Groups acting on biregular trees with prescribed local action

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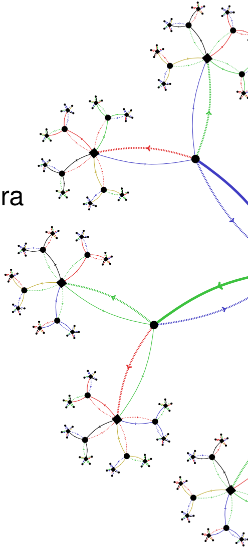
University of Lincoln



Trees, dynamics and
locally compact groups

Heinrich Heine University Düsseldorf

June 2018

Preliminaries

tldc groups

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- By solution to Hilbert's 5th problem, every connected lc group is the inverse limit of Lie groups

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- A transitive group that admits no nontrivial decomposition like this is called primitive

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- $\mathcal{S}$: non-discrete compactly generated, topologically simple tdlc groups
- $\mathcal{P}$: closed & subdegree-finite permutation groups that are primitive but not regular

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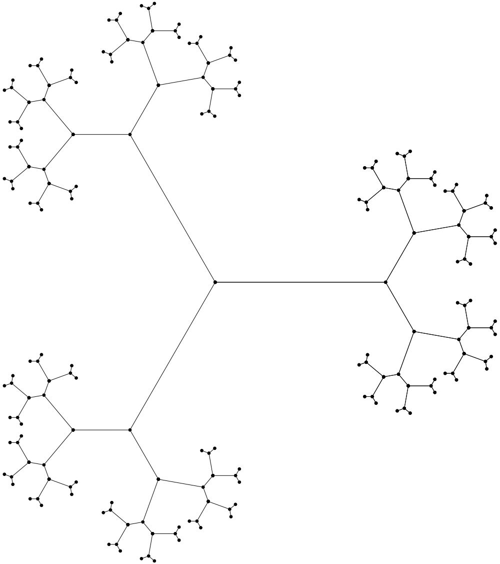

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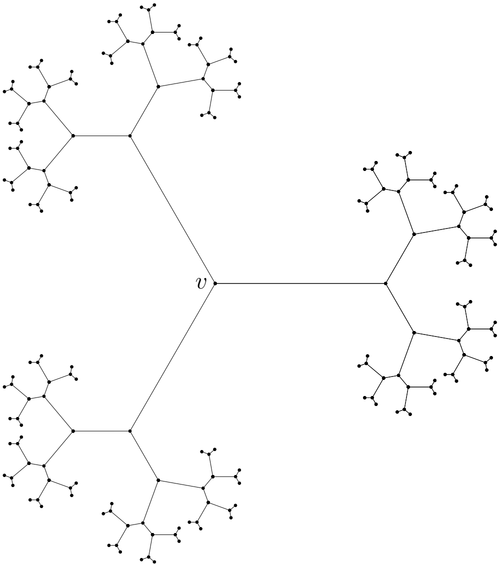

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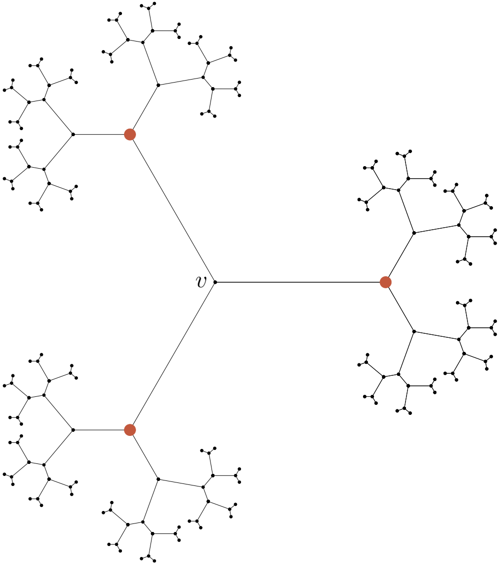

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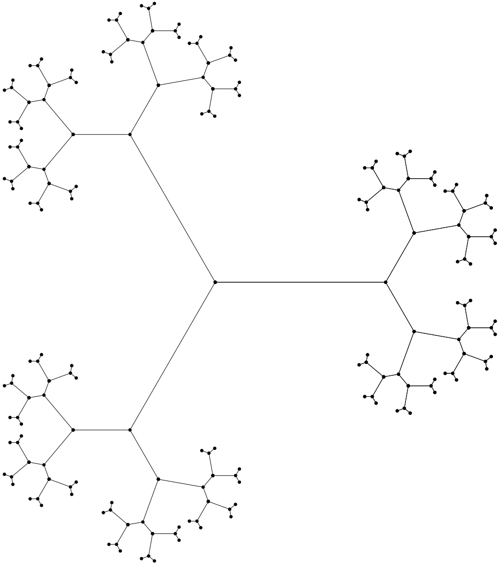

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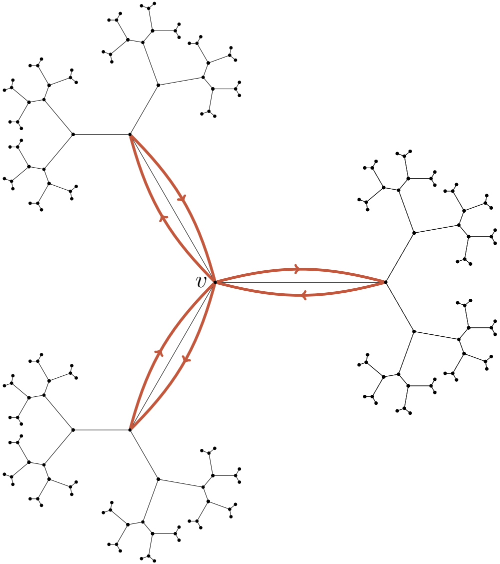

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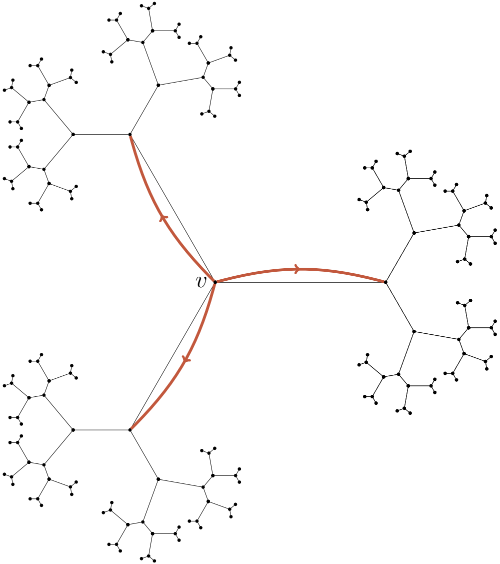

