Self-similar groups: old and new results

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In 1998 Volodya Nekrashevych and I collaborated on the paper "Automorphisms of the binary tree: state-closed subgroups and dynamics of 1/2 endomorphisms" which appeared in print in 2004. Over the past 20 years this paper stimulated the development of many ideas about self-similarity in groups, some of which are treated here.

1 Self-similarity

A group G is *self-similar* if it is a state-closed subgroup of the automorphism group of an infinite regular one-rooted m-tree $\mathcal{T}_m$; in particular, G is residually finite. If the action of G on the first level of $\mathcal{T}_m$ is transitive we say that G is a transitive self-similar group . A group acting on the tree $\mathcal{T}_m$ is *finite-state* provided each of its elements has a finite number of states. An *automata group* is a finitely generated self-similar and finite-state group.

Self-similar and automaton representations are known for groups ranging from the torsion groups of Grigorchuk and of Gupta-Sidki to Arithmetic groups (Kapovich, 2012) and to non-abelian free groups (Glasner-Mozes, 2005; Aleshin-Vorobets, 2007). Two softwares for computation in self- similar groups are available in GAP, by Bartholdi and by Muntyan-Savchuk.

The logic of self-similar and automaton groups is complex. Two almost simultaneous results on unsolvability, shown in 2017: (1) P. Gillibert proved that deciding the order of an element in an automaton group unsolvable; (2) L. Bartholdi and I. Mitrofanov proved that the word problem in self-similar groups unsolvable.

2 Virtual Endomorphisms

We use the notion of virtual endomorphisms to produce transitive self-similar actions. This concept often corresponds to contraction which had already appeared in Lie Groups and in Dynamical Systems; eg. $2\mathbb{Z} \rightarrow \mathbb{Z}$ defined by $2n \longmapsto n$.

Given a general group G, consider a *similarity pair* (H, f) where H a subgroup of G of finite index m and $f : H \rightarrow G$ a homomorphism called a *virtual endomorphism* of G. If f is a monomorphism and the image H^f is also of finite index in G then H and H^f are commensurable in G and f is a *virtual automorphism*.

Given the pair (H, f) we produce by a generalized Kaloujnine-Krasner construction (abbreviated by *K-K),* a transitive state-closed representation of G on the m-tree (or simply of degree m) as follows:

let $T = \{t_0 (= e), t_1, ..., t_{m-1}\}$ be a right transversal T of H in G and $\sigma : G \rightarrow Perm(T)$ be the transitive permutational representation of G on T induced from the action of the group on the right cosets of H. For each $g \in G$, we obtain: (1) its image g^σ under σ,(2) an m-tuple of elements $(h_0, ..., h_{m-1})$ of H, called co-factors of g, defined by

$$h_i = (t_i g) \left((t_i)^{g^\sigma} \right)^{-1}.$$

Then, the Kaloujnine-Krasner theorem gives us a homomorphism of G into the wreath product

$$H wr_{(T)} G^\sigma$$

defined by

$$\varphi_1 : g \mapsto (h_i \mid 0 \leq i \leq m-1)\, g^\sigma.$$

This homomorphism is regarded as a first approximation of a representation of G on the m-ary tree. We use the virtual endomorphism $f : H \rightarrow G$ to iterate the process infinitely:

$$\varphi : g \mapsto \left(\left((h_i)^f\right)^{\varphi} \mid 0 \le i \le m-1\right) g^{\sigma}.$$

The kernel of φ, called the f-core of H, is the largest subgroup K of H which is normal in G and is f-invariant (in the sense $K^f \le K$). When the kernel of φ is trivial, the similarity pair (H, f) and f are said to be *simple*. A transitive state-closed group G of degree m determines a pair (G_0, π_0) where G_0 is the stabilizer of the 0-vertex and the projection π_0 is simple. On the other hand, a similarity pair (H, f) for G where $[G, H] = m$ and f *simple* provides by the *K-K* construction a faithful transitive state-closed representation φ of G of degree m such that $[G^{\varphi}, H^{\varphi}] = m$.

Problem 1 *There are just two faithful transitive state-closed representations of the cyclic group $G = \langle a \rangle$ of order 2 on the binary tree $a \mapsto \sigma = (e, e)\, s$ with s the permutation $(0, 1)$ and $a \mapsto \delta = (\delta, \delta)\, s$. On the other hand, K-K produces the unique representation $a \mapsto \sigma = (e, e)\, s$. What is the exact relationship between self-similar representations and those produced by K-K?*

3 Abelian groups

Two papers by Nekrachevych-S (2004). and Brunner-S (2010) develop a fairly general study of self-similar abelian groups.

