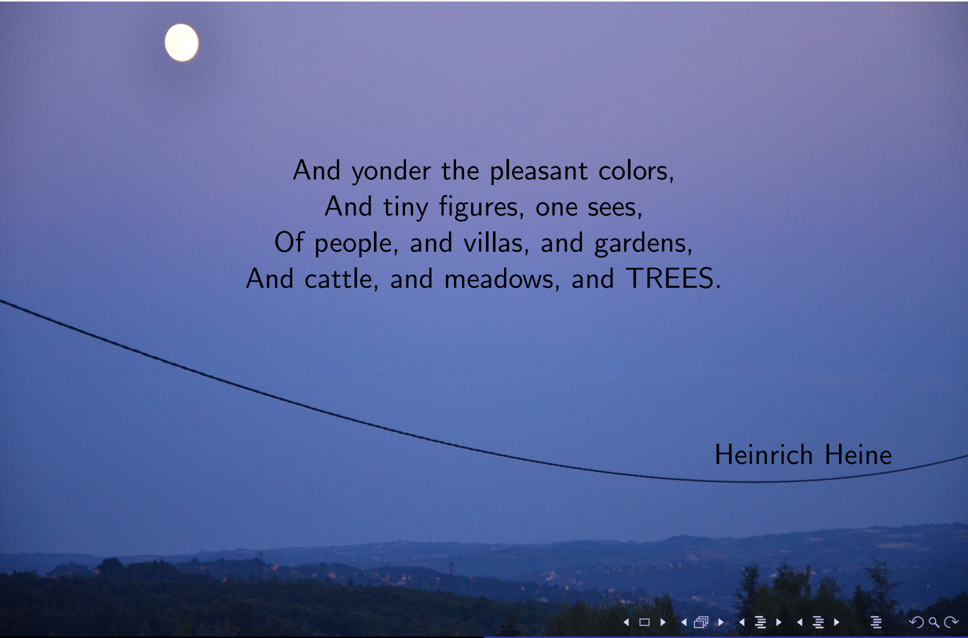
from "Homeward bound - III"



And yonder the pleasant colors,
And tiny figures, one sees,
Of people, and villas, and gardens,
And cattle, and meadows, and TREES.

Heinrich Heine

Group tree shifts I

Zoran Šunić
Hofstra University

* * *

Düsseldorf (Heinrich-Heine-Universität), June 25, 2018

Regular rooted ternary tree X^*

Regular rooted ternary tree X^*

Regular rooted ternary tree X^*

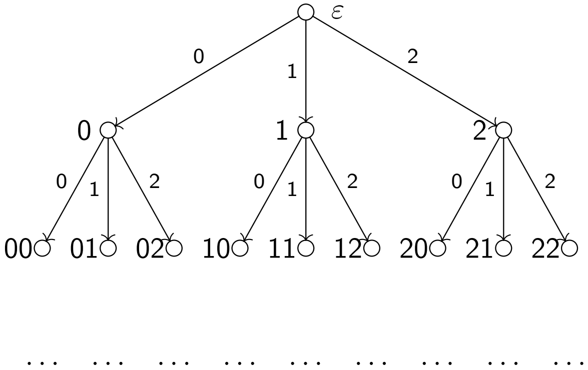
$$X = \{0, 1, 2\}$$

level $0 = X^0$

level $1 = X$

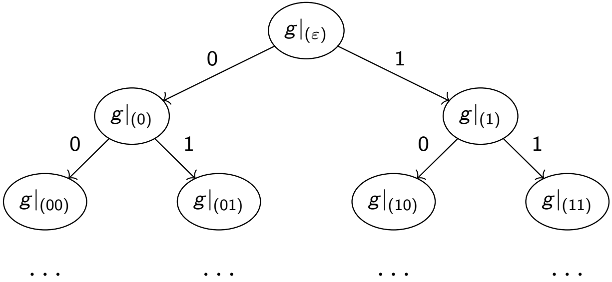
level $2 = X^2$

. . .



Representation of tree automorphisms through portraits

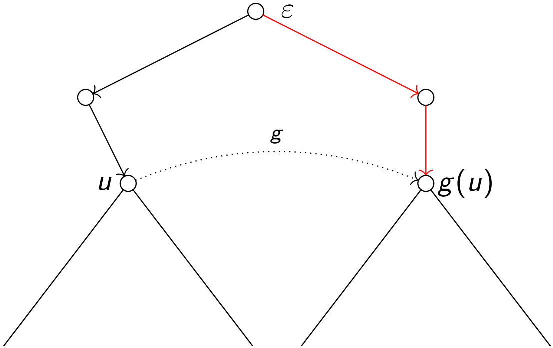
automorphims	$\longleftrightarrow$	portraits
$\mathrm{Aut}(X^*)$	$\longleftrightarrow$	$\Sigma(X)^{X^*}$
$g : X^* \to X^*$	$\longleftrightarrow$	$g : X^* \to \Sigma(X)$



From an automorphism to a portrait

$$g|_{(u)} = \text{the permutation of } X \text{ induced by } g \text{ at } u$$

$$g(ux) = g(u)x'$$
$$g|_{(u)}(x) = x'$$
$$x \mapsto x'$$



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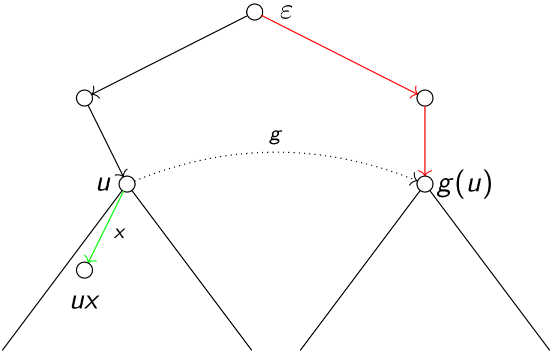

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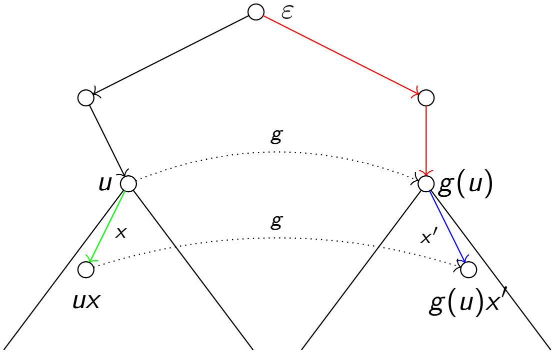
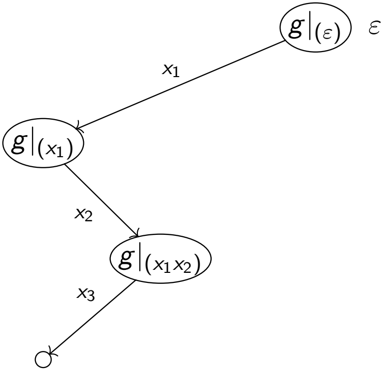


From a portrait to an automorphism

$$g(x_1x_2x_3\dots) = g|_{(\varepsilon)}(x_1)\ g|_{(x_1)}(x_2)\ g|_{(x_1x_2)}(x_3)\dots$$



