Some properties of group actions on zero-dimensional spaces

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Let X be a locally compact Hausdorff topological space and write $\mathcal{CO}(X)$ for the set of compact open subsets of X. Suppose that X is **zero-dimensional**, meaning that $\mathcal{CO}(X)$ forms a base for the topology.

Let $S \subseteq \mathrm{Homeo}(X)$, such that $\mathrm{id}_X \in S$, $S = S^{-1}$ and $\{sU \mid s \in S\}$ is finite for every $U \in \mathcal{CO}(X)$. Let S^n be the set of products of at most n elements of S, and let $G = S^\infty = \langle S \rangle$.

Fix some $U \in \mathcal{CO}(X)$. Write $U_0 = U$;

for $n \in (0, +\infty]$, $U_n = \bigcap_{g \in S^n} gU$ and $U_{-n} = \bigcup_{g \in S^n} gU$.

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The space $U_{-\infty} = \bigcup_{g \in G} gU$ is open in X (so locally compact) and G-invariant. We think of $U_{-\infty}$ as partitioned into a 'core' $U_{+\infty}$ (compact, but not necessarily open) and a sequence of 'shells' $W_n := U_n \setminus U_{n+1}$ indexed by the integers (each of which is compact and open).

Lemma

(i) There exist $a, b \in [-\infty, +\infty]$ with $a \leq 0 \leq b$ such that $U_a = U_{-\infty}$, $U_b = U_\infty$ and W_m is nonempty exactly when $m \in [a, b)$.

(ii) Every G-orbit intersecting $U_n \setminus U_{+\infty}$ also intersects W_m for all $m \in [a, n]$.

(iii) There is a G-orbit Gx that intersects all of the nonempty shells.

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Proof

(i) Suppose for some $b \geq 0$ that $W_b = \emptyset$, i.e. $U_b = U_{b+1}$, and let $m \geq 0$. Then

$$U_{b+m} = \bigcap_{g \in S^m} gU_b = \bigcap_{g \in S^m} gU_{b+1} = U_{b+m+1}.$$

Hence $U_{b+1} = U_{b+2} = \cdots = U_{+\infty}$. The proof in the negative direction is similar.

(ii) Let $x \in U_n \setminus U_{+\infty}$. Then $x \in W_{n'}$ for some $n' \geq n$, and hence there exists $g \in S$ such that $gx \notin U_{n'}$ (otherwise we would have $x \in U_{n'+1}$), but $gx \in U_{n'-1}$ (since $x \in U_{n'}$). Thus $gx \in W_{n'-1}$. Repeat to get images of x in W_m for all $m \leq n'$.

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(iii) Define $P_n = (\bigcup_{g \in S^n} g^{-1} U_n) \setminus U_1$. Then P_n is a compact subset of U. Let I be the set of $n \geq 0$ such that $W_n \neq \emptyset$. Given part (ii) it is enough to show $\bigcap_{n \in I} P_n \neq \emptyset$.

Suppose $x \in P_n$. Then $\exists g \in S, h \in S^{n-1} : ghx \in U_n$, so $hx \in U_{n-1}$ and hence $x \in P_{n-1}$. Thus $(P_n)_{n \in I}$ is a descending sequence.

Suppose $\bigcap_{n \in I} P_n = \emptyset$. Then by compactness $P_n = \emptyset$ for some $n \in I$, that is, $g^{-1} U_n \subseteq U_1$ for all $g \in S^n$. But then $U_n \subseteq \bigcap_{g \in S^n} gU_1 = U_{n+1}$, so $W_n = \emptyset$, contradicting the choice of n.

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Alternative incarnation of (iii) (think of $G = X$ acting by conjugation on itself, and U a vertex stabilizer):

Lemma/Corollary

Let Γ be a connected locally finite graph and let G be a closed vertex-transitive group of automorphisms of Γ. Then exactly one of the following holds:

(i) There is a finite set $v_1, \dots, v_n$ of vertices, such that $\bigcap_{i=1}^{n} G_{v_i} = \{1\}$.

(ii) There is a horoball H in Γ, such that the pointwise fixator of H in G is nontrivial.

Here we define a **horoball** to be a set of the form $\{v \in V\Gamma : \exists n : d(v, v_n) \leq n\}$, where $(v_n)_{n \geq 0}$ is a set of vertices forming a geodesic ray.

