Generalized Grigorchuk's Overgroups

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Trees, dynamics and locally compact groups
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Overview

1. Introduction
2. Generalization
3. Growth of generalized overgroup $\tilde{G}_\omega$
4. Grigorchuk's Space of Marked Groups

Growth of a Group

G - group, A - set of generators (finite and closed under inverse)

Definition (Growth)

Growth $\gamma_G : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ is given by,

$$\gamma_G(n) = |\{g \in G : g = a_1 a_2 \ldots a_k, k \leq n, a_1, \ldots, a_k \in A\}|$$

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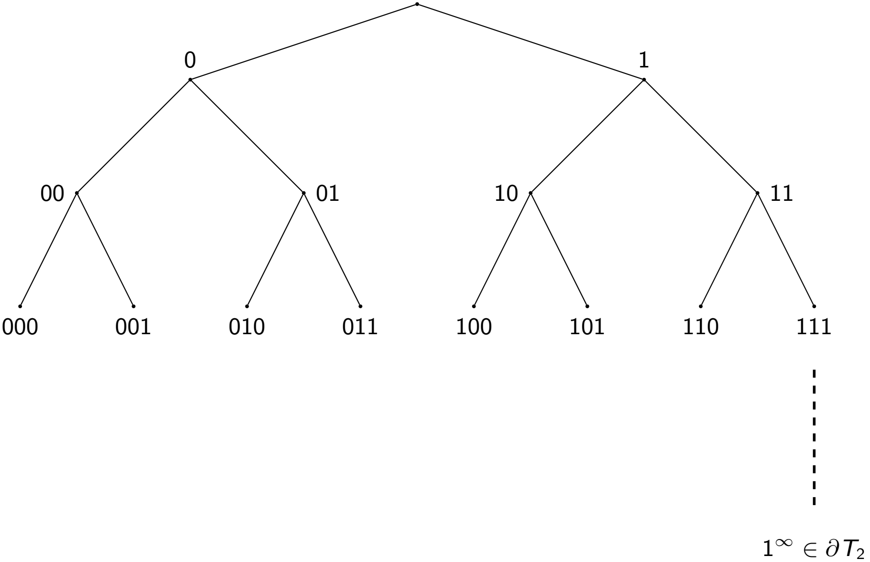
Polynomial Growth	Intermediate Growth	Exponential Growth
$\mathbb{Z}^d$ $(d \geq 1)$, $\mathbb{D}_\infty$ Virtually Nilpotent	First Grigorchuk group G_ω, $\tilde{G}_\omega$	F_r $(r \geq 2)$, non elementary hyperbolic groups

Sub-exponential (Polynomial and Intermediate Growth)

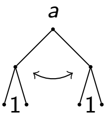
Super-polynomial (Intermediate and Exponential Growth)

T_2



0
1
00
01
10
11
000
001
010
011
100
101
110
111
$1^\infty \in \partial T_2$

Some elements in $Aut(T_2)$



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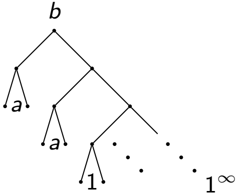

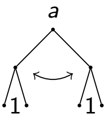

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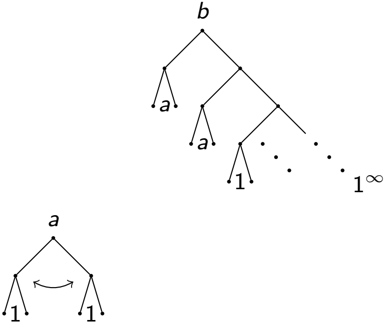

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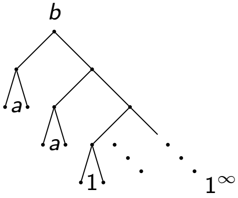

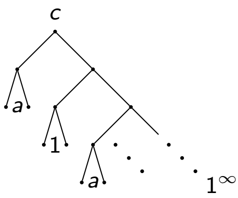

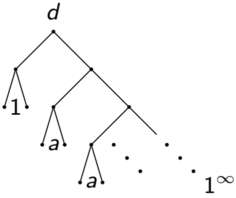

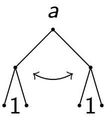

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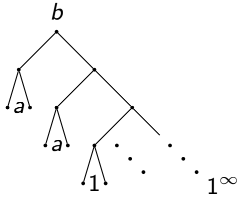

