Generating Lamplighter-like Groups with Bireversible Automata

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September 17, 2017

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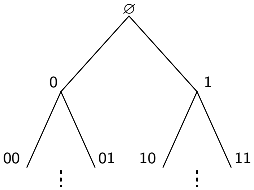

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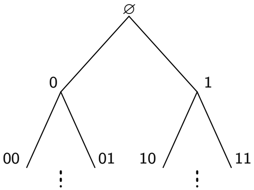

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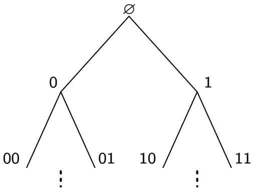

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Any such function on the finite words uniquely determines a function on X^ω, the set of infinite words. Likewise, any "prefix relation preserving" function on X^ω defines an automorphism of the tree.

Finite State (Mealy) Automata

Definition

A finite state *(Mealy) automaton* is $\mathcal{A}$ is a 4-tuple $\mathcal{A} = (Q, X, \delta, \lambda)$ where Q is finite a set of states, X is a finite alphabet, $\delta\colon Q \times X \to Q$ is the *transition function*, and $\lambda\colon Q \times X \to X$ is the *output function*. For each $q \in Q$ and $x \in X$, we will use the notation $\lambda_q(x)$ to mean $\lambda(q, x)$.

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λ_q extends to a function X^* and X^ω:

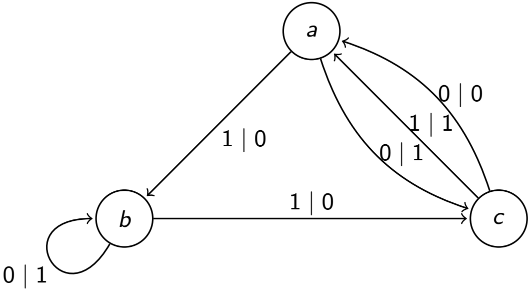
$$\lambda_q(x_0, x_1, \ldots, x_n) = \lambda_q(x_0)\lambda_{\delta(q,x_0)}(x_1, \ldots, x_n)$$

and

$$\lambda_q(x_0, x_1, \ldots) = \lim_{n\to\infty} \lambda_q(x_0, x_1, \ldots, x_n).$$

Finite State (Mealy) Automata

Definition by example: The Alëshin Automaton



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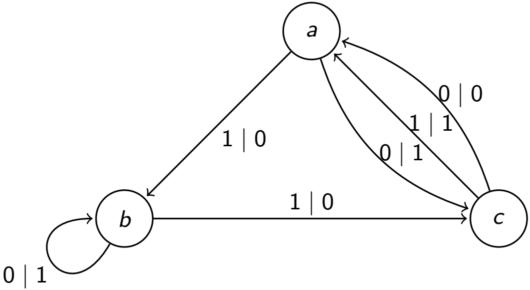

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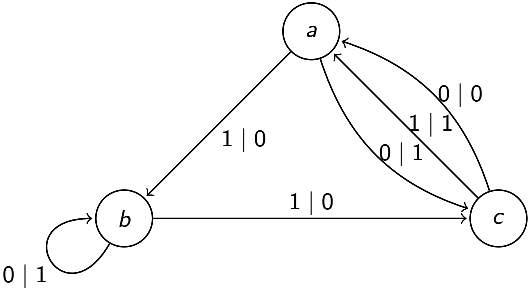

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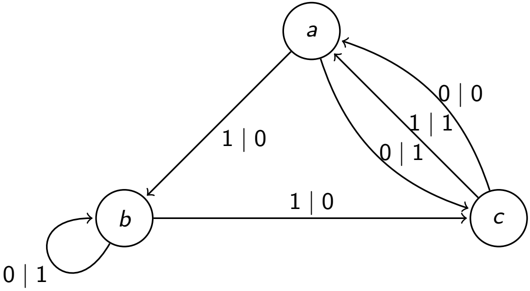

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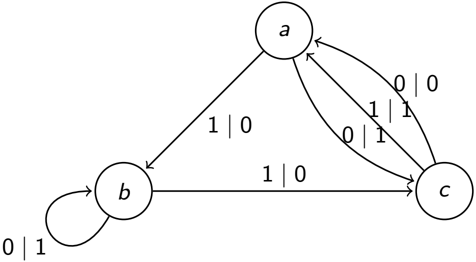
The group generated by the states of $\mathcal{A}$ is called the automaton group for $\mathcal{A}$ and denoted $\mathcal{G}(\mathcal{A})$.

