Galois coverings of Schreier graphs

of groups generated by bounded automata

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(Joint work with D D'Angeli, H Bhate & D Sheth)

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Ihara zeta function

Let $Y = (V, E)$ be a connected graph and let $t \in \mathbb{C}$, with $|t|$ sufficiently small. Then the Ihara zeta function $\zeta_Y(t)$ of graph Y is defined as

$$\zeta_Y(t) = \prod_{[C]\ prime\ cycle\ in\ Y} (1 - t^{\nu(C)})^{-1}, \tag{1}$$

where $[C]$ in Y is an equivalence class of tailless, back-trackless primitive cycles C in Y and length of C is $\nu(C)$.

Example: Cycle Graph

Let Y be a cycle graph with n vertices. As there are only two primes,

$$\zeta_Y(t) = (1 - t^n)^{-2}.$$

Ihara-Bass determinant formula

The Ihara-Bass's Theorem establishes the connection between $\zeta_Y(t)$ and the adjacency matrix A of the graph Y which is given as

Theorem (Ihara and Bass)

Let Q be the diagonal matrix with jth diagonal entry q_j such that $q_j + 1 =$ degree of jth vertex of Y and r be the rank of fundamental group of Y, $r - 1 = |E| - |V|$. Then Ihara determinant formula is

$$\zeta_Y(t)^{-1} = (1 - t^2)^{r-1} \det(I - At + Qt^2).$$

Unramified and d-sheeted coverings

- All graphs are connected and undirected.
- An unramified cover of a graph Y is a surjective graph homomorphism

$$\pi : \widetilde{Y} \to Y$$

  which is a local isomorphism.
- The fiber

$$\pi^{-1}(x) = \{x_1, x_2, x_3, x_4\}.$$

  Here $x_i's$ are representatives of copies of a spanning tree of Y.

Galois covering of a graph

- The group of automorphisms of π is

$$Aut(\pi) = \{\sigma : \widetilde{Y} \to \widetilde{Y} \text{ automorphism } | \pi = \pi \circ \sigma\}.$$

  An automorphism σ is determined by its action on the fiber $\pi^{-1}(x)$ above any vertex x of Y.

- Call $\pi : \widetilde{Y} \to Y$ (or $\widetilde{Y}|Y$) a Galois or normal cover if $Aut(\pi)$ acts transitively on one fiber and hence all fibers. Its Galois group is

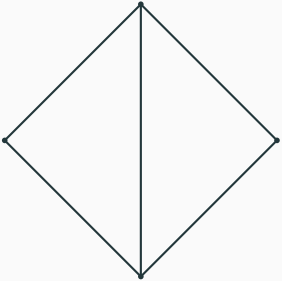
$$\mathbb{G} = G_\pi = Aut(\pi) = G(\widetilde{Y}|Y).$$

- If a fiber $\pi^{-1}(x)$ is a finite set, its cardinality is called the *degree* of π. A finite degree cover $\widetilde{Y}|Y$ is Galois iff

$$|\mathbb{G}| = \deg \pi.$$

  We call σ as Frobenius automorphism.

Examples



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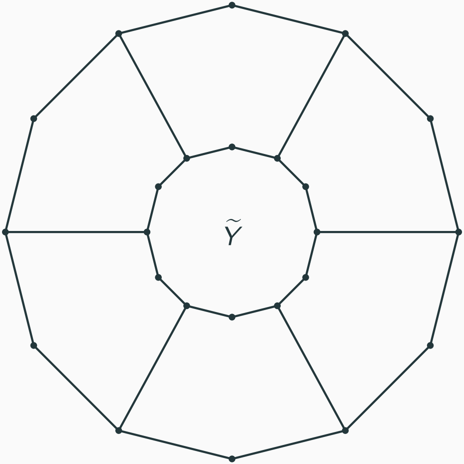
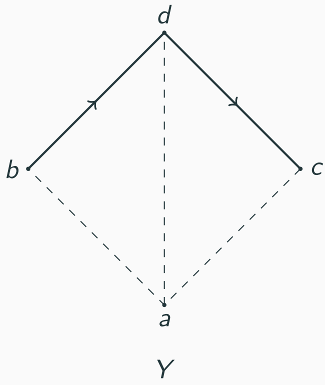