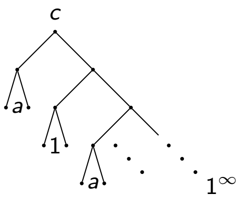

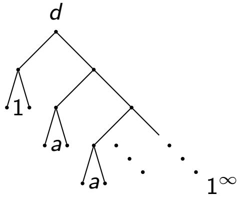

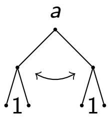

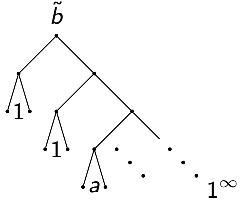

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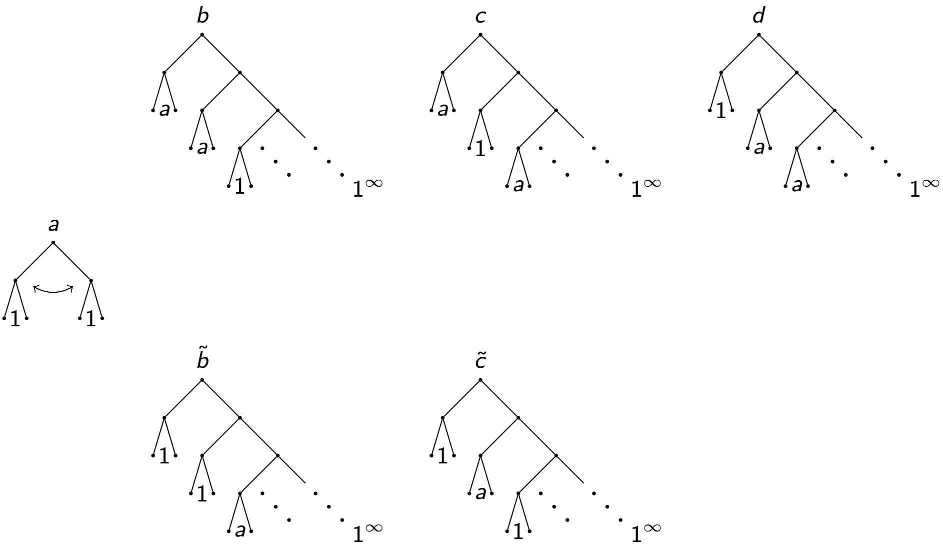

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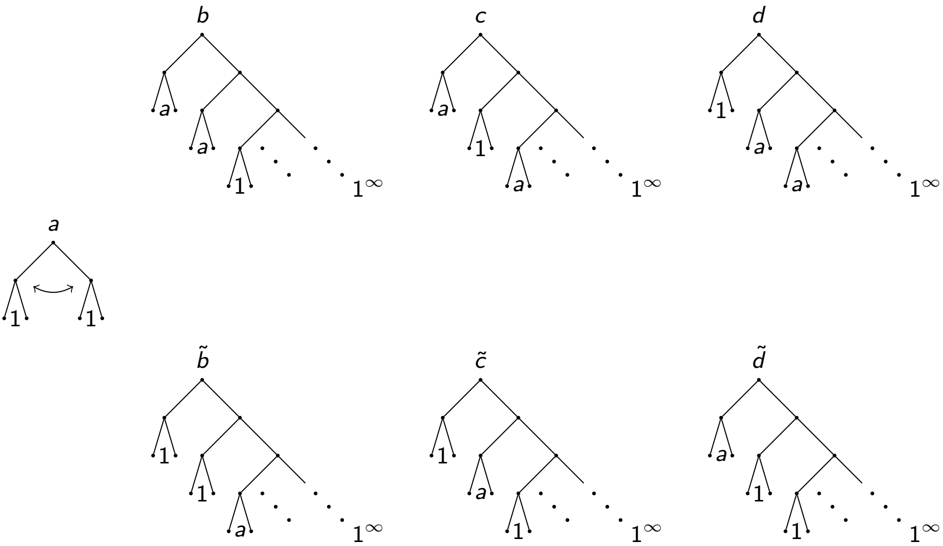

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$e^{n^{\alpha}}$ is an upper bound [Bartholdi,1998] and [Muchnik-Pak,2001]
$e^{n^{\alpha-\epsilon}}$ is a lower bound (for each $\epsilon > 0$) [Erschler-Zheng,2018].