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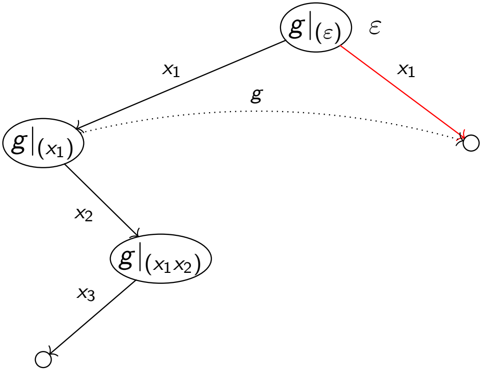
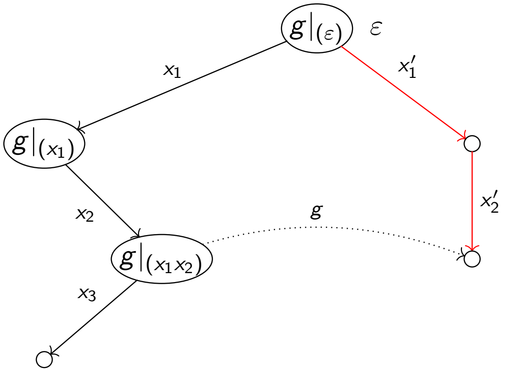


From a portrait to an automorphism

$$g(x_1x_2x_3\dots) = g|_{(\varepsilon)}(x_1)\; g|_{(x_1)}(x_2)\; g|_{(x_1x_2)}(x_3)\dots$$



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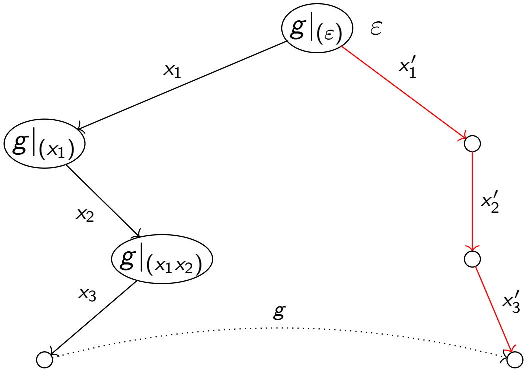
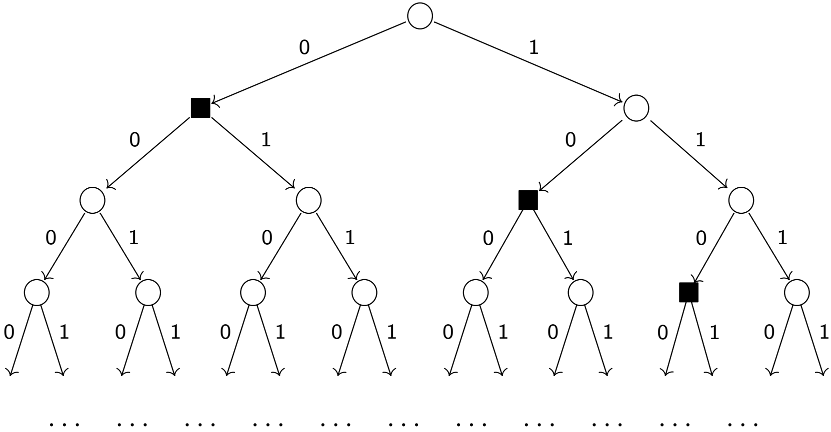


Changing the first symbol behind the first 0

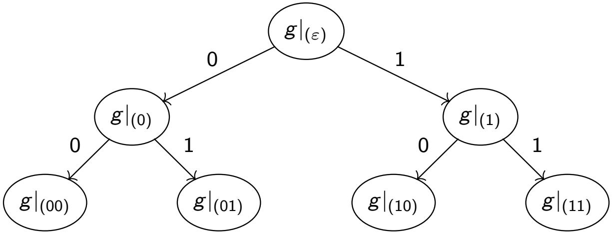


Representation of tree automorphisms through portraits

automorphisms ⟷ portraits

$\mathrm{Aut}(X^*)$ ⟷ $\Sigma(X)^{X^*}$

$g : X^* \to X^*$ ⟷ $g : X^* \to \Sigma(X)$



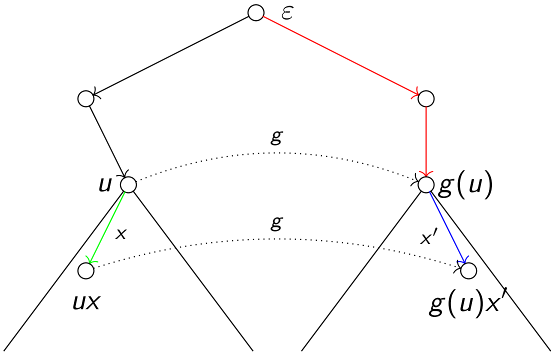
$$g(x_1x_2x_3\dots) = g|_{(\varepsilon)}(x_1)\ g|_{(x_1)}(x_2)\ g|_{(x_1x_2)}(x_3)\dots$$

$$g(ux) = g(u)\ g|_{(u)}(x)$$

Section of an automorphism

$$g|_{(u)} = \text{the permutation of } X \text{ induced by } g \text{ at } u$$

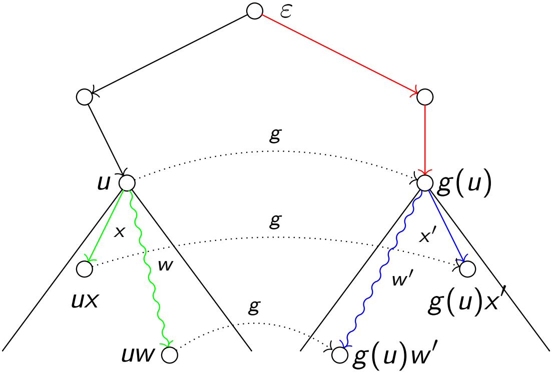
$$g(ux) = g(u)x'$$
$$g|_{(u)}(x) = x'$$
$$x \mapsto x'$$



$$g(ux) = g(u)g|_{(u)}(x)$$

$g|_u$ = the permutation of X^* induced by g at u

$$g(uw) = g(u)w'$$
$$g|_u(w) = w'$$
$$w \mapsto w'$$



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Section of an automorphism

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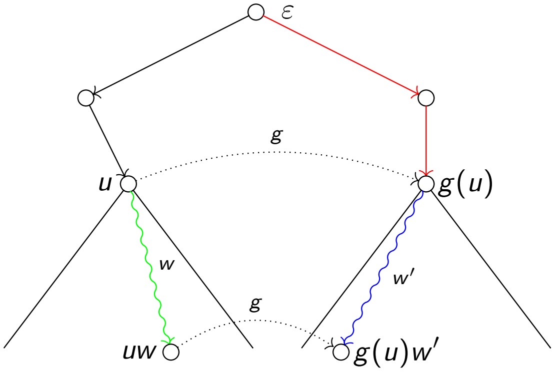


$$g(uw) = g(u)g|_u(w)$$

Thus

$$g|_{(u)} = \text{the permutation of } X \text{ induced by } g \text{ at } u$$

$$g(ux) = g(u)g|_{(u)}(x)$$

$$* * *$$

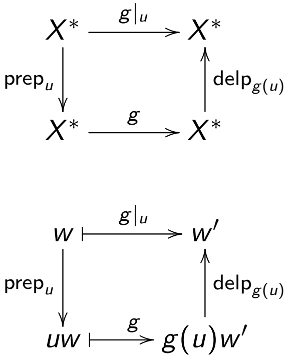
$$g|_u = \text{the automorphism of } X^* \text{ induced by } g \text{ at } u$$