Examples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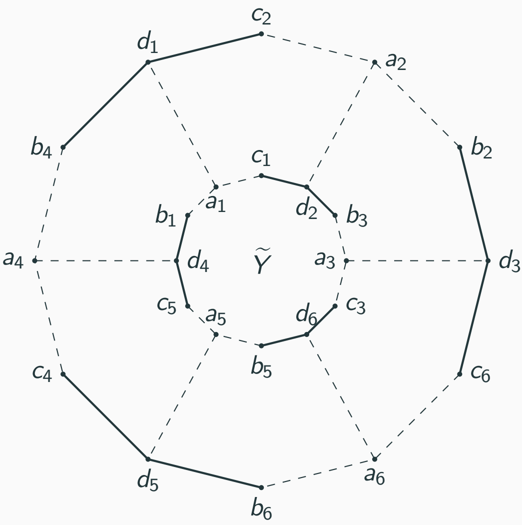
c_2
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$\widetilde{Y}$
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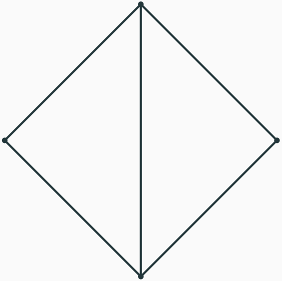
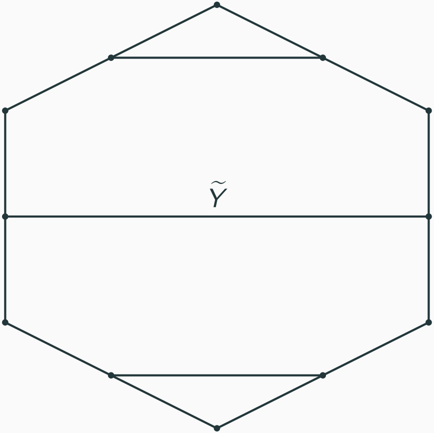
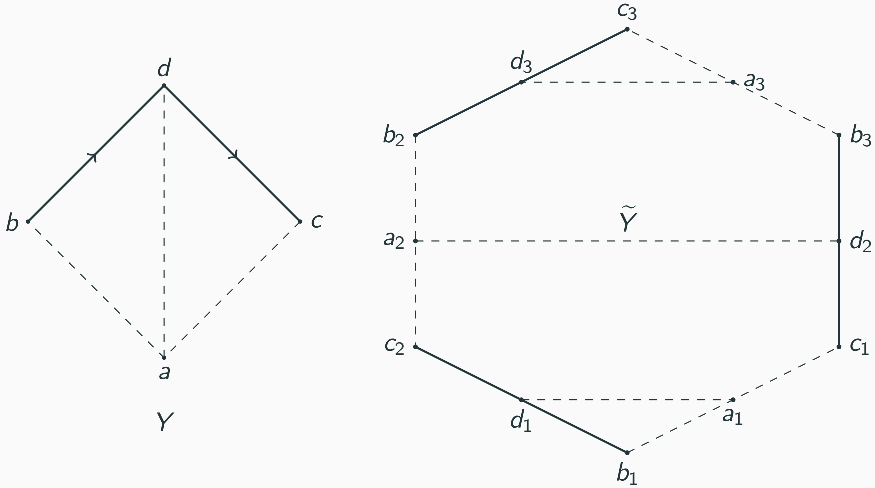
Examples



Y



Examples



- Suppose $\widetilde{Y}$ is normal covering of Y with Galois group $\mathbb{G}$.
  The adjacency matrix of $\widetilde{Y}$ can be block diagonalized where the blocks are of the form

$$A_\rho = \sum_{g \in \mathbb{G}} A(g) \otimes \rho(g),$$

each taken $d_\rho (= \dim \text{irr rep } \rho)$ times and $m \times m$ matrix $A(g)$ for $g \in \mathbb{G}$ is the matrix whose i, j entry is

$$A(g)_{i,j} = \text{ the number of edges in } \widetilde{Y} \text{ between } (i, id) \text{ to } (j, g),$$

where id denotes the identity in $\mathbb{G}$ and m is the number of vertices of the graph Y.