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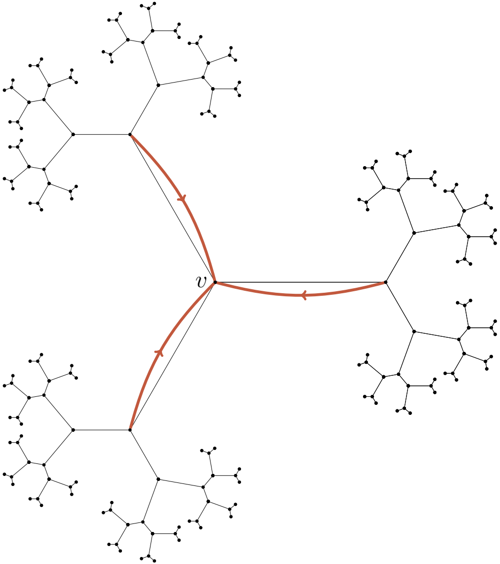

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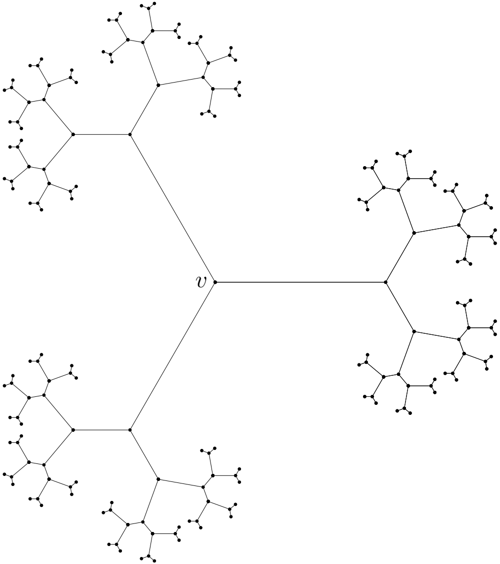

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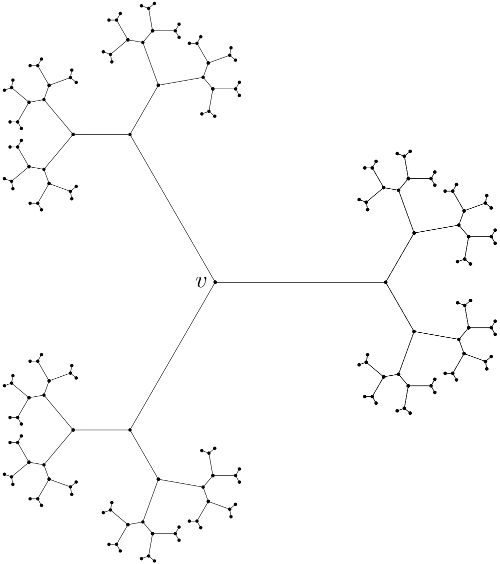

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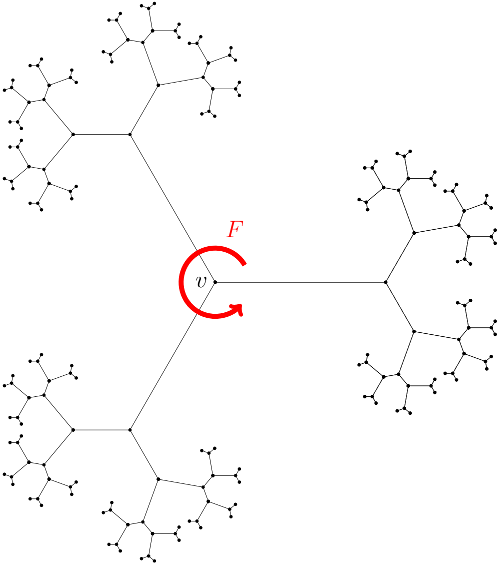

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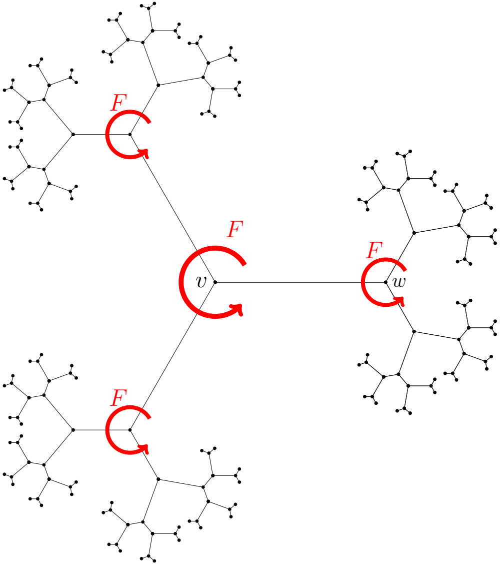

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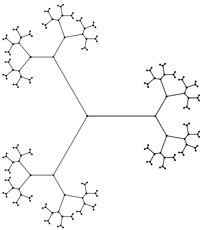

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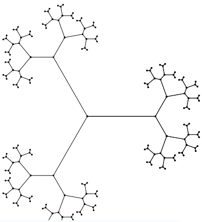

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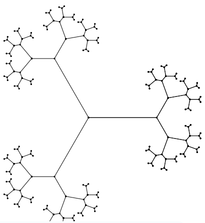

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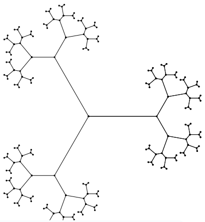

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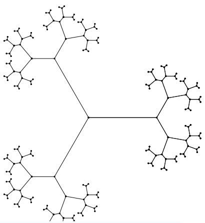

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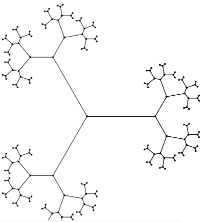

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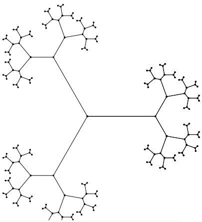