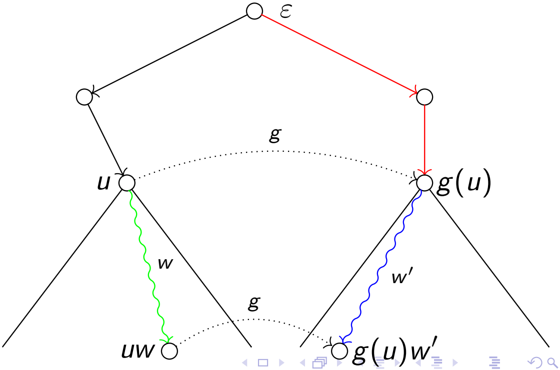
$$g(uw) = g(u)g|_u(w)$$

$$g^{\sigma_u} = g|_u$$

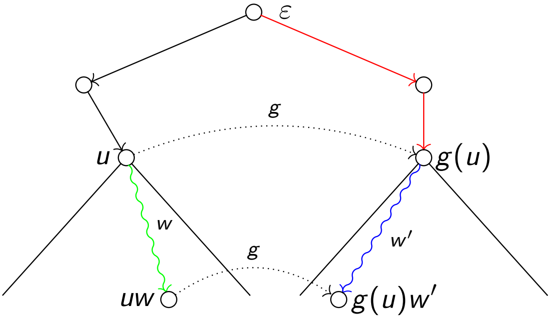
where $g|_u$ is the unique automorphism of X^* for which

$$g(uw) = g(u)g|_u(w).$$



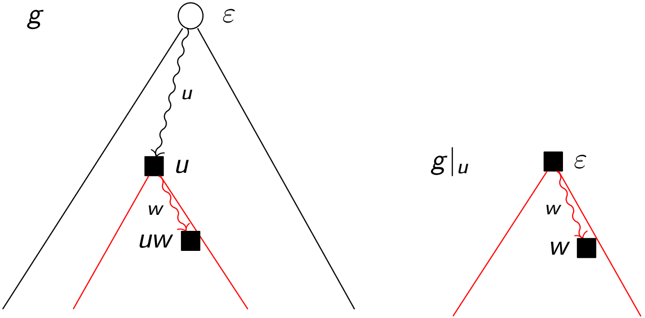


Section at u as a map $\sigma_u : \Sigma^{X^*} \to \Sigma^{X^*}$



$$\sigma_u : \Sigma^{X^*} \to \Sigma^{X^*}$$
$$(g^{\sigma_u})|_{(w)} = g|_{(uw)}$$
$$(g|_u)|_{(w)} = g|_{(uw)}$$

Section at u as a map $\sigma_u : \Sigma^{X^*} \to \Sigma^{X^*}$



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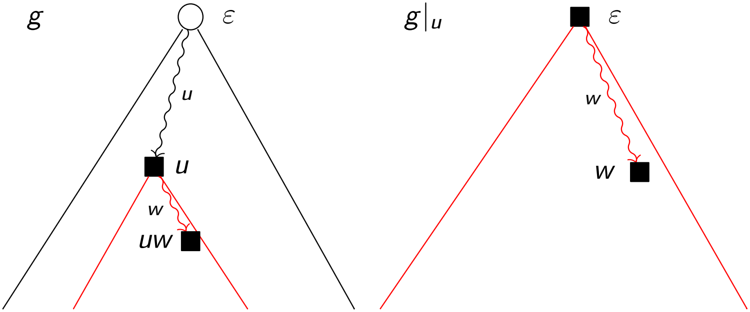

$$\sigma_u : \Sigma^{X^*} \to \Sigma^{X^*}$$

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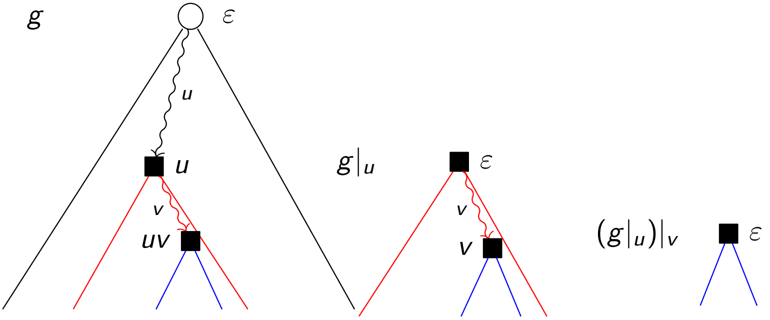
Calculus of sections: section of a section is a section



$$(g|_u)|_v = g|_{uv}$$

$$(g^{\sigma_u})^{\sigma_v}|_{(w)} = (g^{\sigma_u})|_{(vw)} = g|_{(uvw)} = (g^{\sigma_{uv}})|_{(w)}$$

$$(g^{\sigma_u})^{\sigma_v} = g^{\sigma_{uv}}$$

We stared with an action of $\mathrm{Aut}(X^*)$ on the tree X^*.

$$\mathrm{Aut}(X) \curvearrowright X^*$$

But now we see that the tree (the semigroup X^*) acts on $\mathrm{Aut}(X^*)$

$$\mathrm{Aut}(X) \curvearrowleft X^*$$

The tree acts back: $(g^{\sigma_u})^{\sigma_v} = g^{\sigma_{uv}}$

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The first action could be left or right, even at this moment it is still our choice, but the second one is right and this has nothing to do with our choice for the action of $\mathrm{Aut}(X^*)$.

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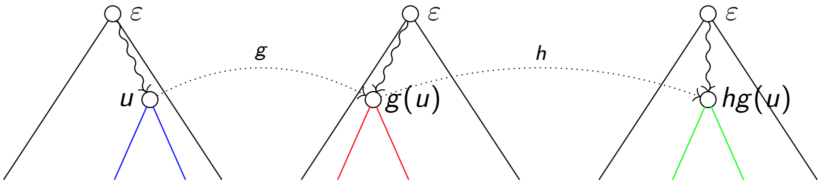
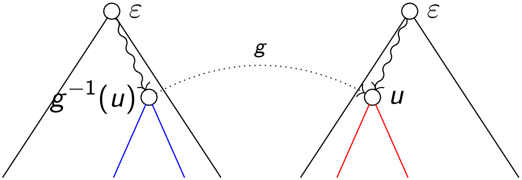
The first action could be left or right, even at this moment it is still our choice, but the second one is right and this has nothing to do with our choice for the action of $\mathrm{Aut}(X^*)$.
This is because we thought of the tree X^* as the right Cayley graph of the semigroup X^*. In other words, this is because we chose to write words over X left-to-right.