- By setting $Q_\rho = Q \otimes I_{d_\rho}$, with $d_\rho =$ degree of ρ, we have the following analogue

$$L(t, \rho, \widetilde{Y}|Y)^{-1} = (1-t^2)^{(r-1)d_\rho} \det(I - tA_\rho + t^2 Q_\rho).$$

Thus we have zeta functions of $\widetilde{Y}$ factors as follows

$$\zeta_{\widetilde{Y}}(t) = \prod_{\rho \in \widehat{\mathbb{G}}} L(t, \rho, \widetilde{Y}|Y)^{d_\rho}.$$

Assumptions

- Let G be a group generated by bounded automaton $\mathcal{A}$ with generating set $S = \{s_1, \cdots, s_m\}$.
- G has level transitive action on the regular rotted tree T_d.
- Recall for every $s \in S$ we have $s = (s|_{x_1}, \cdots, s|_{x_d})\psi_s$, where $\psi_s \in S_d$ and $s|_x =$ the restriction s at x where $x \in X = \{x_1, \cdots, x_d\}$.
- We call ψ_s as root permutation associated to state s.
- Denote $\Psi_G =$ group generated by root permutations ψ_s

$$\Psi_G = \ < \psi_s : s \in S > .$$

Post critical sequences

Definition
A left-infinite sequence $\cdots x_2 x_1$ *over* X *is called post critical if there exists a left-infinite path* $\cdots e_2, e_1$ *in the Moore diagram of* $\mathcal{A}$ *avoiding the trivial state labeled by* $\cdots x_2 x_1 | \cdots y_2 y_1$ *for some* $y_i \in X$*.*

G is a group generated by bounded automaton iff the set of post critical sequences say $\mathcal{P}_{\mathcal{A}}$ is finite.

Schreier and Tile graphs

Let G be a group generated by bounded automaton $\mathcal{A}$. The levels X^r of the tree X^* are invariant under the action of the group G.

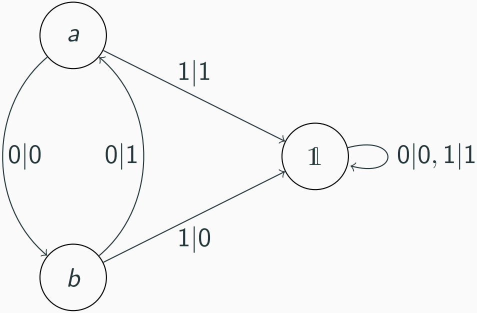
Definition
The Schreier graph Γ_r of the action of G on X^r, is a graph with vertex set X^r and two vertices v and u are adjacent if and only if there exists $s \in S$ such that $s(v) = u$.

Definition
The tile graph Γ'_r of the action of G on X^r, is a graph with vertex set X^r and two vertices v and u are adjacent if and only if there exists $s \in S$ such that $s(v) = u$ and $s|_v = \mathbb{1}$.

The tile graph is therefore a subgraph of the Schreier graph.
In our case, tile graphs are always connected.

Example: Basilica group

The Basilica group B[1]: $a = (b, \mathbb{1})e, b = (a, \mathbb{1})\psi_b$ where $\psi_b = (0, 1)$ and e is the identity in S_2.



Post critical sequences:

$\mathcal{P} = \{(0)^{-\omega}, (10)^{-\omega}, (01)^{-\omega}\}$

[1] R. I. Grigorchuk and A. Żuk, On a torsion-free weakly branch group defined by a three state automaton, *I. J. Algebra and Computation* **12** (2002) 223–246.

