphisms from T to T^i. Use this isomorphism to identify the vertices in $H_i \cup H_{i+1} \cup \cdots$ with their images in T^i. All the vertices that originally belonged to T will now have in-valency 2 and out-valency 2. Continue in this way with the horocycles of the trees T^i where the vertices do not have in-valency 2, and so on.
The resulting graph is highly-arc-transitive and has just one end.

Lamplighter graph

This graph is a Cayley graph for the Lamplighter group $\mathbb{Z} \operatorname{wr} \mathbb{Z}_2$.

If instead, we start with the directed tree where each vertex has in-valency 1 and out-valency 3 and perform the same construction we again end up with a highly-arc-transitive digraph where each vertex has in-valency 2 and out-valency 3. The underlying undirected graph is the Diestel-Leader graph constructed by Diestel and Leader in 2001. It was shown by Eskin, Fisher and Whyte in 2012 that this graph is not quasi-isometric to any Cayley-graph of a finitely generated group.

Application to permutation group theory

Question (Peter M. Neumann) Does there exist a primitive permutation group such that there is a finite suborbit paired with an infinite one?

Phrased in terms of graphs then this question comes: Does there exist a vertex- and arc-transitive digraph with infinite in-valency and finite out-valency such that the automorphism group acts primitively on the vertex set.

Application to permutation group theory

David M. Evans constructed in 2001 such a digraph.

His construction uses "gluing" like in the above example together with techniques from model theory.

His example is highly-arc-transitive.

Tidy subgroups

Let G be a totally disconnected locally compact group.

For an element $x \in G$ and a compact open subgroup U we define

$$U_+ = \bigcap_{n \geq 0} x^n U x^{-n}, \qquad U_- = \bigcap_{n \geq 0} x^{-n} U x^n,$$

$$U_{++} = \bigcup_{n \geq 0} x^n U_+ x^{-n}, \qquad \text{and} \qquad U_{--} = \bigcap_{n \geq 0} x^{-n} U_- x^n.$$

Willis's original definition (1994)

A compact open subgroup U is tidy for x if

T1 $U = U_+U_- = U_-U_+$,

and

T2 U_{++} and U_{--} are closed subgroups of G.

The scale function $s : G \to \mathbb{N}$ is defined with the formula

$$s(x) = |xU_+x^{-1} : U_+|.$$

Another approach (Willis, 2001)

The scale function $s : G \to \mathbb{N}$ is defined with the formula

$$s(x) = \min |xUx^{-1} : U \cap (xUx^{-1})|$$

where the minimum is taken over all compact open subgroups in G.

A compact open subgroup U is tidy for x if and only if

$$|xUx^{-1} : U \cap (xUx^{-1})| = s(x).$$

Graph theoretical interpretation

Let x be an element in G and U a compact open subgroup. Define a digraph Γ such that G/U and the arc set is $G(\alpha, x\alpha)$ where $\alpha = u \in G/U$.
The digraph Γ might not be connected.

Condition **T1** is equivalent to the graph Γ being highly-arc-transitive.

The subgroup U is tidy for x if and only if Γ is highly-arc-transitive and the set of descendants of a vertex is a tree.

Graph theoretical properties of highly-arc-transitive digraphs

In their 1993 paper Cameron, Praeger and Wormald put forward many questions and conjectures concerning highly-arc-transitive digraphs.

In some cases results about the structure of such graphs lead directly to results about groups and *vice versa*.

Perhaps you can help me to see more connections.

Out-spread

Let Γ be a locally finite highly-arc-transitive digraph and α a vertex in Γ. Define p_n as the number of vertices β such that there exists a directed path starting with α and ending with β of length n. The out-spread of Γ is defined as

$$\limsup_{n\to\infty} p_n^{1/n}.$$

Cameron, Praeger and Wormald discuss this concept. But if x is a group element such that $(\alpha, x\alpha)$ is an arc in Γ then the scale $s(x^{-1})$ is equal to the out-spread.

Property Z

A digraph Γ is said to have Property Z if there is a surjective digraph homomorphism $\varphi : \Gamma \to \vec{\mathbb{Z}}$.

Praeger proved the following in 1991:

Theorem

(Praeger) Let Γ be a locally finite arc-transtive digraph with unequal in- and out-valencies. Then Γ has Property Z.

Theorem

(Árnadóttir & Möller) Let Γ be a locally finite arc-transtive digraph with coprime in- and out-valencies. Then Γ is highly-arc-transitive and the set of descendants of a vertex spans a tree.