Calculus of sections: section of a composition is a composition of sections



$$(hg)|_u = h|_{g(u)} \; g|_u$$

Calculus of sections: section of the inverse is the inverse of a section

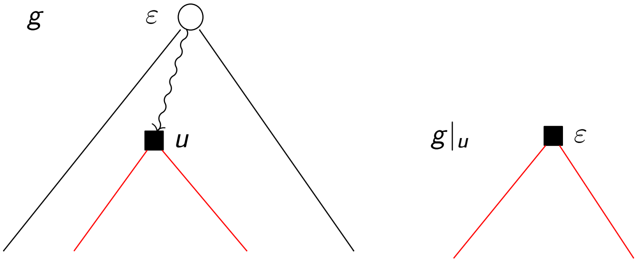


$$(g^{-1})|_u = (g|_{g^{-1}(u)})^{-1}$$

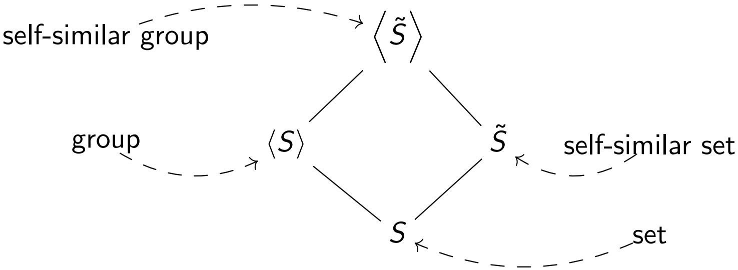
Definition

A subset (in particular, a subgroup) S of $\mathrm{Aut}(X^*)$ is self-similar if it is closed under all section maps, that is, it is union of orbits of the action of X^* on $\mathrm{Aut}(X^*)$.

$$g \in S \implies g|_u \in S$$



Closures w.r.t. the group and self-similarity structure



S = set of tree automorphisms

$\langle S \rangle$ = the smallest subgroup of $\mathrm{Aut}(X^*)$ containing S

$\tilde{S}$ = the smallest self-similar subset of $\mathrm{Aut}(X^*)$ containing S

$\langle \tilde{S} \rangle$ = the smallest self-similar subgroup of $\mathrm{Aut}(X^*)$ containing S

If S is self-similar, so is the group $\langle S \rangle$

Theorem

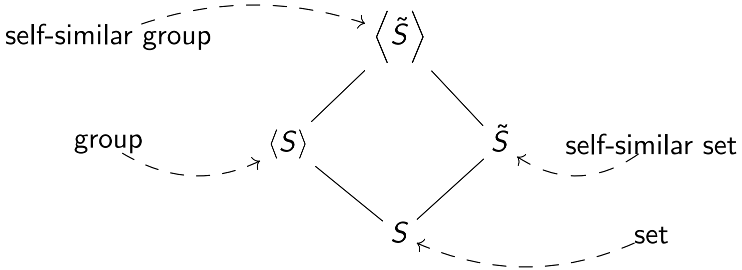
The subgroup generated by a self-similar set is self-similar itself.

Proof.

Because sections of compositions are compositions of sections and sections of the inverse are inverses of the sections. □

$$\begin{aligned} S &= \text{set of tree automorphisms} \\ \langle S \rangle &= \text{the smallest subgroup of } \mathrm{Aut}(X^*) \text{ containing } S \\ \tilde{S} &= \text{the smallest self-similar subset of } \mathrm{Aut}(X^*) \text{ containing } S \\ \langle \tilde{S} \rangle &= \text{the smallest self-similar subgroup of } \mathrm{Aut}(X^*) \text{ containing } S \end{aligned}$$

Closures w.r.t. the group and self-similarity structures



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Topological (metric) structure on Σ^{X^*}

Two portraits are "close" when they agree on "many" levels:

$$d(g,h) = \inf\{\ d_n \mid (\forall m \leq n)(\forall u \in X^m)\ g|_{(u)} = h|_{(u)}\ \}$$

where $d_n \searrow 0$ (popular choices: $\frac{1}{n+1}$, $\frac{1}{e^n}$, ...).

- The topology is the direct product topology on Σ^{X^*}, where Σ is discrete (thus Σ^{X^*} is compact metric space and so are all of its closed subsets).

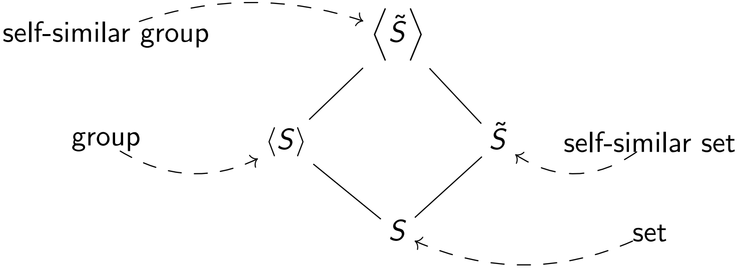
- With this metric Σ^{X^*} is a Cantor set (note that $|\Sigma| \geq 2$).

Convergence in Σ^{X^*}

$g_n \to g$ when g_n agrees with g on "more and more" levels:

$$(\forall m)(\exists n_0)(\forall n \geq n_0)(\forall u \text{ with } |u| \leq m)\ g_n|_{(u)} = g|_{(u)}.$$

Closures w.r.t. the group and self-similarity structure



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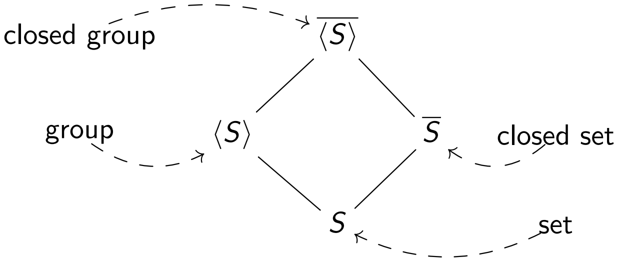
Closures w.r.t. the group and topological structures



$$S = \text{set of tree automorphisms}$$

$$\langle S \rangle = \text{the smallest subgroup of } \mathrm{Aut}(X^*) \text{ containing } S$$

$$\overline{S} = \text{the smallest closed subset of } \mathrm{Aut}(X^*) \text{ containing } S$$

$$\overline{\langle S \rangle} = \text{the smallest self-similar subgroup of } \mathrm{Aut}(X^*) \text{ containing } S$$

If S is a group, so is its closure $\overline{S}$

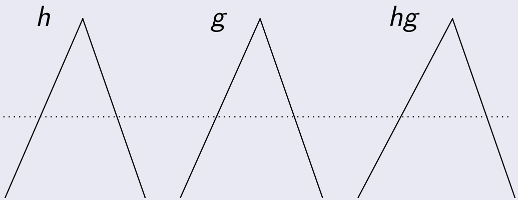
Theorem

The closure of a subgroup is a subgroup itself.