Schreier graphs of Basilica group

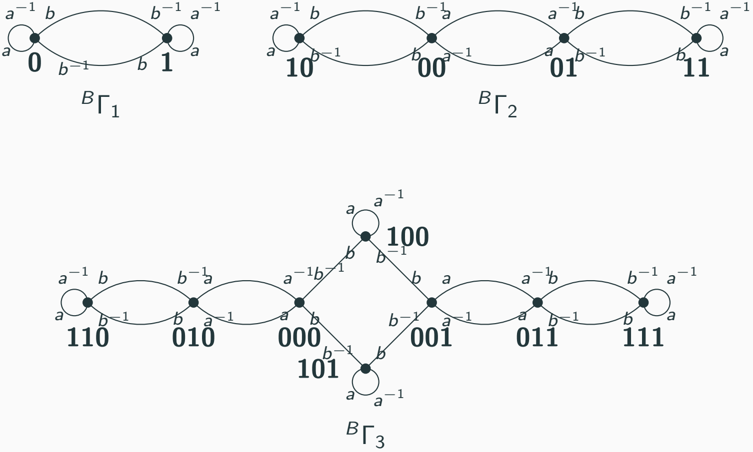
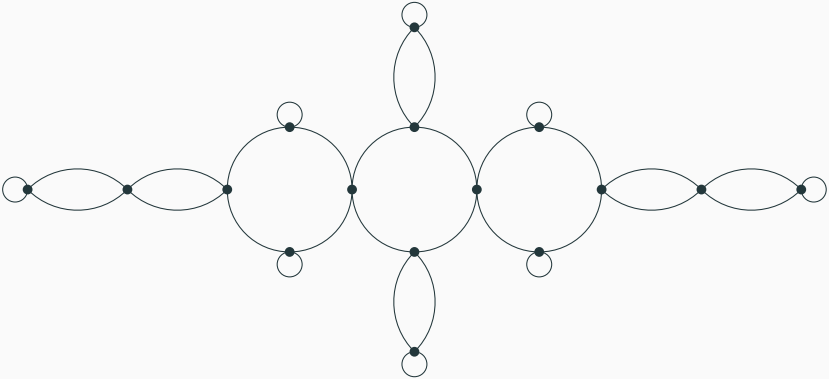


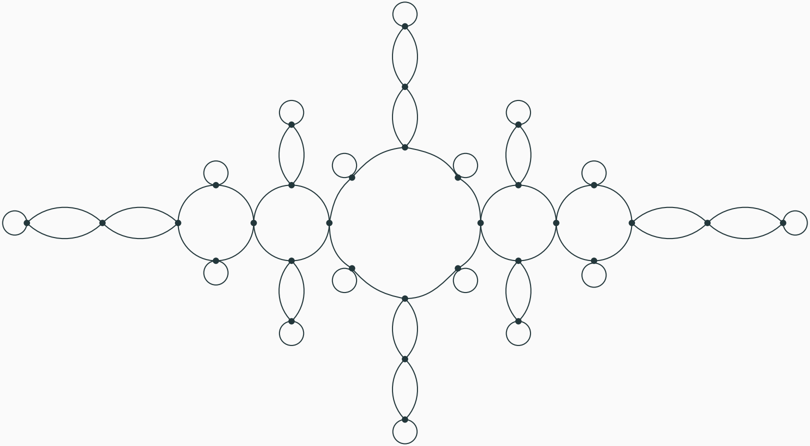
Figure 1: The graphs ${}^{B}\Gamma_1, {}^{B}\Gamma_2$ and ${}^{B}\Gamma_3$ are the Schreier graphs of the Basilica group (B) over X, X^2 and X^3 respectively.

Schreier graphs of Basilica group



$^{B}\Gamma_4$

Schreier graphs of Basilica group



$^{B}\Gamma_5$

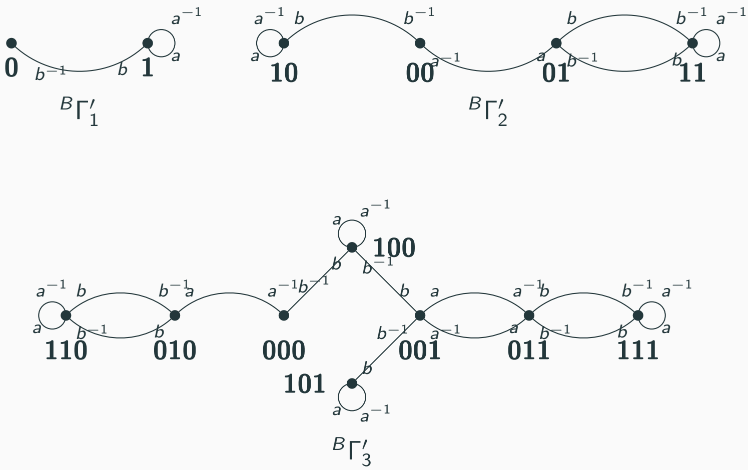
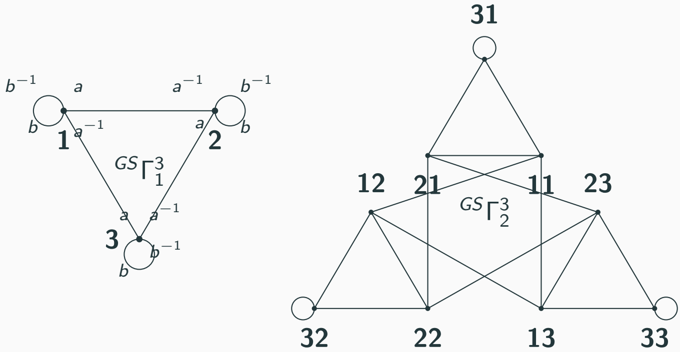