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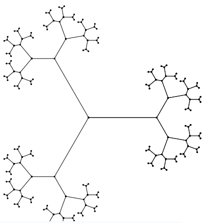

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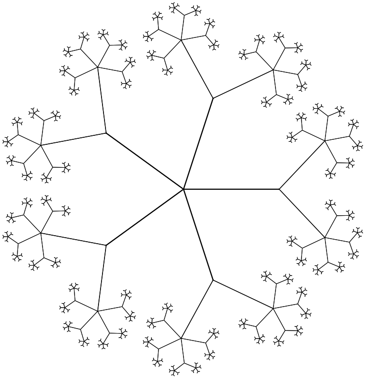
- $\mathcal{U}(F)$ is closed and transitive

A universal group for biregular trees

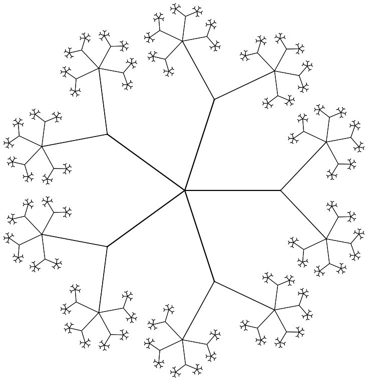


A universal group for biregular trees

Now T is a biregular tree

- vertices at even distance have same valency
- Example 1: the $(5,3)$-biregular tree

  Example 2: the $(\aleph_0,3)$-biregular tree



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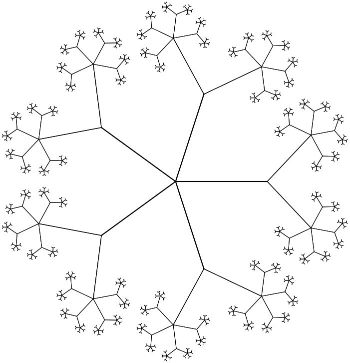

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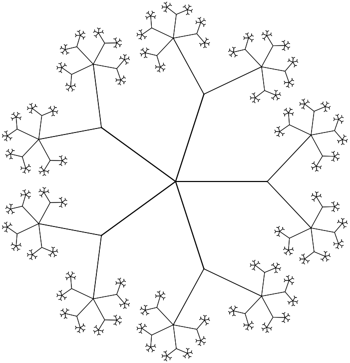

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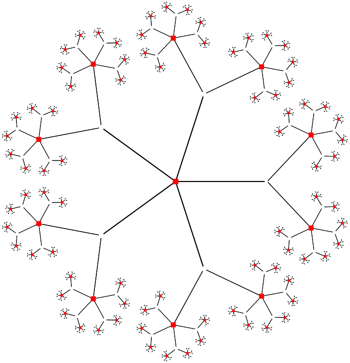

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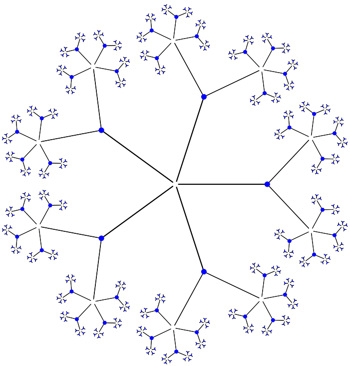

Given disjoint sets X, Y of any cardinality ≥ 2, let T be the $(|X|, |Y|)$-biregular tree

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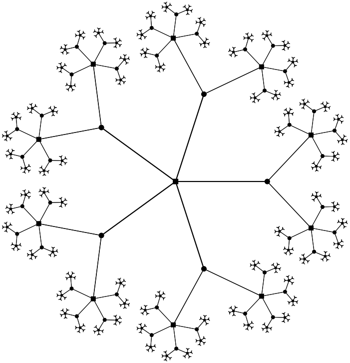

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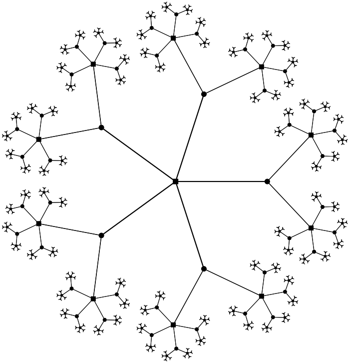

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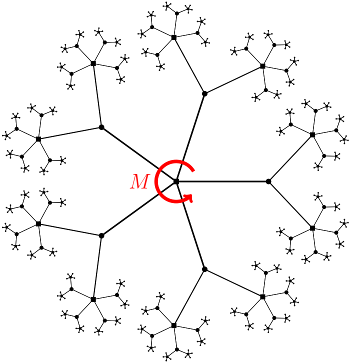

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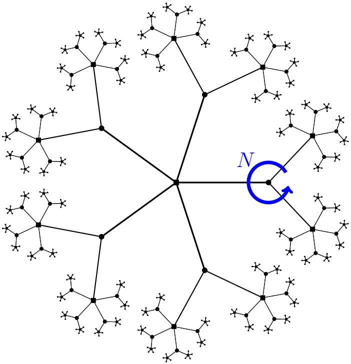

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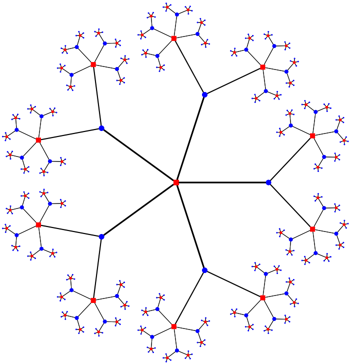

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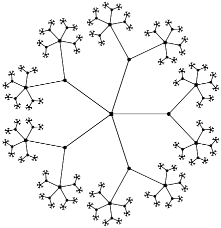

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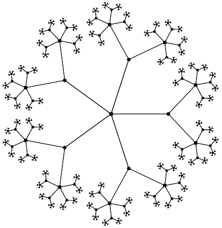

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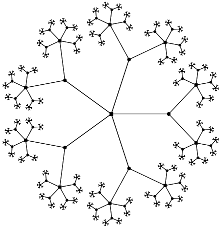

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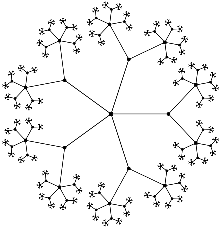

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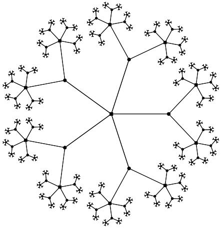