Theorem

$\mathrm{Aut}(X^*)$ *is a topological group.*

Proof.



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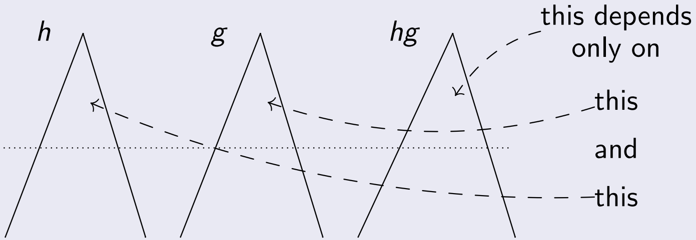

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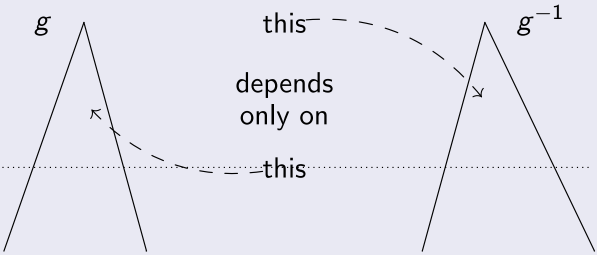

□

Now, let us forget groups (distraught gasp in the audience)

Now, let us forget groups (distraught gasp in the audience)

Fell he down, and wildly gasp'd he,
And his latest sigh was – "Mumma."

Heinrich Heine

Theorem

The action $\mathrm{Aut}(X^*) \curvearrowleft X^*$ *is by continuous maps.*

Now, let us forget groups (distraught gasp in the audience)

Fell he down, and wildly gasp'd he,
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Heinrich Heine

Theorem

The action $\Sigma^{X^*} \curvearrowleft X^*$ is by continuous maps.

What was Σ^{X^*} again?

X = finite alphabet (tree alphabet)

X^* = rooted tree

Σ = finite alphabet (decoration alphabet)

$\Sigma^{X^*} = \{g : X^* \to \Sigma\}$ = rooted trees over X decorated by Σ

$g|_{(u)}$ = the decoration (label) from Σ at the vertex u

σ_u acts on the right on Σ^{X^*} by $(g^{\sigma_u})|_{(v)} = (g^{\sigma_u})|_{(uv)}$

$$d(g,h) = \inf\{\ d_n \mid (\forall m \leq n)(\forall u \in X^m)\ g|_{(u)} = h|_{(u)}\ \}$$

X^* acts by continuous maps

Theorem

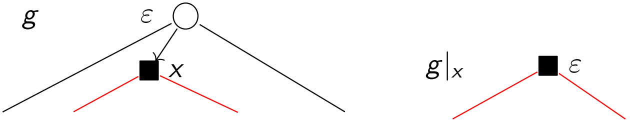
The action $\Sigma^{X^*} \curvearrowleft X^*$ is by continuous maps.

Proof.

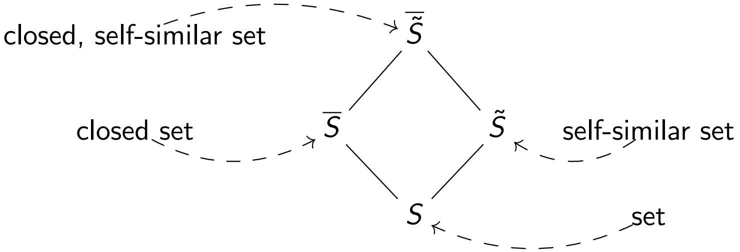
If (the labels defined by) g and h agree on n levels, then (the labels defined by) $g|_x$ and $h|_x$ agree on at least $n-1$ levels.

$$d(g,h) \leq d_n \implies d(g|_x, h|_x) \leq d_{n-1}$$

Recall that $d_n \searrow 0$.
Thus, $g \mapsto g|_x$ is continuous. □



Closures w.r.t. the topological and self-similarity structures



$S =$ set of tree automorphisms

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$\overline{\tilde{S}} =$ the smallest closed, self-similar subset of $\mathrm{Aut}(X^*)$ containing S

If S is self-similar, so is its closure $\overline{S}$

Theorem

The closure of a self-similar set is self-similar itself.

Proof.

Direct corollary of the fact that the action of X^* is by continuous maps.

If S is self-similar, so is its closure $\overline{S}$

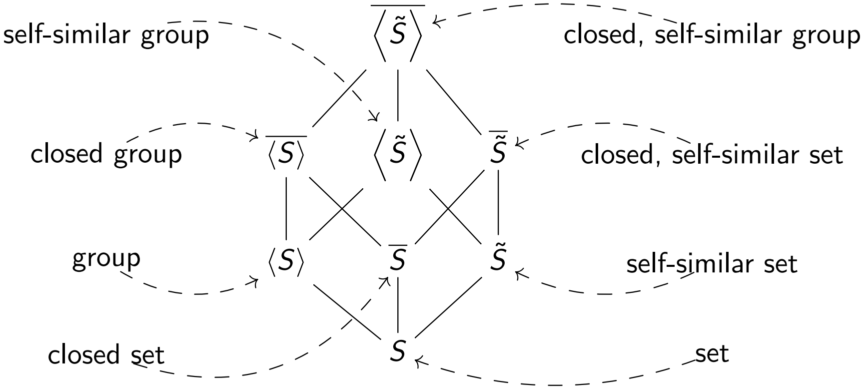
Theorem

The closure of a self-similar set is self-similar itself.

Proof.

Direct corollary of the fact that the action of X^* is by continuous maps.
(In general, if $f : Y \to Y$ is continuous, then $y \in \overline{A}$ implies $f(y) \in \overline{f(A)}$. Thus, if $g \in \overline{S}$ then $g^{\sigma_u} \in \overline{S^{\sigma_u}} \subseteq \overline{S}$). □

Closures w.r.t. group, metric, and self-similarity structure



Group tree shift is ...

Group tree shift is ...

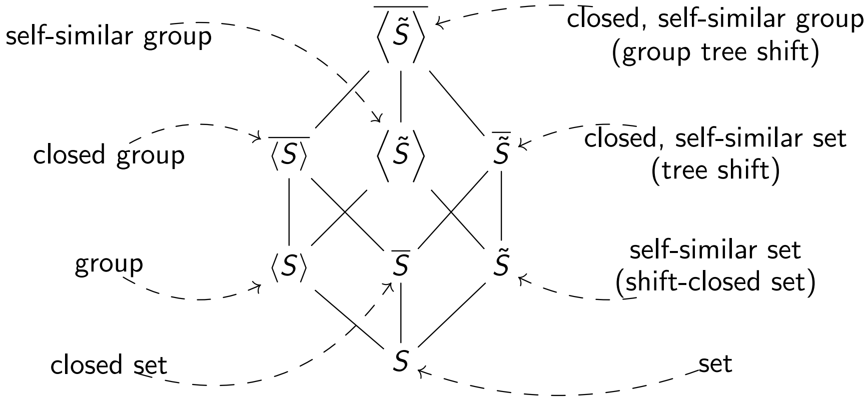
Definition

A group tree shift is a closed, self-similar group.