Duals and inverses

For an $\mathcal{A} = (Q, X, \delta, \lambda)$, its *dual automaton* $\partial\mathcal{A}$ is given by (X, Q, λ, δ), i.e., the alphabet and states are interchanged and the output and transition functions are interchanged.



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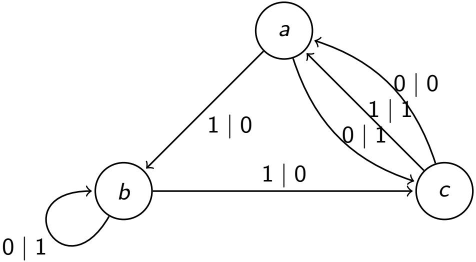

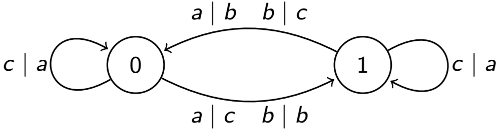

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If $\mathcal{A}$, $\partial\mathcal{A}$, and $\partial\mathcal{A}^{-1}$ are invertible, $\mathcal{A}$ is called *bireversible*. In this case, all 8 possible automata are invertible.

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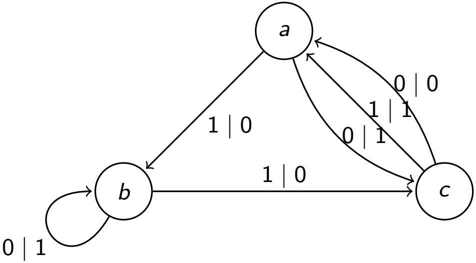

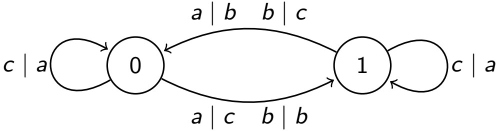

Why bireversible automaton groups?

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Bireversible automaton groups act essentially freely on the boundary of the rooted tree hence, one can potentially compute spectral measures for their random walks via the action on the tree.

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Various automaton representations of lamplighter like groups have been found (Grigorchuk, Żuk, Nekrashevych, Sushanksii, Sidki, Savchuk, Steinberg, Silva, Suniǩ, Juschenko, Wesolek, Bartholdi.....)

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Likewise, Ahmed and Savchuk realized $(\mathbb{Z}/2\mathbb{Z})^2 \wr \mathbb{Z}$ as a bireversible automaton group (2018).

Rational power series approach

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Then for any $f \in R[[t]]$ define two functions μ_f and α_f on R^ω given by

$$\mu_f : g(t) \mapsto f(t)g(t)$$

$$\alpha_f : g(t) \mapsto f(t) + g(t)$$

Exercise: For f invertible, these define automorphisms of T_R and for $\mu_f \alpha_h \mu_{f^{-1}} = \alpha_{fh}$.

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Proposition (S, Steinberg)

Let $f(t) = r\left(\frac{1-at}{1-bt}\right)$ where $r \in R^\times$ and $a, b \in R$. Then μ_f is finite state with set of states $\{\alpha_{-sra}\mu_f\alpha_{sb} : s \in R\}$. Moreover, for any $s \in R$, the state $\alpha_{-sra}\mu_f\alpha_{sb}$ permutes the degree zero terms via:

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We can associate to f a finite state automaton $\mathcal{A}_f$ with states $\{\alpha_{-sra}\mu_f\alpha_{sb} : s \in R\}$. Transition and output functions are

$$\delta(\alpha_{-sra}\mu_f\alpha_{sb}, \tilde{s}) = \alpha_{-(sb+\tilde{s})ra}\mu_f\alpha_{(sb+\tilde{s})b}$$

and

$$\lambda(\alpha_{-sra}\mu_f\alpha_{sb}, \tilde{s}) = r(\tilde{s} + (b-a)s)$$

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(Sketch) $\alpha_{-sra}\mu_f\alpha_{sb} = \alpha_{-sra+sbf}\mu_f = \alpha_{s(-ar+bf)}\mu_f$ so we can consider the generating set $\{\alpha_{s(-ar+bf)}, \mu_f : s \in R\}$.