Idea of proof based on the modular function

Set $G = \mathrm{Aut}\,\Gamma$. Fix a vertex α. For a group element g we define

$$\psi(g) = \frac{|G_\alpha(g\alpha)|}{|G_{g\alpha}\alpha|}.$$

Praeger shows that this is a group homomorphism (attributes this to Peter M. Neumann). This is the modular function on G.
Let α be a vertex in Γ and x an element in G such that $(\alpha, x\alpha)$ is an arc in Γ. Then $\psi(x) =$(out-valency)/(in-valency) and

$$\frac{|G_\alpha(x^n\alpha)|}{|G_{x^n\alpha}\alpha|} = \psi(x^n) = \psi(x)^n = \frac{(\mathrm{out-valency})^n}{(\mathrm{in-valency})^n}.$$

Applied to groups

This extension of Praeger's result is essentially equivalent to:

Theorem

Let G be a totally disconnected locally compact group. Let x be an element in G and U a compact open subgroup. If the indicies $|xUx^{-1} : U \cap (xUx^{-1})|$ and $|x^{-1}Ux : U \cap (x^{-1}Ux)|$ are coprime then U is tidy for x.

Highly-arc-transitive digraphs with prime out-valency

Möller, Potočnik and Seifter have recently shown that if Γ is a locally finite highly-arc-transitive digraph with prime out-valency then either the digraph has precisely two ends or the set of descendants is a tree.

Translated into group theory: Let G be a totally disconnected locally compact group, x an element in G and U a compact open subgroup of G. If condition **T1** is satisfied and $|U : U \cap (xUx^{-1})|$ is a prime then either $s(x) = 1$ or U is tidy for x.

Sharply k-arc-transitive digraphs

A digraph is said to be *sharply k-arc-transitive* if it is k-arc-transitive but not $(k+1)$-arc-transitive.

Möller, Potočnik and Seifter have recently constructed a variety of examples of sharply k-arc-transitive digraphs for any value of k. In the automorphism groups of some of these example we have the situation that if U is the stabilizer of a vertex α and x a group element such that $(\alpha, x\alpha)$ is an arc then

$$|U : U \cap (x^n U x^{-n})| = |U : U \cap (xUx^{-1})|^n,$$

for $n = 1, 2, \ldots, k$, but not equal if $n > k$.

Comparison with tidy subgroups

In comparison, a open compact subgroup U of a totally disconnected locally compact group G is tidy for an element x if and only if

$$|U : U \cap (x^n U x^{-n})| = |U : U \cap (xUx^{-1})|^n,$$

for all $n \geq 1$.

Reachability

Say two arcs in an arc-transitive digraph Γ are related if they have a common initial vertex or a common terminal vertex. Take the transitive closure of this to get an equivalence relation on the arcs of Γ. In the paper by Cameron, Praeger and Wormald this is called the *reachability relation*.

For and arc e we let $\Delta = \Delta(e)$ denote the subdigraph spanned by the equivalence class of e. When $\mathrm{Aut}\,\Gamma$ is arc-transitive then any two such subdigraphs are isomorphic. This subdigraph is either bipartite or equal to the whole of Γ (the reachability relation is universal).

These seem to be central in the study of arc-transitive digraphs. Is there a group theoretic interpretation?

Property Z

Cameron, Praeger and Wormald ask if every highly-arc-transtive digraph with a non-universal reachability relation has property Z.

Evans in 1997 constructed a non-locally finite counterexample example.

Malnič, Marusič, Seifter and Zgrablič constructed a locally finite counterexample that is surprisingly simple.

Universality of the reachability relation

Does there exist a locally finite highly-arc-transitive digraph with universal reachability relation?

YES! Example constructed by DeVos, Mohar and Šámal in 2015

If the out-valency is a prime then NO! Proved by Malnič, Marusič, Möller, Seifter, Trofimov and Zgrablič in 2005.

Two-ended highly-arc-transitive digraphs

Cameron, Praeger and Wormald ask if it is true that whenever you have a highly-arc-transitive digraph with two ends then the subdigraph Δ is a complete bipartite digraph.

NO! Example constructed by DeVos, Mohar and Šámal in 2015 and independently by C. Neumann in 2013.

But if the digraph has prime out-valency then YES, proved by Potočnik, Möller and Seifter.

The only unresolved conjecture/question from Cameron, Praeger and Wormald

They proof:

Lemma 2.8. Let M be a connected locally finite digraph which is either vertex-transitive or 1-arc-transitive, and let $\theta : M \to M$ be a covering projection. Then θ is an isomorphism.

I think the only unresolved conjecture/question from the paper by Cameron, Praeger and Wormald is

Question 2.10. Is Lemma 2.8 still true if the hypothesis of vertex-transitivity or 1-arc-transitivity is deleted? That is, must a covering projection of a connected locally finite digraph be an isomorphism?

Thank you for your attention!