'Tis still the same heroic lot,
'Tis still the same old noble stories;
The names are changed, the natures not

Heinrich Heine

Closures w.r.t. group, metric, and self-similarity structure



Examples (finally!) of closed, self-similar groups

- $\mathrm{Aut}(X^*)$
- $\mathrm{Aut}_p(X^*)$ = p-ary automorphisms,
  where p is a prime, $X = \{0, 1, \ldots, p-1\}$, and all vertex permutations are powers of the standard cycle $(012\ldots p-1)$.
- More generally, $\mathrm{Aut}_{\Sigma'}(X^*) = (\Sigma')^{X^*}$ where Σ' is a subgroup of Σ.
- ?

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- ?

Perhaps we should try closures of arbitrary self-similar groups.

Examples of self-similar groups

- $\mathrm{Aut}_f(X^*)$ = finitary automorphisms = automorphisms whose vertex permutations are trivial below some level = automorphisms with only finitely many nontrivial vertex permutations (this is a countable, locally finite group)
- More generally, for $\Sigma'' \leq \Sigma' \leq \Sigma$, automorphisms whose vertex permutations come from Σ', but only finitely many are outside of Σ''.
- $\mathrm{Aut}_{f.r.}(X^*)$ = automorhisms finitary along rays = automorphisms that, along every ray, have only finitely many nontrivial vertex permutations
- More generally, for $\Sigma'' \leq \Sigma' \leq \Sigma$, automorphisms whose vertex permutations come from Σ', but, along every ray, only finitely many are outside of Σ''.
- ?

Examples of self-similar groups

- $\mathrm{Aut}_f(X^*)$ = finitary automorphisms = automorphisms whose vertex permutations are trivial below some level = automorphisms with only finitely many nontrivial vertex permutations (this is a countable, locally finite group)
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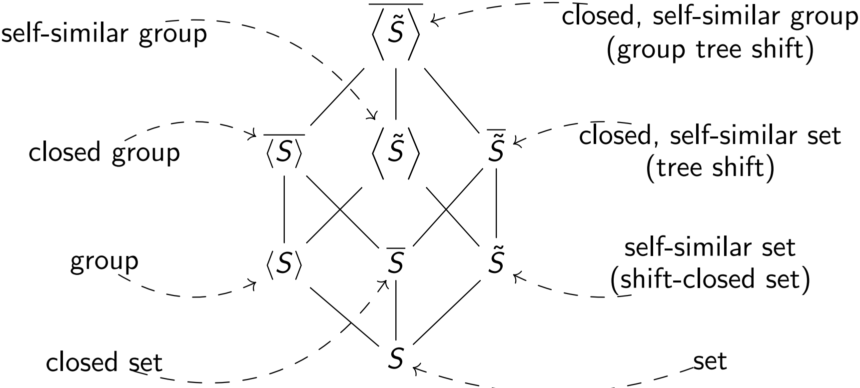
- ?

Note that $\mathrm{Aut}_f(X^*)$ and $\mathrm{Aut}_{f.r}(X^*)$ are dense in $\mathrm{Aut}(X^*)$ and, in the more general cases, the closures are equal to $\mathrm{Aut}_{\Sigma'}(X^*)$.

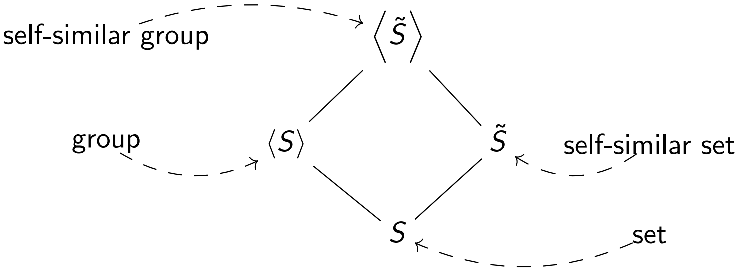
Closures w.r.t. group, metric, and self-similarity structure



So, we still have only finitely many examples, $\text{Aut}_{\Sigma'}(X^*)$, one for each subgroup of the symmetric group Σ.

Perhaps we should try closures of arbitrary self-similar groups.

Moreover, in order to obtain "small" examples, start with "small" self-similar sets.

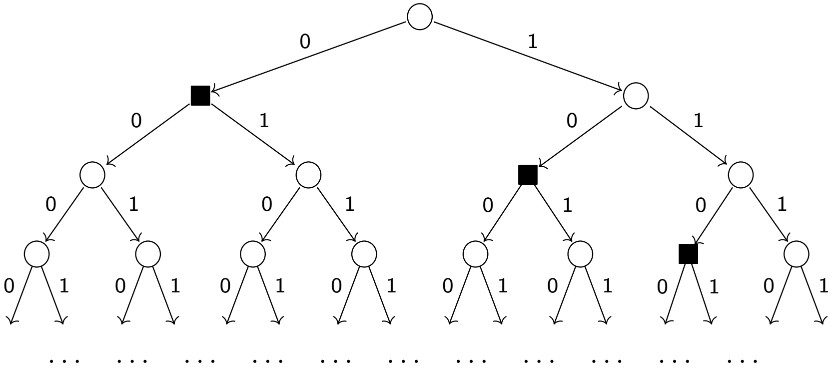


g = a tree automorphism

g is of finite order if $\langle g \rangle$ is finite

g is finite state if $\tilde{g}$ is finite (its orbit under X^* is finite)

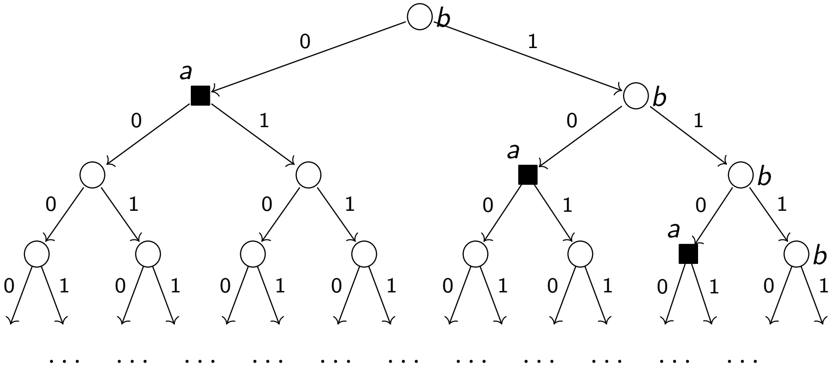
$b \in \mathrm{Aut}(X^*)$ changes the first symbol after the first 0