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If $b - a$ a unit, then $(-ar + bf)f^m$ is linearly independent over R (in fact if and only if). And so,
$N = \langle \alpha_{s(-ar+bf)f^m} : m \in \mathbb{Z}\rangle \cong \bigoplus_{\mathbb{Z}} R^+$.

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N is normal and torsion and so intersects $\langle \mu_f \rangle \cong \mathbb{Z}$ trivially and so $\mathbb{G}(\mathcal{A}_f) \cong R^+ \wr Z$. □

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where $r \in R^\times$, $a, b \in R$ and $a-b$ is a unit. Then $\mathcal{A}_f$ has $|R|$ states. Moreover,

1. *$\mathcal{A}_f$ is reversible if and only if b is a unit.*
2. *$(\mathcal{A}_f)^{-1}$ is reversible if and only if a is a unit.*
3. *$\mathcal{A}_f$ is bireversible if and only if both a and b are units.*

Commutative rings with a, b and a-b units

Theorem (S, Steinberg)

Let A be a finite abelian group. Then there is a finite commutative ring R with $R^+ \cong A$ and two elements $a, b \in R^\times$ with $a - b \in R^\times$ if and only if $A \cong A_1 \oplus A_2$ where A_1 has odd order and $A_2 \cong (\mathbb{Z}/2\mathbb{Z})^{a_1} \oplus (\mathbb{Z}/2^2\mathbb{Z})^{a_2} \oplus \cdots \oplus (\mathbb{Z}/2^t\mathbb{Z})^{a_t}$ with $a_i \neq 1$ for all $1 \leq i \leq t$.

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Corollary

For any finite abelian group described in the last theorem, there exists a bireversible automaton generating $A \wr \mathbb{Z}$.

Examples

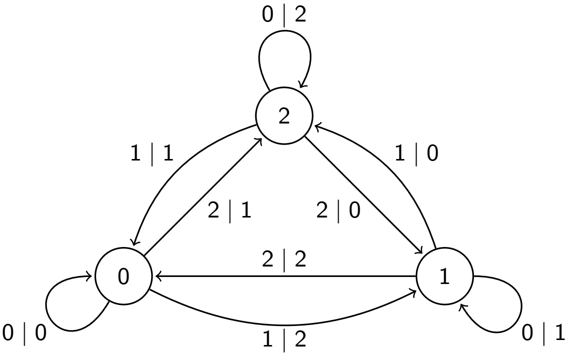

Figure: The bireversible automaton given by Bodarenko, D'Angeli, Rodaro for $\mathbb{Z}/3\mathbb{Z} \wr \mathbb{Z}$ and corresponding to $f(t) = 2\left(\frac{1-2t}{1-t}\right)$.(Observed by Bondarenko and Savchuk)

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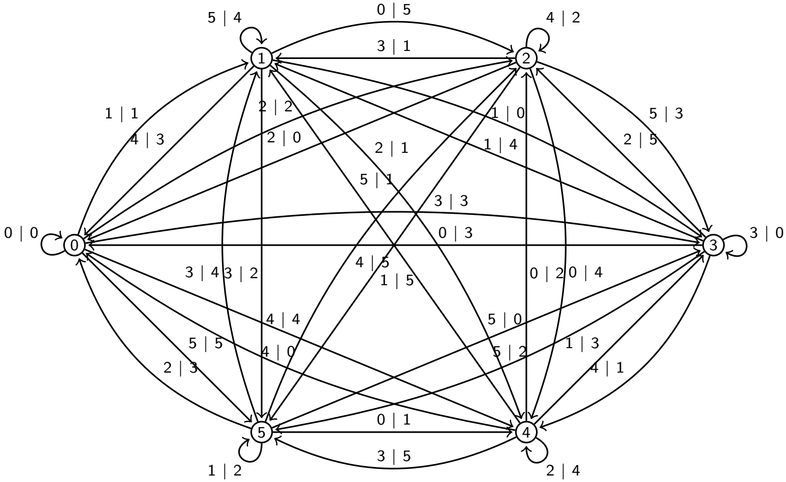