Figure 2: The graphs ${}^{B}\Gamma_1', {}^{B}\Gamma_2'$ and ${}^{B}\Gamma_3'$ are the Tile graphs of the Basilica group (B) over X, X^2 and X^3 respectively.

Examples

Gupta-Sidki p group[2]: $a = (b, b^{-1}, \mathbb{1}, \cdots, \mathbb{1}, a)e, b = (\mathbb{1}, \cdots, \mathbb{1})\psi_b$, where $\psi_b = (1, \cdots, p)$ and $e \in S_p$ and $\mathcal{P} = \{(p)^{-w}, (p)^{-w}1, (p)^{-w}2\}$.

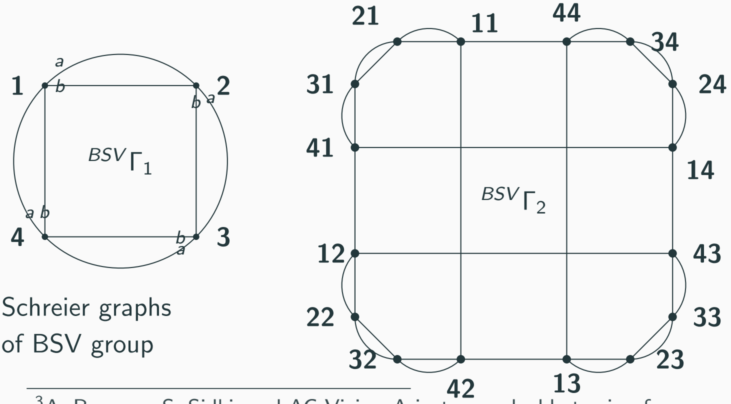


Schreier graphs of Gupta-Sidki p = 3 group (GS)

[2] N. Gupta and S. Sidki, On the Burnside problem for periodic groups, *Mathematische Zeitschrift*, **182** (1983) 385–388.

Examples

Brunner-Sidki-Vieira (BSV)-group[3]:
$a = (\mathbb{1}, \cdots, \mathbb{1}, a^{-1})\psi_a, b = (\mathbb{1}, \cdots, \mathbb{1}, b)\psi_b$ where
$\psi_a = \psi_b = (1, 2, \cdots, n) \in S_n$ and $\mathcal{P} = \{(4)^{-w}, (41)^{-w}, (14)^{-w}\}$.

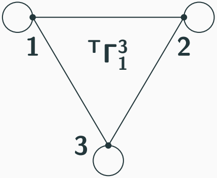


Schreier graphs of BSV group

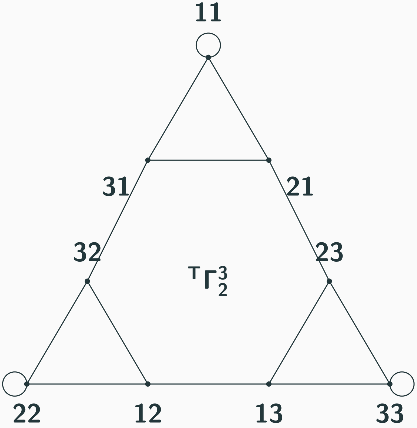
[3]A. Brunner, S. Sidki, and AC Vieira, A just-nonsolvable torsion-free group defined on the binary tree, *Journal of Algebra*, **211** (1999) 99–114.

Examples

Tower of Hanoi group H_n for $n = 3$
$a = (\mathbb{1}, \mathbb{1}, a)(1,2), b = (\mathbb{1}, b, \mathbb{1})(1,3), c = (c, \mathbb{1}, \mathbb{1})(2,3)$ and
$\mathcal{P} = \{(1)^{-w}, (2)^{-w}, (3)^{-w}\}$.