$b \in \mathrm{Aut}(X^*)$ changes the first symbol after the first 0

$a \in \mathrm{Aut}(X^*)$ changes the first symbol in any word

$id \in \mathrm{Aut}(X^*)$ changes nothing

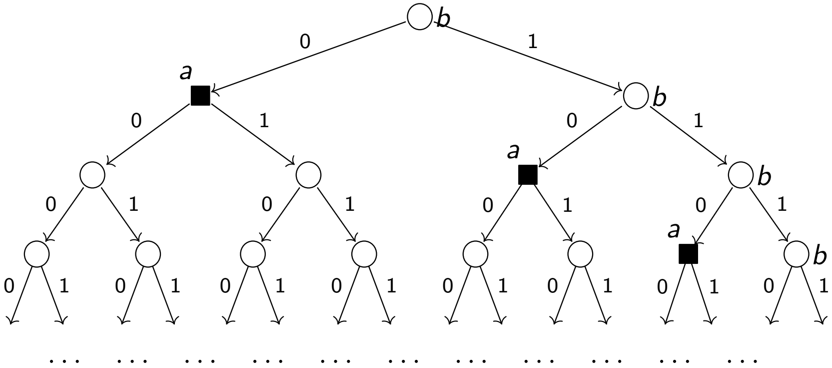


The portrait of an automorphism as a Moore automaton

b : do nothing, but be on a lookout for 0 and if you see it call a

a : change the first symbol and call id

id : do nothing



The portrait of an automorphism as a Moore automaton

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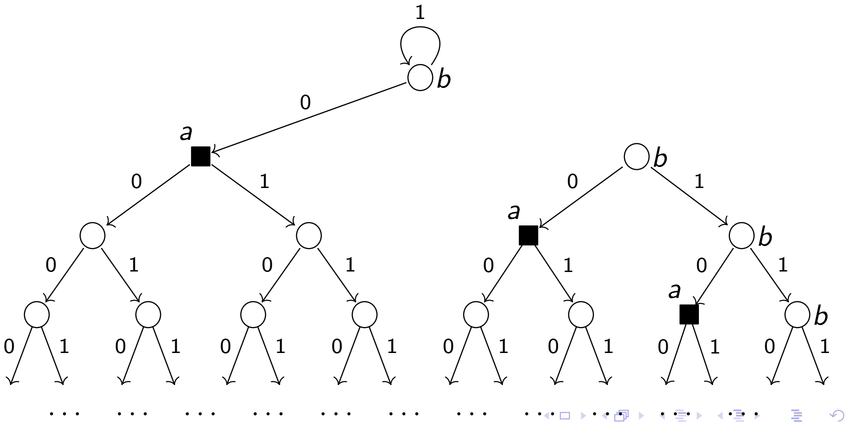

The portrait of an automorphism as a Moore automaton

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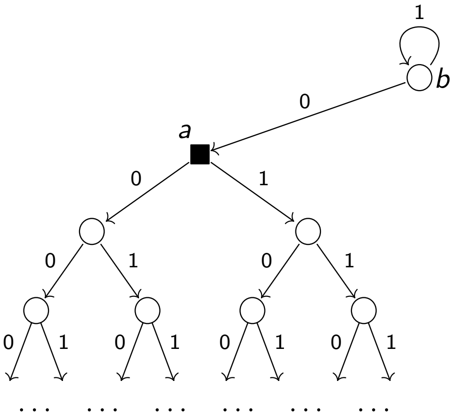
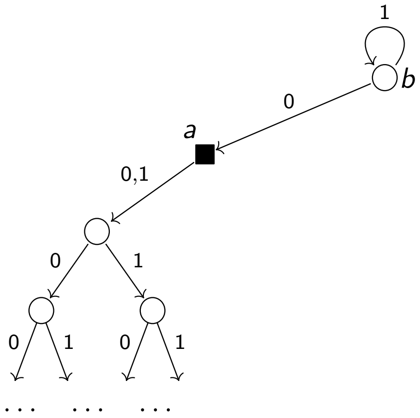


b : do nothing, but be on a lookout for 0 and if you see it call a

a : change the first symbol and call id

id : do nothing

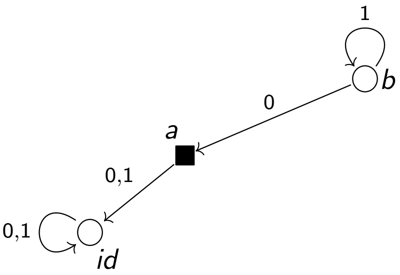


The portrait of an automorphism as a Moore automaton

b : do nothing, but be on a lookout for 0 and if you see it call a

a : change the first symbol and call id

id : do nothing, ever

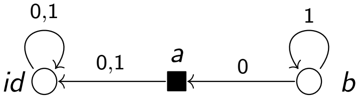


$$\tilde{b} = \{b,\ a,\ id\}$$

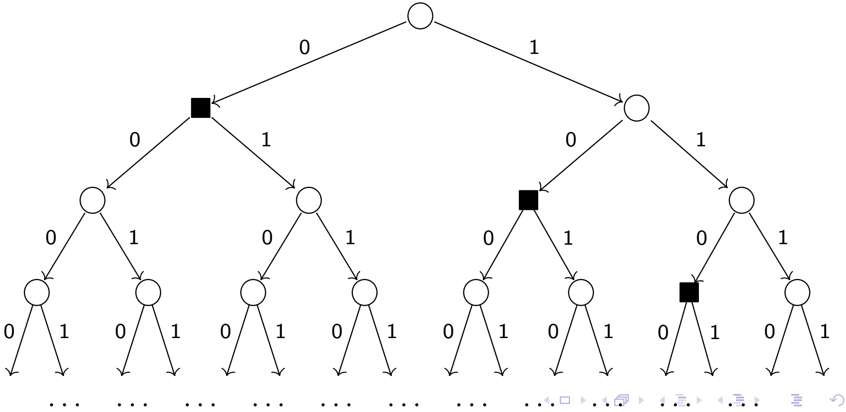
An automaton is

- a finite self-similar set of tree automorphisms.
- a finite set of tree automorphisms closed under taking sections.
- a finite set of finite-state tree automorphisms.
- a finite X^*-invariant set of tree automorphisms
- a finite union of finite orbits of the action $\mathrm{Aut}(X^*) \curvearrowleft X^*$
- a finite graph S with vertices labeled by permutations in $\Sigma(X)$ and,for each $x \in X$ and $s \in S$ one outgoing edge starting at s labeled by x.

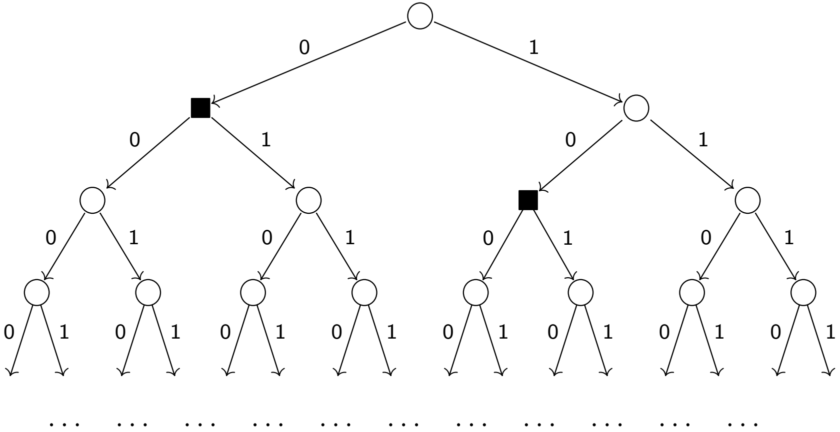
(additional words sometimes used: finite, Moore, transducer, ...)