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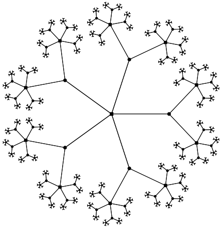

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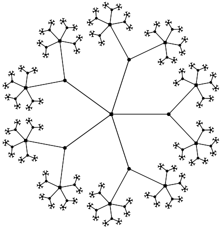

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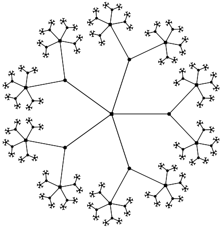

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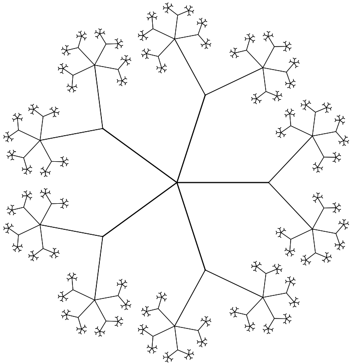
A legal colouring of T by X and Y



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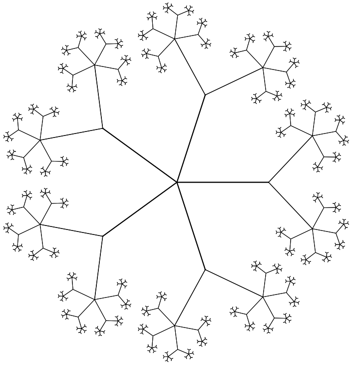

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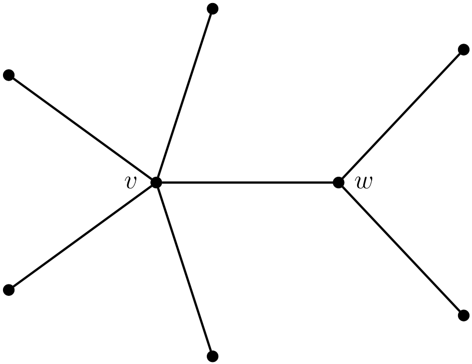

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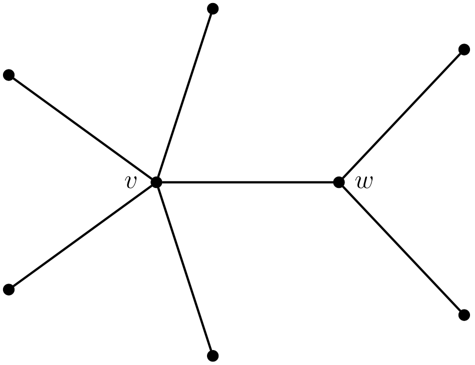

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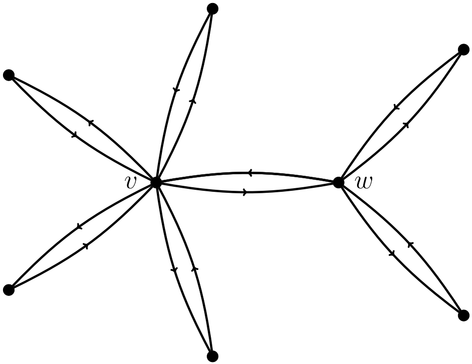

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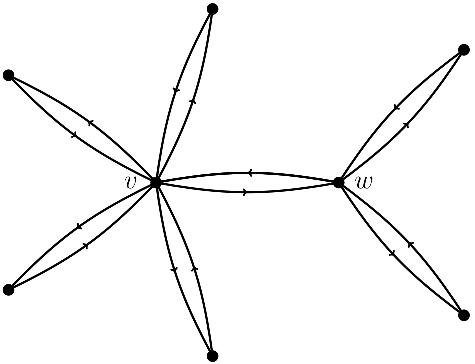

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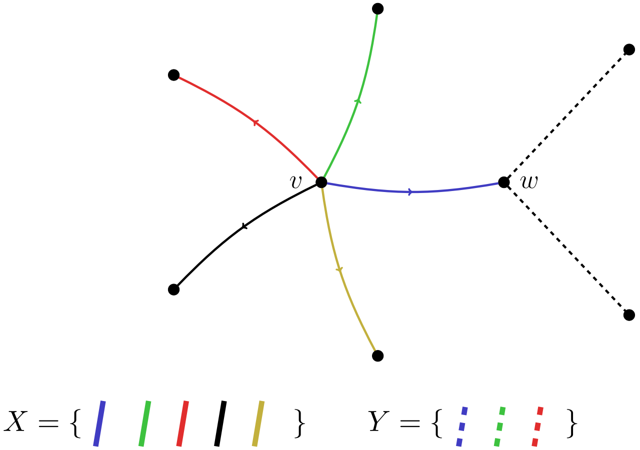

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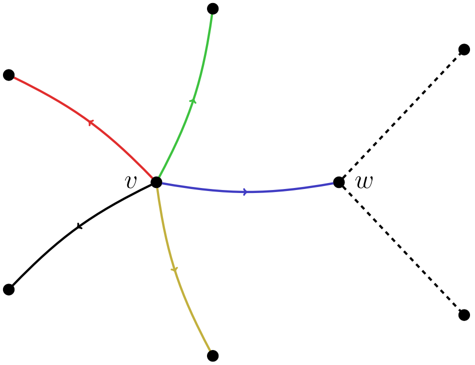

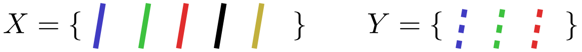

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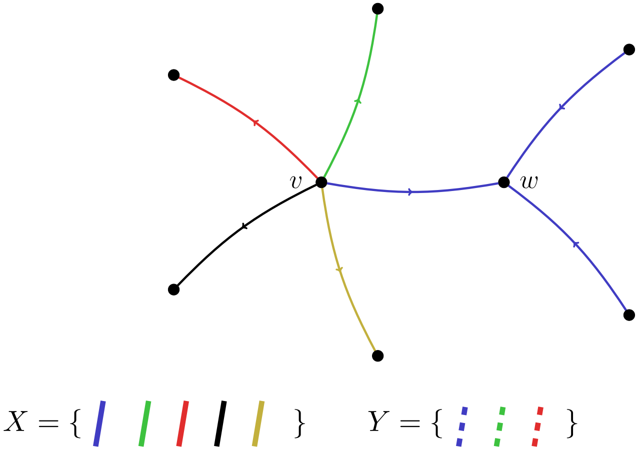