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- Intermediate growth [Bartholdi-Gri,2000]
- Plays an important role in Yaroslav Vorobets recent work topological full group on a minimal Cantor system

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Then $b_\omega, c_\omega, d_\omega$ are given by first, second and third rows, respectively.
Define $\tilde{b}_\omega, \tilde{c}_\omega, \tilde{d}_\omega$ accordingly.

$$G_\omega = \langle a, b_\omega, c_\omega, d_\omega \rangle, \tilde{G}_\omega = \left\langle a, \tilde{b}_\omega, \tilde{c}_\omega, \tilde{d}_\omega \right\rangle$$

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When $\omega = (012)^\infty$, $G_\omega = G$ and $\tilde{G}_\omega = \tilde{G}$.

G_ω - main result

Theorem (Gri,1984)

- *If ω is eventually constant, then G_ω is of polynomial growth and hence virtually abelian.*
- *If ω is not eventually constant, then G_ω is of intermediate growth.*

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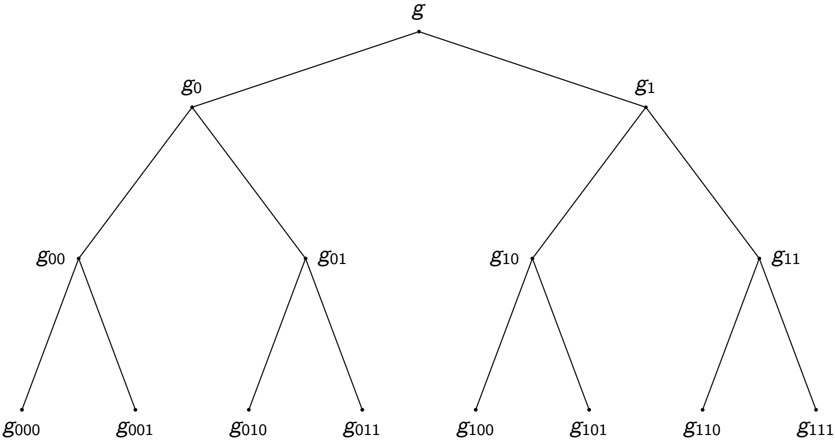
$\Omega_1 = \Omega - (\Omega_0 \cup \Omega_2)$ - e.g. $012(01)^\infty$

$Stab_{\tilde{G}_\omega}(3)$



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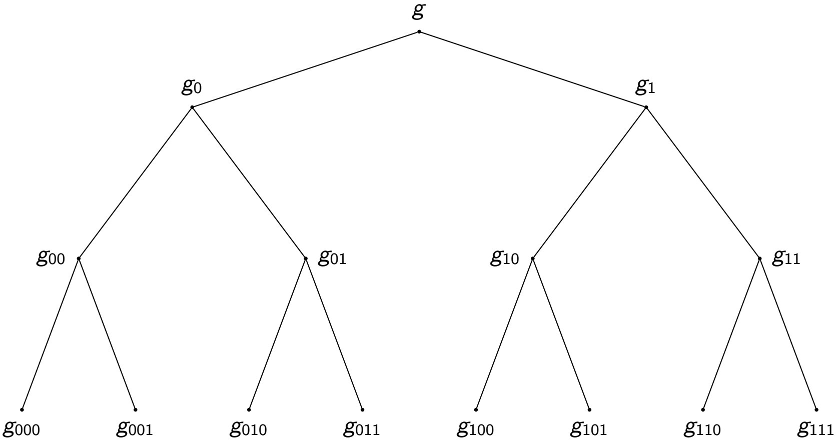

Strong contraction property

$|g_{000}| + |g_{001}| + \ldots + |g_{111}| \leq C|g| + D$ for some constant $C < 1$ and $D \geq 0$.