$b|_{(u)}$ = the label on the state reached by reading u starting from b



Grigorchuk automaton $\{id, a, b, c, d\}$



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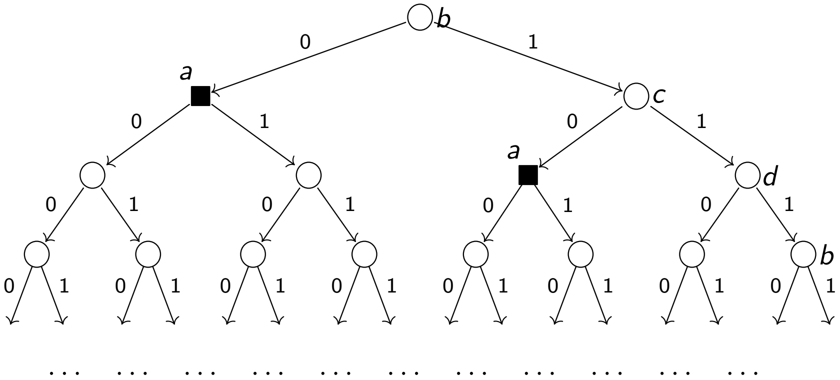

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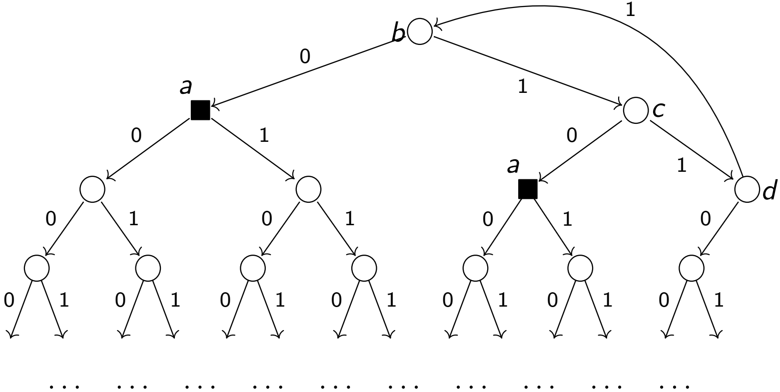

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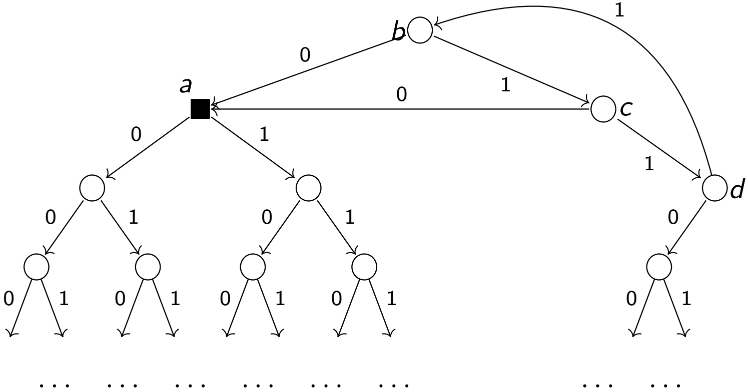
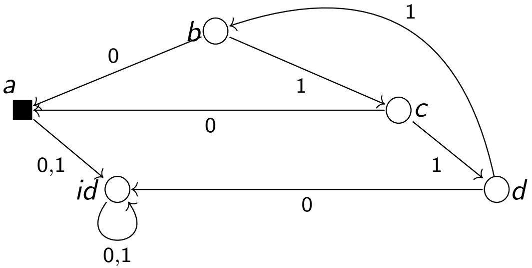


Grigorchuk automaton $\{id, a, b, c, d\}$



$\mathrm{Aut}_{fs}(X^*)$ = finite state automorphisms

Proposition

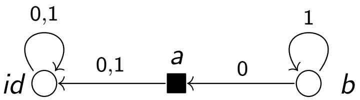
(a) $\mathrm{Aut}_{fs}(X^*) \leq \mathrm{Aut}(X^*)$.
(b) $\mathrm{Aut}_{fs}(X^*)$ *is self-similar.*
(c) $\mathrm{Aut}_{fs}(X^*)$ *is dense in* $\mathrm{Aut}(X^*)$.

Proof.

(a) Because sections of a product (inverse) are products (inverses) of sections.
(b) Because sections of sections are sections.
(c) Because it contains $\mathrm{Aut}_f(X^*)$. □

Examples are now easy to generate, but ...

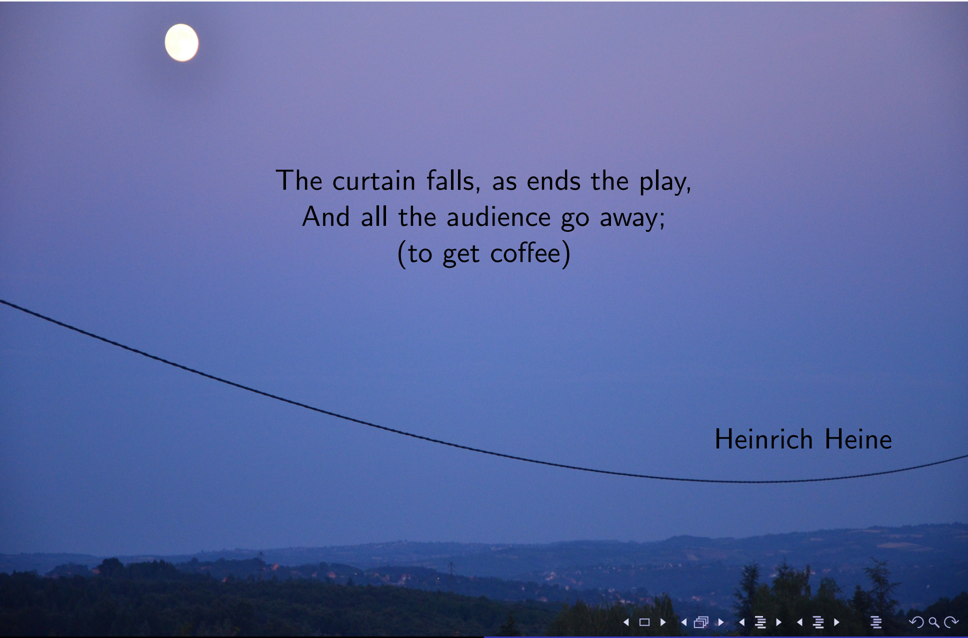
How does the closure of the group generated by



look like? How to describe/characterize/recognize its elements?

from "Lazarus – 18"

The curtain falls, as ends the play,
And all the audience go away;



The curtain falls, as ends the play,
And all the audience go away;
(to get coffee)

Heinrich Heine