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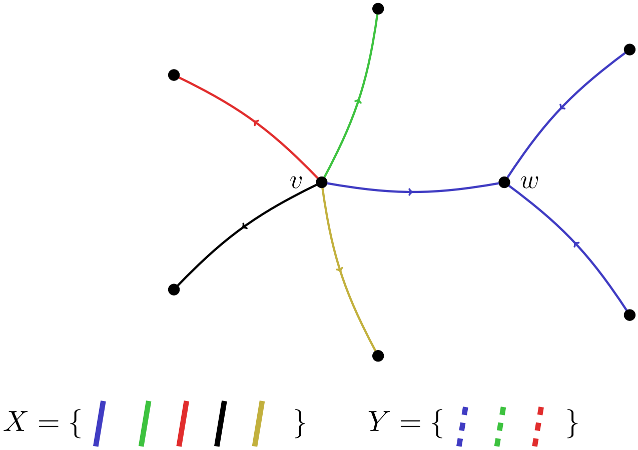

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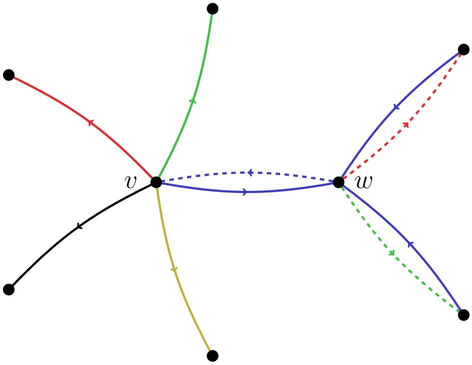

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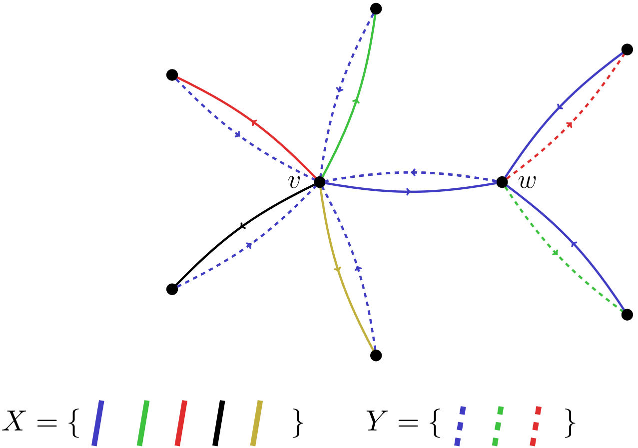

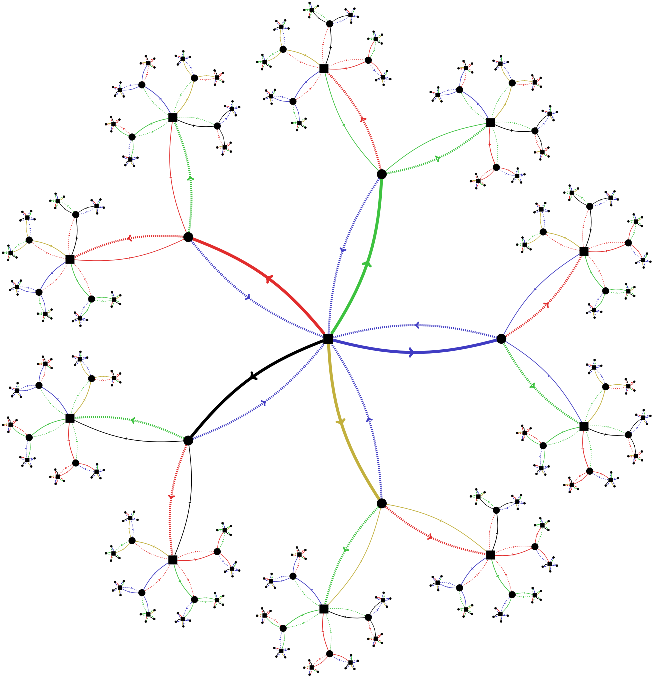

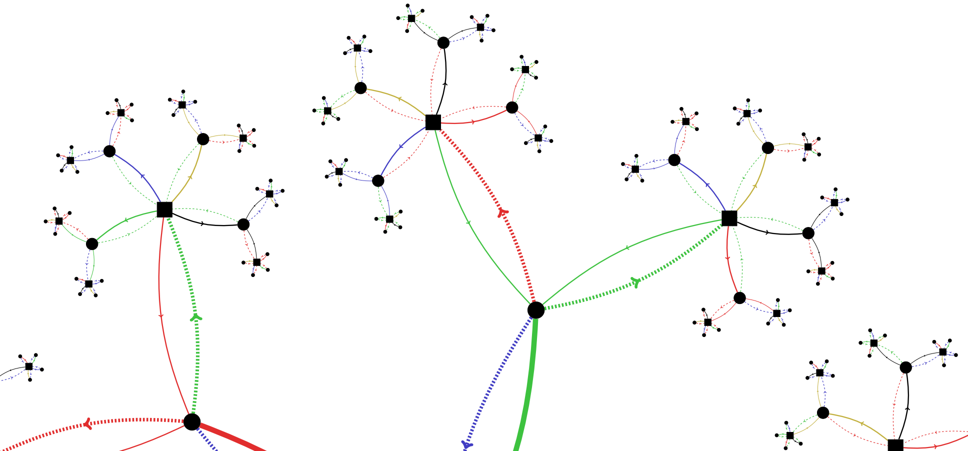

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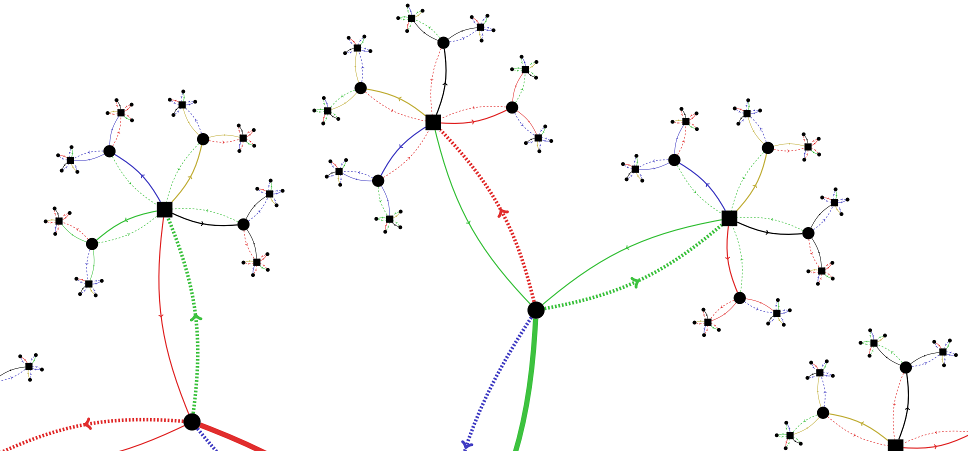