Figure: An automaton which generates $\mathbb{Z}/6\mathbb{Z} \wr \mathbb{Z}$ for $f = \dfrac{1-3t}{1-2t}$ that is not reversible and whose inverse is also not reversible.

Take $\mathcal{O} = \mathbb{Z}_2[\zeta]$ with ζ a third root of unity and $R = \mathcal{O}/4\mathcal{O} \cong \mathbb{Z}/4\mathbb{Z}[\zeta]$. Taking $r = 1$, $a = 1$, and $b = 2 + \zeta$, we find that a, b, and $b - a$ are all units with inverses 1, $3 + \zeta$, and $1 + \zeta$ respectively. $R^+ \cong (\mathbb{Z}/4\mathbb{Z})^2$.

state \ letter	0	1	2	3	ζ	$1+\zeta$	$2+\zeta$	$3+\zeta$	2ζ	$1+2\zeta$	$2+2\zeta$	$3+2\zeta$	3ζ	$1+3\zeta$	$2+3\zeta$	$3+3\zeta$
0	0	1	2	3	x	$1+\zeta$	$2+\zeta$	$3+\zeta$	2ζ	$1+2\zeta$	$2+2\zeta$	$3+2\zeta$	3ζ	$1+3\zeta$	$2+3\zeta$	$3+3\zeta$
1	$2+\zeta$	$3+\zeta$	ζ	$1+\zeta$	$2+2\zeta$	$3+2\zeta$	2ζ	$1+2\zeta$	$2+3\zeta$	$3+3\zeta$	3ζ	$1+3\zeta$	2	3	0	1
2	2ζ	$1+2\zeta$	$2+2\zeta$	$3+2\zeta$	3ζ	$1+3\zeta$	$2+3\zeta$	$3+3\zeta$	0	1	2	3	ζ	$1+\zeta$	$2+\zeta$	$3+\zeta$
3	$2+3\zeta$	$3+3\zeta$	3ζ	$1+3\zeta$	2	3	0	1	$2+\zeta$	$3+\zeta$	ζ	$1+\zeta$	$2+2\zeta$	$3+2\zeta$	2ζ	$1+2\zeta$
ζ	$3+\zeta$	ζ	$1+\zeta$	$2+\zeta$	$3+2\zeta$	2ζ	$1+2\zeta$	$2+2\zeta$	$3+3\zeta$	3ζ	$1+3\zeta$	$2+3\zeta$	3	0	1	2
$1+\zeta$	$1+2\zeta$	$2+2\zeta$	$3+2\zeta$	2ζ	$1+3\zeta$	$2+3\zeta$	$3+3\zeta$	3ζ	1	2	3	0	$1+\zeta$	$2+\zeta$	$3+\zeta$	ζ
$2+\zeta$	$3+3\zeta$	3ζ	$1+3\zeta$	$2+3\zeta$	3	0	1	2	$3+\zeta$	ζ	$1+\zeta$	$2+\zeta$	$3+2\zeta$	2ζ	$1+2\zeta$	$2+2\zeta$
$3+\zeta$	1	2	3	0	$1+\zeta$	$2+\zeta$	$3+\zeta$	ζ	$1+2\zeta$	$2+2\zeta$	$3+2\zeta$	2ζ	$1+3\zeta$	$2+3\zeta$	$3+3\zeta$	3ζ
2ζ	$2+2\zeta$	$3+2\zeta$	2ζ	$1+2\zeta$	$2+3\zeta$	$3+3\zeta$	3ζ	$1+3\zeta$	2	3	0	1	$2+\zeta$	$3+\zeta$	ζ	$1+\zeta$
$1+2\zeta$	3ζ	$1+3\zeta$	$2+3\zeta$	$3+3\zeta$	0	1	2	3	ζ	$1+\zeta$	$2+\zeta$	$3+\zeta$	2ζ	$1+2\zeta$	$2+2\zeta$	$3+2\zeta$
$2+2\zeta$	2	3	0	1	$2+\zeta$	$3+\zeta$	ζ	$1+\zeta$	$2+2\zeta$	$3+2\zeta$	2ζ	$1+2\zeta$	$2+3\zeta$	$3+3\zeta$	3ζ	$1+3\zeta$
$3+2\zeta$	ζ	$1+\zeta$	$2+\zeta$	$3+\zeta$	2ζ	$1+2\zeta$	$2+2\zeta$	$3+2\zeta$	3ζ	$1+3\zeta$	$2+3\zeta$	$3+3\zeta$	0	1	2	3
3ζ	$1+3\zeta$	$2+3\zeta$	$3+3\zeta$	3ζ	1	2	3	0	$1+\zeta$	$2+\zeta$	$3+\zeta$	ζ	$1+2\zeta$	$2+2\zeta$	$3+2\zeta$	2ζ
$1+3\zeta$	3	0	1	2	$3+\zeta$	ζ	$1+\zeta$	$2+\zeta$	$3+2\zeta$	2ζ	$1+2\zeta$	$2+2\zeta$	$3+3\zeta$	3ζ	$1+3\zeta$	$2+3\zeta$
$2+3\zeta$	$1+\zeta$	$2+\zeta$	$3+\zeta$	ζ	$1+2\zeta$	$2+2\zeta$	$3+2\zeta$	2ζ	$1+3\zeta$	$2+3\zeta$	$3+3\zeta$	3ζ	1	2	3	0
$3+3\zeta$	$3+2\zeta$	2ζ	$1+2\zeta$	$2+2\zeta$	$3+3\zeta$	3ζ	$1+3\zeta$	$2+3\zeta$	3	0	1	2	$3+\zeta$	ζ	$1+\zeta$	$2+\zeta$