Example 2 *Let*

$$\begin{aligned} G &= \mathbb{Z}^d = \langle x_1, x_2, ..., x_d \rangle , \\ H &= \langle m x_1, x_2, ..., x_d \rangle , \\ f &: m x_1 \mapsto x_2 \mapsto x_3 \mapsto ... \mapsto x_d \mapsto x_1. \end{aligned}$$

Then with respect to this data, G is represented as a transitive automaton group on the m-ary tree:

$$\begin{aligned} \alpha_1 &= (e, e, ..., e, \alpha_2)\,\sigma, \text{ where } \sigma = (0, 1, ..., m-1) , \\ \alpha_2 &= (\alpha_3, ..., \alpha_3) , ..., \alpha_d = (\alpha_1, ..., \alpha_1) . \end{aligned}$$

The class of abelian state-closed groups A is closed under topological closure and also under diagonal closure (by adding the diagonals $a^z = (a, a, ..., a)$ for all $a \in A$). These facts allow exponentiation of elements of A by $\sum_{0 \leq i \leq m} \alpha_i z^i \in \mathbb{Z}_2[z]$ which translates abelian state-closed groups language to a commutative algebra one over $\mathbb{Z}_2$.

A faithful transitive self-similar representations of $\mathbb{Z}^\omega$ using transcendentals in $\mathbb{Z}_2$:

Theorem 3 *(Bartholdi-S, 2018) Let θ be a transcendental unit in $\mathbb{Z}_2$. Consider the ring $R = \mathbb{Z}[1/(2\theta)]$. Let G be the additive group $G = R \cap \mathbb{Z}_2$ and $H = G \cap 2\mathbb{Z}_2$. Define $d : 2\mathbb{Z}_2 \to \mathbb{Z}_2$ by $a \mapsto a/(2\theta)$ and $f = d|_H : H \to G$. Then, G is isomorphic to $\mathbb{Z}^\omega$ and the pair (H, f) is simple. However there does not exist a faithful automaton representation of $\mathbb{Z}^\omega$.*

Problem 4 *Is there a faithful transitive self-similar representations of $(\mathbb{Z}_2)^\omega$?*

4 Nilpotent groups

(with A. Berlatto, 2007)

Theorem 5 *Let G be a general nilpotent group, H a subgroup of finite index m in G, $f \in Hom(H, G)$ and $L = f$-$core(H)$. Then,*

$$\ker(f) \leq \sqrt[H]{L} = \{h \in H : h^n \in L \text{ for some } n\},$$

the isolator of L in H.

Denote finitely generated torsion-free nilpotent groups of class c by $\mathfrak{T}_c$-groups.

Corollary 6 *Let G be an $\mathfrak{T}_c$-group and (H, f) a simple similarity pair for G. Then, f is an almost automorphism of G. In the Malcev completion of G, the virtual endomorphism f becomes an automorphism of G.*

Class 2 **groups are rich in self similarity**:

Theorem 7 *Let G be an $\mathfrak{T}_2$-group and H a subgroup of finite index in G. Then there exists a subgroup K of finite index in H which admits a simple* **epimorphism** $f : K \rightarrow G$.

Given an integer $m > 1$, let $l(m)$ be the number of prime divisors of m (counting multiplicities) and $s(G)$ the derived length of G.

Theorem 8 *Let G be an $\mathfrak{T}_c$-group and H a subgroup of finite index m in G. If*

$f : H \rightarrow G$ is simple then $s(G) \leq l(m)$.

There is no such bound for the nilpotency class $c(G)$:

Example 9 *There exists an ascending sequence of simple triples* (G_n, H_n, f_n) *where the* G_n*'s are metabelian* 2*-generated* $\mathfrak{T}_c$*-groups with* $[G_n, H_n] = 4$ *and nilpotency class* $c = n$.

On non-existence:

Problem 10 *J. Dyer (1970) constructed a rational nilpotent Lie algebra with nilpotent automophism group. The construction yields an* $\mathfrak{T}_c$*-group which does not admit a faithful transitive self-similar representation. The group is* 2*-generated* $\mathfrak{T}_6$*-group with Hirsch length* 9*. Are there* $\mathfrak{T}_3$*-groups which are not self-similar?*

5 Metabelian groups

Self-similar representations of metabelian groups is the next central issue of study. The following treats those of split type.

Theorem 11 *(Kochloukova-S) Let X be a finitely generated abelian group and B be a finitely generated, right $\mathbb{Z}X$-module of Krull dimension* 1 *such that $C_X(B) = \{x \in X \mid B(x-1) = 0\} = 1$. Then $G = B \rtimes X$ admits a faithful transitive self-similar representation.*

The strategy of the proof : Show that there exists δ in $\mathbb{Z}X$ such that δB is of finite index in B and such that the map $f(\delta b) = b$ for all b in B is core-free. Define the subgroup $H = (\delta B) \rtimes X$ and extend f by $f|_X = id_X$. Then f defines a simple virtual endomorphism $f : H \to G$.

Under certain additional conditions, the group G from the above theorem is **finitely presented** and **of type** FP_m. The conditions come from the Bieri-Strebel theory of m-tame modules and its relation to the FP_m-Conjecture for metabelian groups.