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Hypotheses: Let X be a locally compact zero-dimensional space, $S \subseteq \mathrm{Homeo}(X)$ such that $S = S^{-1}$ and $\{sU \mid s \in S\}$ is finite for every $U \in \mathcal{CO}(X)$, and $G = \langle S \rangle$.

Theorem (Auslander–Glasner–Weiss; R.)

Let $U \in \mathcal{CO}(X)$ and write $U_{+\infty} = \bigcap_{g \in G} gU$. Then the following are equivalent:

(i) Given $x \in U$ and $y \in U_{+\infty}$ such that $y \in \overline{Gx}$, then $x \in \overline{Gy}$.

(ii) For all $V \in \mathcal{CO}(U)$, there is a finite subset F of G such that $V_{+\infty} = \bigcap_{g \in F} gV$.

(iii) U_{∞} is open and there is a G-invariant quotient map $\phi : U_{+\infty} \to Y$, such that G acts trivially on Y and minimally on each fibre of ϕ.

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Distal action: if $(g_i x, g_i y) \to (z, z)$ as $i \to \infty$, then $x = y$.
In particular, if $\overline{Gy}$ is compact and $y \in \overline{Gx}$, then $\overline{Gx} = \overline{Gy}$.

Corollary

Suppose that G acts distally on X and that every orbit has compact closure. Then $\{gV \mid g \in G\}$ is finite for every $V \in \mathcal{CO}(X)$. In particular, the action of G is equicontinuous.

(If X is the Cantor set, then $G \leq \mathrm{Homeo}(X)$ acts equicontinuously if and only if there is a compatible G-invariant metric on X, or equivalently X is the boundary of some locally finite rooted tree on which G acts by automorphisms.)

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A locally compact group G is **distal** (as a topological group) if it acts distally on itself by conjugation; equivalently, no conjugacy class of G accumulates at the identity. For example: nilpotent groups; discrete groups; compact groups; any residually distal group is distal.

t.d.l.c. group = "totally disconnected locally compact group". T.d.l.c. groups are zero-dimensional; in fact the cosets of compact open *subgroups* form a base for the topology (Van Dantzig).

Corollary (Willis; Caprace–Monod; R.)

Let G be a compactly generated t.d.l.c. group. Then G is distal if and only if the cosets of open *normal* subgroups of G form a base for the topology.

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Proposition (Caprace–Monod; R.–Wesolek)

Let G be a compactly generated t.d.l.c. group and let U be a compact open subgroup of G.

(i) Let $(K_i)_{i \in \mathbb{N}}$ be a sequence of closed normal subgroups such that $K_i \to \{1\}$ as $i \to \infty$. Then for i large enough, $K_i \cap U$ is normal in G.

(ii) Suppose that $\bigcap_{g \in G} gUg^{-1} = \{1\}$ and that G has no nontrivial discrete normal subgroup. Then every nontrivial closed normal subgroup of G contains a minimal one.

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Let G be a t.d.l.c. group and let H be a compactly generated group of automorphisms of G. Write $\mathrm{Res}_G(H)$ for the intersection of all open H-invariant subgroups of G.

Theorem (R.)

(i) There is an H-invariant open subgroup of the form $V\mathrm{Res}_G(H)$ for some compact open subgroup V of G. Moreover, $\mathrm{Res}_G(H)$ is normal in $V\mathrm{Res}_G(H)$.

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Let G be a group acting faithfully on a space X, and given $Y \subseteq X$, write $\mathrm{rist}_G(Y)$ for the set of elements that fix $X \setminus Y$ pointwise. The action is **micro-supported** if $\mathrm{rist}_G(Y) \neq \{1\}$ for every nonempty open Y.

Theorem (Caprace–R.–Willis)

Let G be a compactly generated t.d.l.c. group with faithful continuous action by homeomorphisms on the Cantor set X. Suppose that G has a compact open subgroup U, such that U is micro-supported on X and $\bigcap_{g \in G} gUg^{-1} = \{1\}$. Then there is a partition of X into clopen sets $B_1, \dots, B_n$ such that for every $A \in \mathcal{CO}(X) \setminus \{\emptyset\}$, there is $g \in G$ and $1 \leq i \leq n$ such that $B_i \subseteq gA$.

If G is topologically simple, then the action is also minimal, and consequently G is not amenable.

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