sub-exponential growth

Lemma

Let $\epsilon > 0$, $n_\epsilon \in \mathbb{N}$ such that $n_\epsilon \epsilon > 5/2$. Let $n \geq n_\epsilon$. Let $s \in \mathbb{N}$ such that ω_s is the first time that the third symbol appears in ω. Let $g = W \in \mathcal{D}^\epsilon(n)$ represent a word in $\tilde{H}_\omega^{(s)}$. Then,

$$\sum_{i_1, i_2, \ldots, i_s} |g_{i_1 i_2 \ldots i_s}| \leq \left(1 - \frac{\epsilon}{5}\right) |g| + 2^s - 1.$$

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$\omega \in \Omega_0 \implies \tilde{G}_\omega$ has sub-exponential growth.

$\tilde{G}_\omega$ - main result

Theorem

- *If ω is eventually constant, then $\tilde{G}_\omega$ is of polynomial growth and hence virtually abelian.*
- *If ω is not eventually constant, then $\tilde{G}_\omega$ is of intermediate growth.*

Grigorchuk's Space of Marked Groups

Definition ($\mathcal{M}_k$)

The Grigorchuk's space of marked groups with k generators,

$$\mathcal{M}_k = \{(G, A) : A \text{ is an ordered set of k elements generating } G\}$$

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together with the topology generated by the metric

$$d((G_1, A_1), (G_2, A_2)) = 2^{-n}$$

where n is the largest integer such that the balls of radius n centered at identity of the Cayley graphs of (G_1, A_1) and (G_2, A_2) are identical.

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$(G, \{a, b, c, d\}) \in \mathcal{M}_4$

$(\tilde{G}, \{a, b, c, d, x, \tilde{b}, \tilde{c}, \tilde{d}\}) \in \mathcal{M}_8$

Grigorchuk's Space of Marked Groups

Define $\tilde{\mathscr{Y}}, \tilde{\mathscr{Y}}_0, \tilde{\mathscr{Y}}_1, \tilde{\mathscr{Y}}_2$ as subsets of $\mathcal{M}_8$,

$$\tilde{\mathscr{Y}} = \{(\tilde{G}_\omega, \tilde{A}_\omega)\}_{\omega\in\Omega}$$
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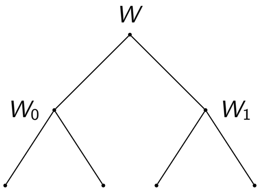
What is

$$cl(\tilde{\mathscr{Y}})$$

Algorithm for the word problem in $\tilde{G}_\omega$

Let W be a reduced word of the alphabet $\{a, b_\omega, c_\omega, d_\omega, x_\omega, \tilde{b}_\omega, \tilde{c}_\omega, \tilde{d}_\omega\}$. Algorithm α is,

- $|W| = 0 \implies W = 1$. (positive)
- $|W|_a \equiv 1 \pmod 2 \implies W \neq 1$. (negative)
- $|W| = 1 \implies$ (stop).
- Replace W with reduced W_0, W_1 and repeat



Modified Overgroup $\tilde{G}_\omega^\sharp$

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Modified overgroup $\tilde{G}_\omega^\sharp$ is the group generated by involutions which commutes with each other (except for $a^\sharp$), $\{a^\sharp, b_\omega^\sharp, c_\omega^\sharp, d_\omega^\sharp, x^\sharp, \tilde{b}_\omega^\sharp, \tilde{c}_\omega^\sharp, \tilde{d}_\omega^\sharp\}$ where all the relations $R^\sharp$ of $\tilde{G}_\omega^\sharp$ are exactly the words R of the alphabet $\{a, b_\omega, c_\omega, d_\omega, x_\omega, \tilde{b}_\omega, \tilde{c}_\omega, \tilde{d}_\omega\}$ such that R gives positive result when applied the algorithm α.

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$\omega \in \Omega_0 \implies \tilde{G}^\sharp_\omega = \tilde{G}_\omega$

$\omega \in \Omega_1 \cup \Omega_2 \implies \tilde{G}^\sharp_\omega \twoheadrightarrow \tilde{G}_\omega$ *with non trivial kernel.*

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$$\omega^{(n)} \to \omega \iff \tilde{G}^\sharp_{\omega^{(n)}} \to \tilde{G}^\sharp_\omega$$

$\tilde{G}^{\sharp}_{\omega}$ for $\omega \in \Omega_2$

Theorem (Benli-Grigorchuk 2014)

$G^{\sharp}_{0^\infty}$ *is virtually* $\mathbb{Z}_2 \wr \mathbb{Z}$.

