from "Ascension"

Saint Peter cried: Alack, alack!
Philosophy's but the trade of a quack.
In truth it is a puzzle to me
Why people study philosophy.
It is such tedious and profitless stuff,
And is moreover godless enough;
In hunger and doubt their votaries dwell,
Till Satan carries them off to hell."

Heinrich Heine

Group tree shifts II

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* * *

Düsseldorf (Heinrich-Heine-Universität), June 27, 2018

Regular rooted ternary tree X^*

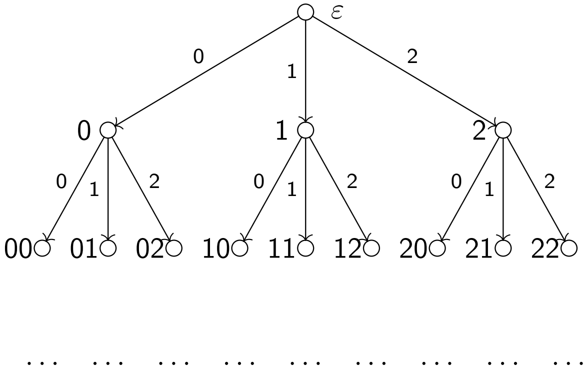
$$X = \{0, 1, 2\}$$

level $0 = X^0$

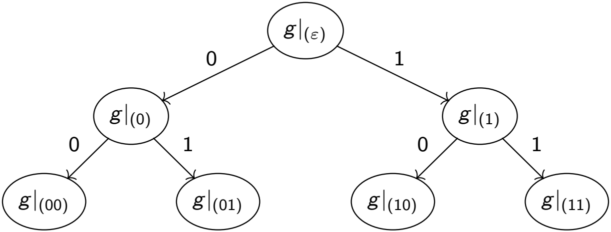
level $1 = X$

level $2 = X^2$

$\cdots$



automorphims	$\longleftrightarrow$	portraits
$\mathrm{Aut}(X^*)$	$\longleftrightarrow$	$\Sigma(X)^{X^*}$
$g : X^* \to X^*$	$\longleftrightarrow$	$g : X^* \to \Sigma(X)$

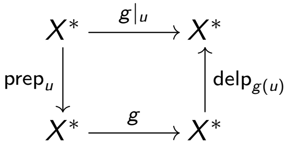


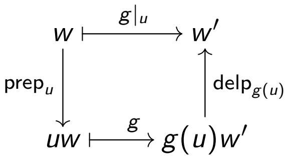
$$g(x_1x_2x_3\dots) = g|_{(\varepsilon)}(x_1)\ g|_{(x_1)}(x_2)\ g|_{(x_1x_2)}(x_3)\dots$$
$$g(ux) = g(u)\ g|_{(u)}(x)$$

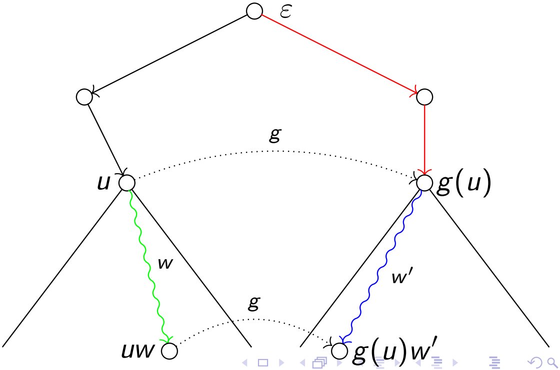
$$g^{\sigma_u} = g|_u$$

where $g|_u$ is the unique automorphism of X^* for which

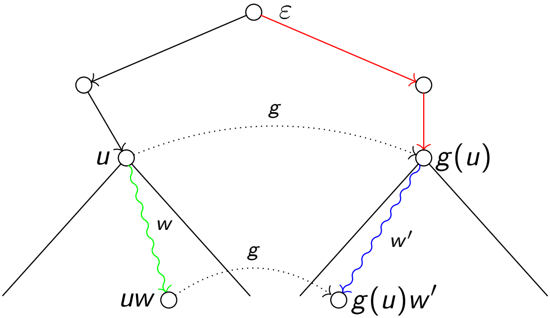
$$g(uw) = g(u)g|_u(w).$$







Section at u as a map $\sigma_u : \Sigma^{X^*} \to \Sigma^{X^*}$



$$\sigma_u : \Sigma^{X^*} \to \Sigma^{X^*}$$
$$(g^{\sigma_u})|_{(w)} = g|_{(uw)}$$
$$(g|_u)|_{(w)} = g|_{(uw)}$$