Schreier graphs of
Tower of hanoi group



Generalized replacement product of graphs

If $e = \{v, v'\}$ is an edge of the k-regular graph Γ which has color say s near v and s' near v' and if K is the set of colors $K = \{1, 2, \cdots, k\}$, then the *rotation map* $\mathbf{Rot}_\Gamma : X^n \times K \to X^n \times K$ is defined by

$$\mathbf{Rot}_\Gamma(v, s) = (v', s'), \quad \text{for all} \quad v, v' \in X^n, \quad s, s' \in K.$$

Definition

The generalized replacement product $\Gamma_n \textcircled{g} \Gamma_r$ is $|S|$-regular graph with vertex set $X^{n+r} = X^n \times X^r$, and whose edges are described by the following rotation map: Let $(v, u) \in X^n \times X^r$

$$\mathbf{Rot}((v,u),s) = ((v, s(u)), s^{-1}), \quad \text{if } s \in S \text{ and } s|_u = \mathbb{1}. \tag{1}$$

$$\mathbf{Rot}((v,u),s) = ((s|_u(v), s(u)), s^{-1}), \quad s \in S, s|_u \neq \mathbb{1}, s|_{uv} = \mathbb{1}. \tag{2}$$

$$\mathbf{Rot}((v,u),s) = ((s|_u(v), s(u)), s^{-1}), \quad s \in S, s|_u \neq \mathbb{1}, s|_{uv} \neq \mathbb{1}. \tag{3}$$

Proposition
If $n, r \geq 1$, then the following holds:

1. *The graphs ${}^G\Gamma_n \circledS {}^G\Gamma_r$, ${}^G\Gamma_{n+r}$ are isomorphic.*
2. *${}^G\Gamma_{n+r}$ is an unramified, d^n sheeted graph covering of ${}^G\Gamma_r$.*

Proposition

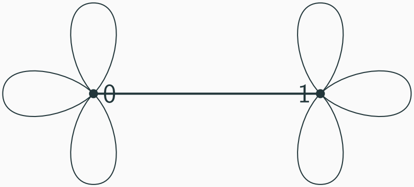
1. *The first rotation map gives the $|X|$ disjoint copies of tile graph ${}^G\Gamma'_r$ indexed by $x \in X$.*
2. *In addition to the first rotation map, the second rotation map adds the edges between the copies of ${}^G\Gamma'_r$ and it produces the tile graph ${}^G\Gamma'_{r+1}$.*
3. *In addition to the first and second rotation maps, the third rotation map adds the edges between the post critical vertices of the tile graph ${}^G\Gamma'_{r+1}$ and it produces the Schreier graph ${}^G\Gamma_{r+1}$.*

Applying the first and second rotation maps to the tile graph ${}^G\Gamma'_r$ is identical to the construction of inflation.

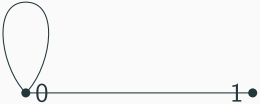
Proposition

Let Γ_n and Γ_r be Schreier graphs of the group generated by bounded automaton S. Then the first and second rotation maps of generalized replacement product $\Gamma_n \text{ⓖ} \Gamma_r$ and the n-th iteration of inflation are equivalent.

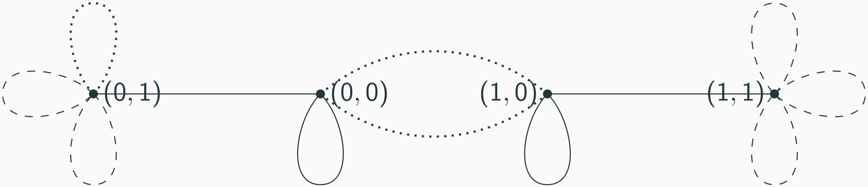
Examples: Generalized replacement product of graphs