Table: The transition table for $f = r\left(\frac{1-at}{1-bt}\right)$ with $r = 1$, $a = 1$, and $b = 2 + \zeta$.

Examples

Take $\mathcal{O} = \mathbb{Z}_2[\zeta]$ with ζ a third root of unity and $R = \mathcal{O}/4\mathcal{O} \cong \mathbb{Z}/4\mathbb{Z}[\zeta]$. Taking $r = 1$, $a = 1$, and $b = 2 + \zeta$, we find that a, b, and $b - a$ are all units with inverses 1, $3 + \zeta$, and $1 + \zeta$ respectively. $R^+ \cong (\mathbb{Z}/4\mathbb{Z})^2$.

state \ letter	0	1	2	3	ζ	$1+\zeta$	$2+\zeta$	$3+\zeta$	2ζ	$1+2\zeta$	$2+2\zeta$	$3+2\zeta$	3ζ	$1+3\zeta$	$2+3\zeta$	$3+3\zeta$
0	0	1	2	3	ζ	$1+\zeta$	$2+\zeta$	$3+\zeta$	2ζ	$1+2\zeta$	$2+2\zeta$	$3+2\zeta$	3ζ	$1+3\zeta$	$2+3\zeta$	$3+3\zeta$
1	$1+\zeta$	$2+\zeta$	$3+\zeta$	ζ	$1+2\zeta$	$2+2\zeta$	$3+2\zeta$	2ζ	$1+3\zeta$	$2+3\zeta$	$3+3\zeta$	3ζ	1	2	3	0
2	$2+2\zeta$	$3+2\zeta$	2ζ	$1+2\zeta$	$2+3\zeta$	$3+3\zeta$	3ζ	$1+3\zeta$	2	3	0	1	$2+\zeta$	$3+\zeta$	ζ	$1+\zeta$
3	$3+3\zeta$	3ζ	$1+3\zeta$	$2+3\zeta$	3	0	1	2	$3+\zeta$	ζ	$1+\zeta$	$2+\zeta$	$3+2\zeta$	2ζ	$1+2\zeta$	$2+2\zeta$
ζ	3	0	1	2	$3+\zeta$	ζ	$1+\zeta$	$2+\zeta$	$3+2\zeta$	2ζ	$1+2\zeta$	$2+2\zeta$	$3+3\zeta$	3ζ	$1+3\zeta$	$2+3\zeta$
$1+\zeta$	ζ	$1+\zeta$	$2+\zeta$	$3+\zeta$	2ζ	$1+2\zeta$	$2+2\zeta$	$3+2\zeta$	3ζ	$1+3\zeta$	$2+3\zeta$	$3+3\zeta$	0	1	2	3