$\tilde{G}^\sharp_\omega$ for $\omega \in \Omega_2$

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$\tilde{G}^\sharp_{0\infty}$ *is virtually* $\mathbb{Z}_2^3 \wr \mathbb{Z}$.

$\tilde{G}^\sharp_\omega$ for $\omega \in \Omega_2$

Theorem (Benli-Grigorchuk 2014)

$G^\sharp_{0\infty}$ is virtually $\mathbb{Z}_2 \wr \mathbb{Z}$.

Theorem

$\tilde{G}^\sharp_{0\infty}$ is virtually $\mathbb{Z}_2^3 \wr \mathbb{Z}$.

Theorem

Let $\omega \in \Omega_2$ and let N be such that $\omega_N \neq \omega_{N+1} = \omega_{N+2} = \ldots$. Then $\tilde{G}^\sharp_\omega$ is commensurable to $(\tilde{G}^\sharp_{0\infty})^{2^N}$ which is virtually $(\mathbb{Z}_2^3 \wr \mathbb{Z})^{2^N}$.

Closure of $\tilde{\mathscr{Y}}$

Define $\tilde{\mathscr{Y}}_1', \tilde{\mathscr{Y}}_2'$ as subsets of $\mathcal{M}_8$ by,

$$\tilde{\mathscr{Y}}_1' = \{(\tilde{G}_\omega^\sharp, \tilde{A}_\omega^\sharp)\}_{\omega \in \Omega_1}$$
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Theorem (Gri84)

$$cl(\mathscr{Y}) = \underbrace{\mathscr{Y}_0 \bigsqcup \mathscr{Y}_1 \bigsqcup \mathscr{Y}_2'}_{\textit{Cantor set}} \bigsqcup \underbrace{\mathscr{Y}_2}_{\textit{isolated points}}$$

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Theorem

$$cl(\tilde{\mathscr{Y}}) = \underbrace{\tilde{\mathscr{Y}}_0 \bigsqcup \tilde{\mathscr{Y}}_1 \bigsqcup \tilde{\mathscr{Y}}_1' \bigsqcup \tilde{\mathscr{Y}}_2'}_{\text{Cantor set}} \bigsqcup \underbrace{\tilde{\mathscr{Y}}_2}_{\text{isolated points}}$$

References

Bar98 Bartholdi, Laurent **The growth of Grigorchuk's torsion group**. *Internat. Math. Res. Notices 1998, no. 20*, 1049-1054.

BG00 Bartholdi, Laurent; Grigorchuk, Rostislav I. **On the spectrum of Hecke type operators related to some fractal groups**. *Tr. Mat. Inst. Steklova* 231 (2000), *Din. Sist., Avtom. i Beskon. Gruppy*, 5-45; translation in *Proc. Steklov Inst. Math. 2000, no. 4(231)*, 1-41.

BG02 Bartholdi, Laurent; Grigorchuk, Rostislav I. **On parabolic subgroups and Hecke algebras of some fractal groups**. *Serdica Math. J. 28 (2002), no. 1*, 47-90.

BG14 Benli, Mustafa G.; Grigorchuk, Rostislav I. **On the condensation property of the lamplighter groups and groups of intermediate growth**. *Algebra Discrete Math. 17 (2014), no. 2*, 222-231.

References

EZ18 Erschler, Anna; Zheng, Tianyi **Growth of periodic Grigorchuk groups**. *arXiv, 1802.09077v1, Feb., 2018*.

Gri84 Grigorchuk, Rostislav I. **Degrees of growth of finitely generated groups and the theory of invariant means**. *Izv. Akad. Nauk SSSR Ser. Mat. 48 (1984), no. 5*, 939-985.

MP01 Muchnik, Roman; Pak, Igor **On growth of Grigorchuk groups**. *Internat. J. Algebra Comput. 11 (2001), no. 1*, 1-17.

Thank You