$^{Gr}\Gamma_1$

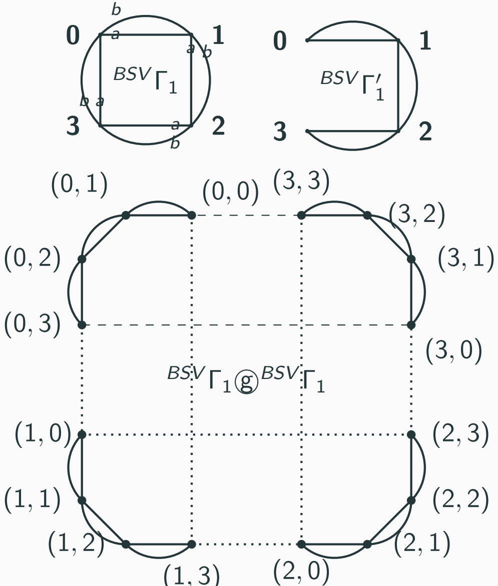


$^{Gr}\Gamma_1'$

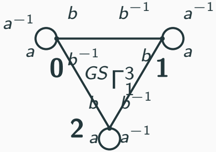


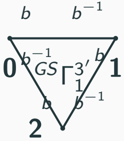
$^{Gr}\Gamma_2$

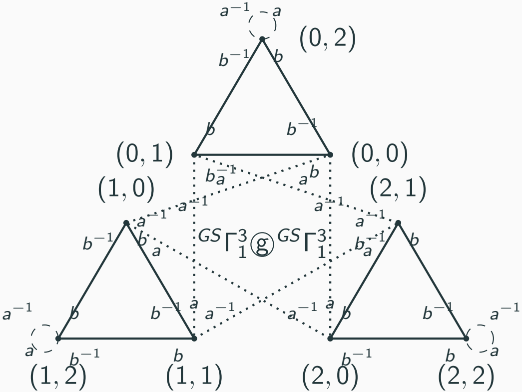
Examples: Generalized replacement product of graphs



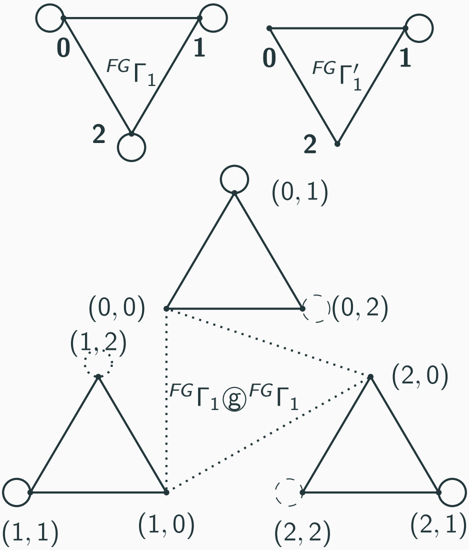
Examples: Generalized replacement product of graphs



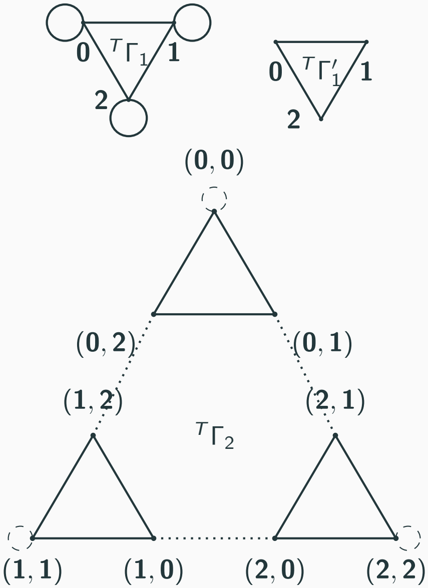




Examples: Generalized replacement product of graphs



Examples: Generalized replacement product of graphs



Given a group generated by bounded automaton, what can be said about the Galois coverings of the corresponding Schreier graphs?

Theorem
If $|\Psi_G| = d$*, then the Schreier graph* ${}^G\Gamma_{n+1}$ *is a Galois covering of* ${}^G\Gamma_n$ *with Galois group* $\mathbb{G} = Gal({}^G\Gamma_{n+1}|{}^G\Gamma_n) \simeq \Psi_G$*.*

In other words,
If $|\Psi_G| = d$*, then the root permutations* ψ_s *are the Frobenius automorphisms associated to* ${}^G\Gamma_{n+1}$ *over* ${}^G\Gamma_n$*.*

Proof Sketch

- Recall that ${}^G\Gamma_{n+r}$ is an unramified q-sheeted covering over ${}^G\Gamma_n$. Take $r = 1$, so we have a covering map $\pi :{}^G \Gamma_{n+1} \to{}^G \Gamma_n$ of degree d. (Use: Generalized replacement product of Schreier graphs.)
- Look at the lifts of every non-tile edge of the graph ${}^G\Gamma_n$ which is of the form $e_{s|_u} = \{u, s(u)\}$, where $s|_u \neq \mathbb{1}$.