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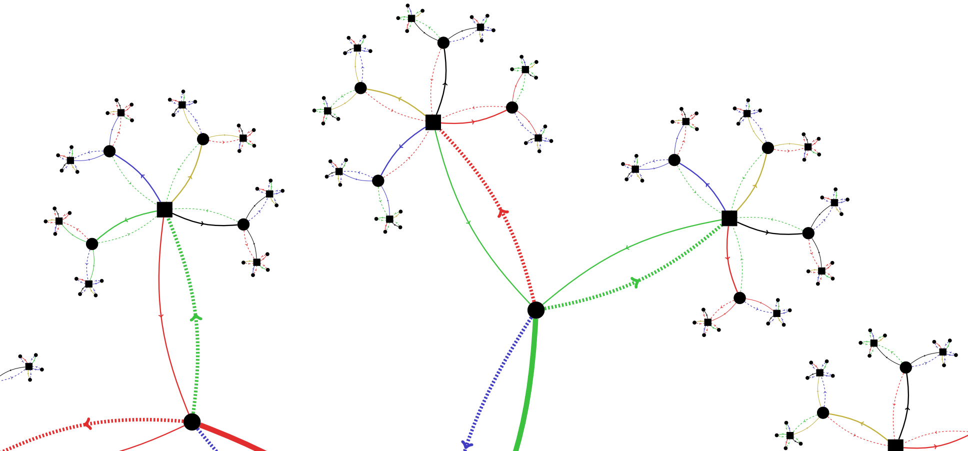

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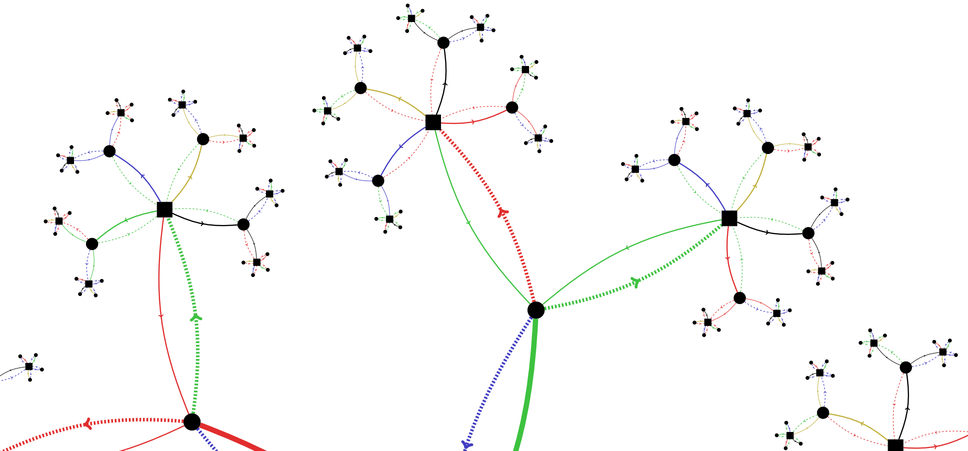

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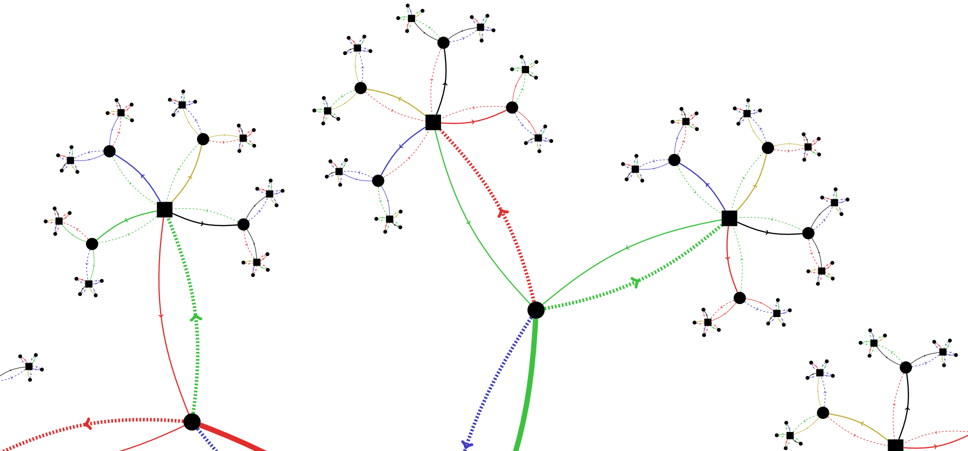

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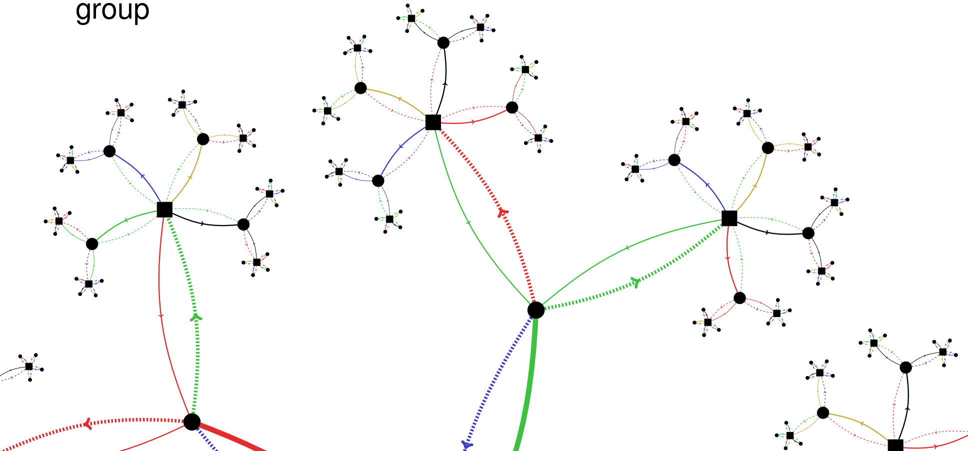
A different choice for $\mathcal{L}$ gives a permutationally isomorphic group