$2+\zeta$	$1+2\zeta$	$2+2\zeta$	$3+2\zeta$	2ζ	$1+3\zeta$	$2+3\zeta$	$3+3\zeta$	3ζ	1	2	3	0	$1+\zeta$	$2+\zeta$	$3+\zeta$	ζ
$3+\zeta$	$2+3\zeta$	$3+3\zeta$	3ζ	$1+3\zeta$	2	3	0	1	$2+\zeta$	$3+\zeta$	ζ	$1+\zeta$	$2+2\zeta$	$3+2\zeta$	2ζ	$1+2\zeta$
2ζ	2	3	0	1	$2+\zeta$	$3+\zeta$	ζ	$1+\zeta$	$2+2\zeta$	$3+2\zeta$	2ζ	$1+2\zeta$	$2+3\zeta$	$3+3\zeta$	3ζ	$1+3\zeta$
$1+2\zeta$	$3+\zeta$	ζ	$1+\zeta$	$2+\zeta$	$3+2\zeta$	2ζ	$1+2\zeta$	$2+2\zeta$	$3+3\zeta$	3ζ	$1+3\zeta$	$2+3\zeta$	3	0	1	2
$2+2\zeta$	2ζ	$1+2\zeta$	$2+2\zeta$	$3+2\zeta$	3ζ	$1+3\zeta$	$2+3\zeta$	$3+3\zeta$	0	1	2	3	ζ	$1+\zeta$	$2+\zeta$	$3+\zeta$
$3+2\zeta$	$1+3\zeta$	$2+3\zeta$	$3+3\zeta$	3ζ	1	2	3	0	$1+\zeta$	$2+\zeta$	$3+\zeta$	ζ	$1+2\zeta$	$2+2\zeta$	$3+2\zeta$	2ζ
3ζ	1	2	3	0	$1+\zeta$	$2+\zeta$	$3+\zeta$	ζ	$1+2\zeta$	$2+2\zeta$	$3+2\zeta$	2ζ	$1+3\zeta$	$2+3\zeta$	$3+3\zeta$	3ζ
$1+3\zeta$	$2+\zeta$	$3+\zeta$	ζ	$1+\zeta$	$2+2\zeta$	$3+2\zeta$	2ζ	$1+2\zeta$	$2+3\zeta$	$3+3\zeta$	3ζ	$1+3\zeta$	2	3	0	1
$2+3\zeta$	$3+2\zeta$	2ζ	$1+2\zeta$	$2+2\zeta$	$3+3\zeta$	3ζ	$1+3\zeta$	$2+3\zeta$	3	0	1	2	$3+\zeta$	ζ	$1+\zeta$	$2+\zeta$
$3+3\zeta$	3ζ	$1+3\zeta$	$2+3\zeta$	$3+3\zeta$	0	1	2	3	ζ	$1+\zeta$	$2+\zeta$	$3+\zeta$	2ζ	$1+2\zeta$	$2+2\zeta$	$3+2\zeta$

Table: The output table for $f = r\left(\frac{1 - at}{1 - bt}\right)$ with $r = 1$, $a = 1$, and $b = 2 + \zeta$.

Thank you!

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