  Define a map $\sigma_{s|_u} :\ {}^G\Gamma_{n+1} \to\ {}^G\Gamma_{n+1}$ such that

$$\sigma_{s|_u}(vx_i) = vs|_u(x_i),\ \forall\ vx_i \in X^{n+1}.$$

- By Self-similarity of G, we have

$$\sigma(e_{s|_u})(x) = \psi_{s|_u}(x), \text{ for all } x \in X.$$

Therefore every such $\sigma(e_{s|_u})$ is an automorphism and they are finite in number.

- Use the facts : $|\Psi_G| = d$ and G has level transitive action to show there are exactly d such automorphisms.

$$\Rightarrow \mathbb{G} \;=\; < \psi_{s|_u} \mid s \in S, u \in X^n \text{ with } s|_u \neq \mathbb{1} > \;=\; \Psi_G.$$

- Every $\sigma(e_{s|_u})$ is compatible with the covering map ϕ.

L and zeta functions of Schreier graphs of FG

The covering $\widetilde{Y} = {}^{FG}\Gamma_2$ over the graph $Y = {}^{FG}\Gamma_1$ is 3-sheeted normal covering. In this case the Galois group is $\mathbb{G} = \ < g = (1,2,3) \mid g^3 = e > \ \simeq \dfrac{\mathbb{Z}}{3\mathbb{Z}}$. We now write all matrices $A(g), g \in \mathbb{G}$.

$$A(e) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad A(g) = A(g^2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The *Artinized* adjacency matrices A_{χ_i}, where χ_i is an irreducible character of $\mathbb{G}$.

$$A_{\chi_1} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad A_{\chi_2} = A_{\chi_3} = A(e).$$

Reciprocals of L functions for $\widetilde{Y}|Y$ are as follows

1) For A_{χ_1}

$$\zeta_Y(t)^{-1} = L(t, A_{\chi_1}, \widetilde{Y}|Y)^{-1} = (1-t^2)^3(t-1)(3t-1)\left(3t^2-t+1\right)^2$$

2) As $A_{\chi_2} = A_{\chi_3}$

$$L(t, A_{\chi_2}, \widetilde{Y}|Y)^{-1} = L(t, A_{\chi_3}, \widetilde{Y}|Y)^{-1}$$

$$= (1-t^2)^3\left(3t^2-t+1\right)^2\left(9t^4-6t^3+t^2-2t+1\right)^2$$

We have

$$\zeta_{\widetilde{Y}}(t)^{-1} = \prod_{\chi_i \in \{\chi_1,\chi_2,\chi_3\}} L(t, A_{\chi_i}, \widetilde{Y}|Y)^{-1}$$

$$= (1-t^2)^9(t-1)(3t-1)\left(3t^2-t+1\right)^4\left(9t^4-6t^3+t^2-2t+1\right)^2.$$

Zeta and L functions of Schreier graphs of Basilica group

Reciprocals of L functions for $\widetilde{Y}|Y = {}^{B}\Gamma_3|{}^{B}\Gamma_2$:
1) For A_1

$$\zeta_{\Gamma_2}(t)^{-1} = L(t, A_1, \widetilde{Y}|Y)^{-1} = (1-t^2)^4(t-1)(3t-1)\left(3t^2+1\right)\left(9t^4-2t^2+1\right).$$

2) For A_σ $L(t, A_\sigma, \widetilde{Y}|Y)^{-1} = (1-t^2)^4\left(3t^2-2t+1\right)$
$\times\left(27t^6-18t^5+3t^4-4t^3+t^2-2t+1\right).$

As $\widetilde{Y}|Y$ is normal covering, we have

$$\zeta_{\Gamma_3}(t)^{-1} = L(t, A_1, \widetilde{Y}|Y)^{-1} L(t, A_\sigma, \widetilde{Y}|Y)^{-1}$$

$$= (1-t^2)^8(t-1)(3t-1)\left(3t^2+1\right)\left(3t^2-2t+1\right)\left(9t^4-2t^2+1\right)$$
$$\left(27t^6-18t^5+3t^4-4t^3+t^2-2t+1\right).$$

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Thank you very much for your attention!