Some properties of $\mathcal{U}(M, N)$

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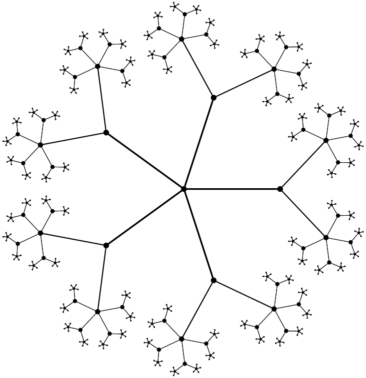

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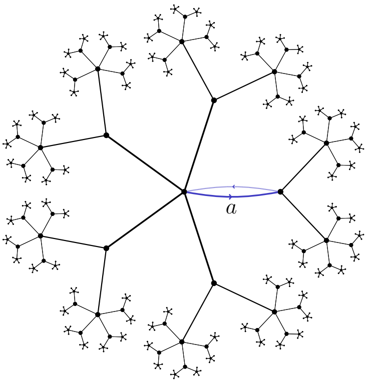

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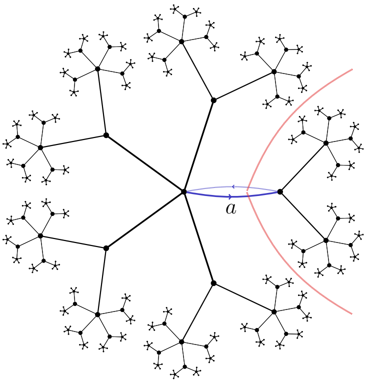

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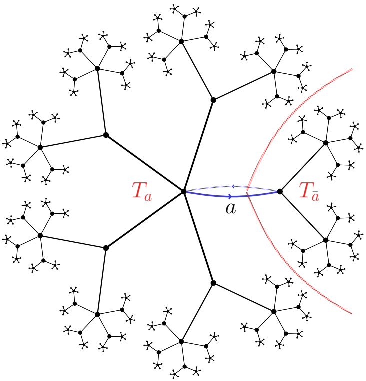

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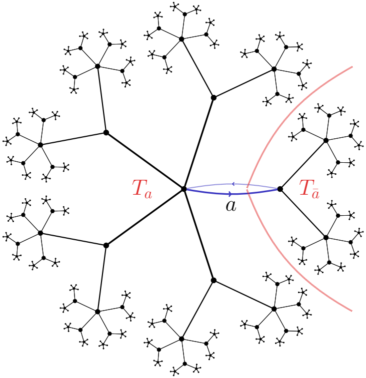

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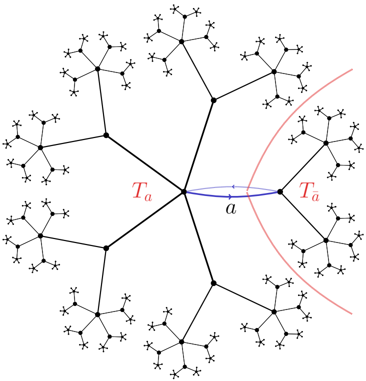

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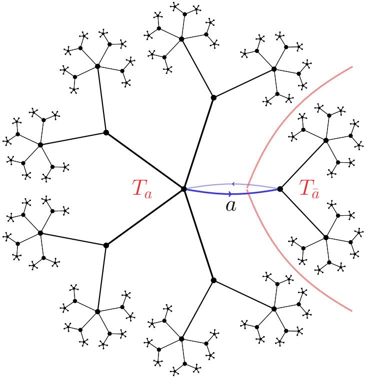

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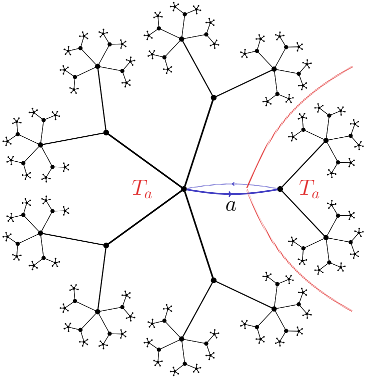

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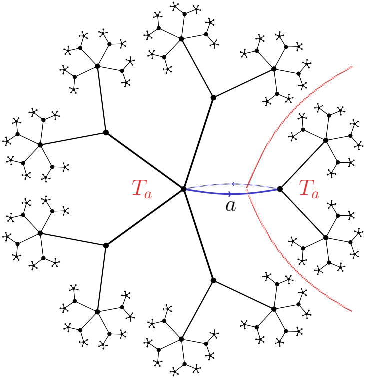

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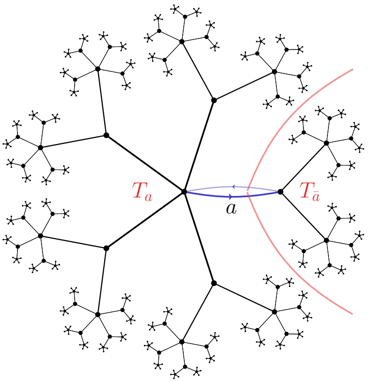

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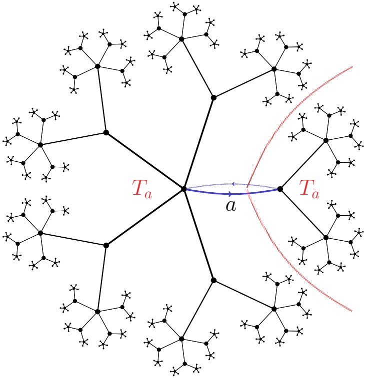

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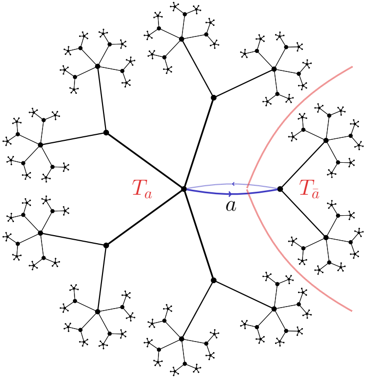