6 Wreath Products

For wreath products of abelian groups, the guiding example is the lamplighter group $G = C_2 wr C$ (Grigorchuk-Zuk). It has a faithful transitive self-similar representation on the binary tree and is generated by s, α where s is the transposition $(0, 1)$ and $\alpha = (\alpha, \alpha s)$.

Since we are dealing with residually finite groups, the following result of Gruenberg is fundamental.

Theorem 12 *The wreath product $G = BwrX$ is residually finite iff B, X are residually finite and either B is abelian or X is finite.*

Theorem 13 *(A. Dantas-S, 2017) Let $G = BwrX$ be a transitive self-similar wreath product of abelian groups. If X is torsion free then B is a torsion group of finite exponent. Thus, $\mathbb{Z}wr\mathbb{Z}$ cannot have a faithful transitive self-similar representation.*

Let p be a prime number, $d \geq 1$ and define the groups $G_{p,d} = C_p wr C^d$.

Theorem 14 *(A. Dantas-S, 2017) Let $d \geq 2$. Then $G_{p,d}$ does not have a faithful transitive self-similar representation on the p-adic tree but has such a representation on the p^2-tree.*

The groups $G_{p,d}$ belong to a general construction:

Theorem 15 *(Bartholdi-S, 2018) Let B be finite abelian group, X a transitive self-similar group. Then, $BwrX$ is a transitive self-similar and is finite-state whenever B and X are.*

Remark 16 *(Savchuk-S., 2016) There are non-trivial extensions of groups of type $BwrX$ which are transitive automaton groups. An example of this is $G = (\langle x, y\rangle \, wr \, \langle t\rangle) \, \langle a\rangle$ where $\langle x, y\rangle$ is the 4-group, t has infinite order and a has order 2.*

7 Linear Groups

Theorem 17 *(Brunner-S, 1998) The affine groups* $\mathbb{Z}^n \rtimes GL(n,\mathbb{Z})$ *are transitive automata groups of degree exponential in* n. *In particular and importantly,* $GL(n,\mathbb{Z})$ *is a finite-state group.*

Problem 18 *Does* $GL(n,\mathbb{Z})$ *admit a faithful self-similar representation for* $n \geq 2$*? Note that Z. Sunik produced in (Kapovich, 2012) a faithful self-similar action of* $PSL(2,\mathbb{Z})$ *on the* 3*-tree.*

Reduction of tree-degree:

Theorem 19 *(Nekrashevych-S, 2004) Let* $B(n,\mathbb{Z})$ *be the (pre-Borel) subgroup of finite index in* $GL_n(\mathbb{Z})$ *consisting of the matrices whose entries above the main diagonal are even integers. The affine linear groups* $\mathbb{Z}^n \rtimes B(n,\mathbb{Z})$ *are realizable faithfully as transitive automata groups acting on the* **binary** *tree.*

With D. Kochloukova, 2017.

Given a commutative algebra A we denote the upper triangular $m \times m$ matrix group, or Borel subgroup, with coefficients in A by $U(m, A)$ and its projective quotient by $PU(m, A)$ which is nilpotent-by-abelian.

Theorem 20 *Let $n \geq 1$ be an integer, p be a prime number and*

$$A = \mathbb{F}_p[x^{\pm 1}, \frac{1}{f_1}, \ldots, \frac{1}{f_{n-1}}]$$

the subring of $\mathbb{F}_p(x)$, where $f_0 = x, f_1, \ldots, f_{n-1} \in F_p[x] \backslash F_p(x-1)$ are pairwise different, monic, irreducible polynomials. Then $PU(m, A)$ is a transitive automaton group of degree p^l, where l is a cubic polynomial of degree m.

The metabelian version is

$$G = A \rtimes Q$$

where $Q = \langle x_0, x_1, ..., x_{n-1} \rangle \cong \mathbb{Z}^n$ and x_i acts on A as multiplication by f_i $(0 \le i \le n-1)$. Let $I = (x-1)\,A$ and let $H = I \rtimes Q$. Then $[G : H] = p$ and the map $f : ((x-1)\,r, q) \mapsto (r, q)$ is a simple epimorphism. The group G is **finitely presented**, of type FP_n, not of type FP_{n+1}.

Similar to $B(n, \mathbb{Z}])$, define $B(n, \mathbb{F}_p[x])$ as the subgroup of $GL_n(\mathbb{F}_p[x])$ consisting of the matrices whose entries above the main diagonal belong to the ideal $(x-1)\mathbb{F}_p[x]$.

Theorem 21 *Let $n \ge 2$ be an integer, p a prime number and G the affine group $\mathbb{F}_p[x]^n \rtimes B(n, \mathbb{F}_p[x])$. Then G is transitive, finite-state and state-closed of degree p.*

Problem 22 *The group here is not finitely generated when $n = 2$, yet is finitely generated for $n \ge 3$. When is it finitely presented?*