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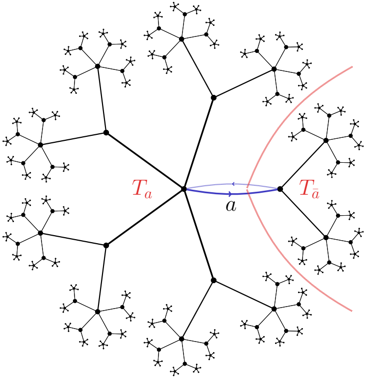

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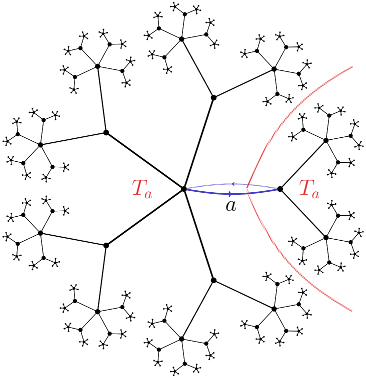

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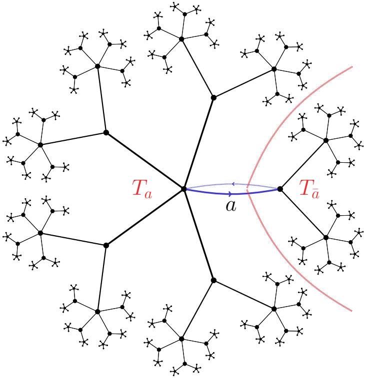

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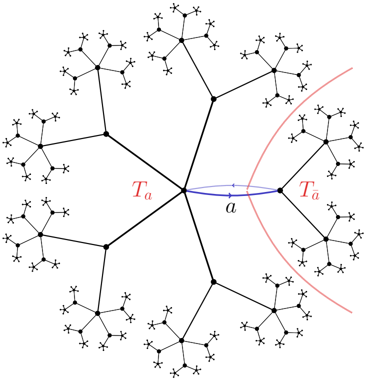

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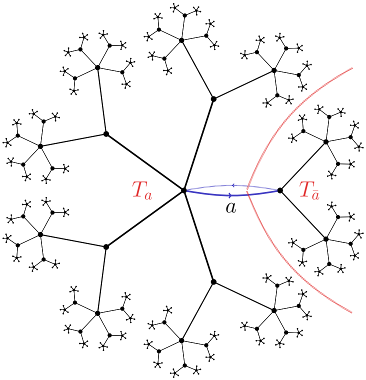

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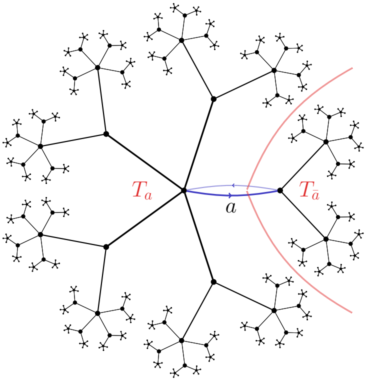

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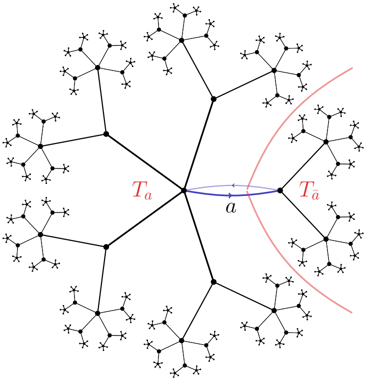

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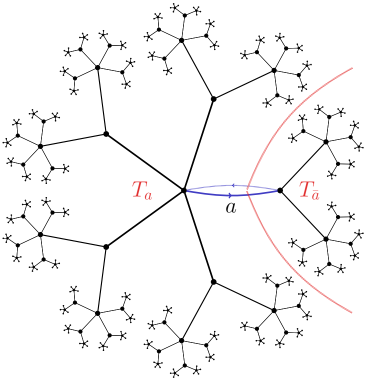

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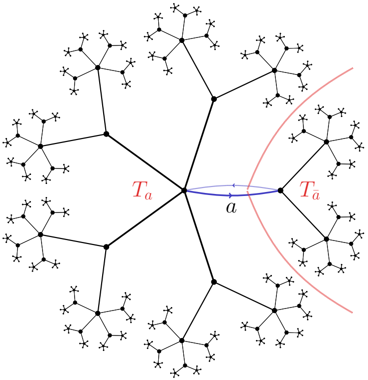

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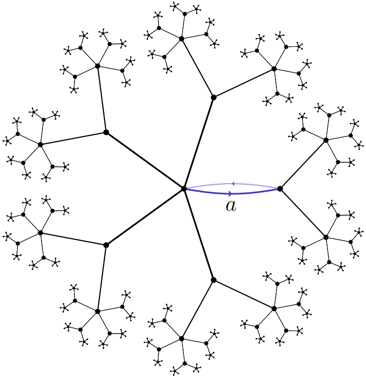

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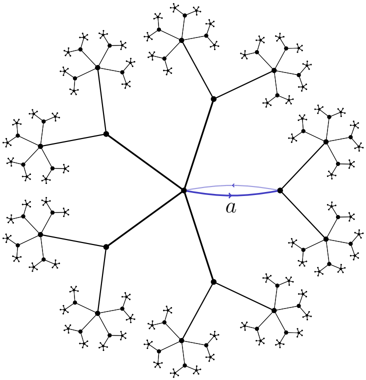

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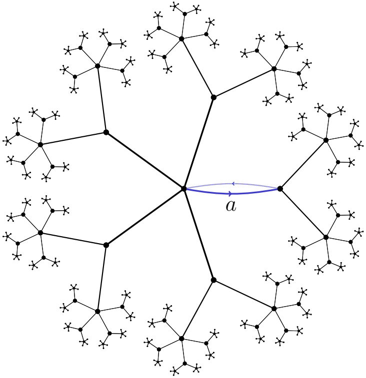

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Corollary. If $M, N \in \mathcal{P}$, then $\mathcal{U}(M, N) \in \mathcal{S}$.

Applications

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Thank you

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M. Burger and S. Mozes, *Groups acting on trees: from local to global structure*, Publications mathé de l'I.H.É.S. (2000)

S. Smith, *A product for permutation groups and topological groups*
Duke Math. J. (2017)