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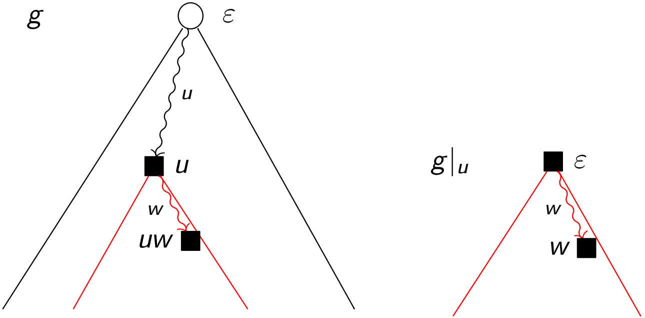

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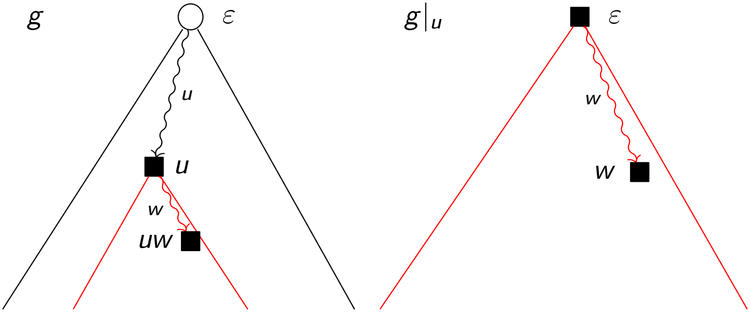


$$\sigma_u : \Sigma^{X^*} \to \Sigma^{X^*}$$

$$(g^{\sigma_u})|_{(w)} = g|_{(uw)}$$

$$(g|_u)|_{(w)} = g|_{(uw)}$$

Calculus of sections: section of a section is a section

$$(g|_u)|_v = g|_{uv}$$

$$(g^{\sigma_u})^{\sigma_v} = g^{\sigma_{uv}}$$

The tree (the semigroup X^* acts back)

$$\mathrm{Aut}(X) \curvearrowleft X^*$$

Calculus of sections

Section of a composition is composition of sections (on the same level)

$$(hg)|_u = h|_{g(u)} \ g|_u$$

Section of the inverse is the inverse of a section (on the same level)

$$(g^{-1})|_u = (g|_{g^{-1}(u)})^{-1}$$

Photo from the last lecture

That's me explaining, for the 12th time, what a section is

Topological (metric) structure on Σ^{X^*}

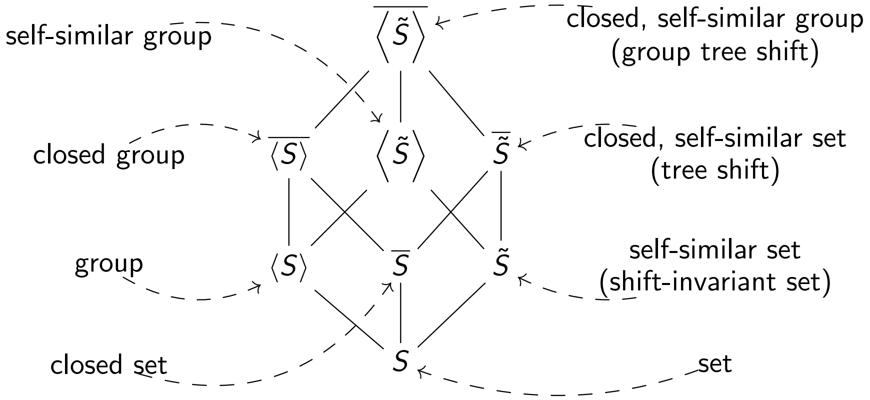
Two portraits are "close" when they agree on "many" levels:

$$d(g,h) = \inf\{ \ d_n \mid (\forall m \leq n)(\forall u \in X^m) \ g|_{(u)} = h|_{(u)} \ \}$$

where $d_n \searrow 0$ (popular choices: $\frac{1}{n+1}$, $\frac{1}{e^n}$, ...).

- The topology is the direct product topology on Σ^{X^*}, where Σ is discrete (thus, Cantor set)

Closures w.r.t. group, metric, and self-similarity structure



The three structures cooperate

Theorem

The subgroup generated by a self-similar set is self-similar.

Theorem

The closure of a subgroup is a subgroup.

Theorem

The closure of a self-similar set is self-similar.

Theorem

$\mathrm{Aut}(X^*)$ *is a topological group.*

Theorem

The action $\Sigma^{X^*} \curvearrowleft X^*$ *is by continuous maps.*

Examples of closed, self-similar groups

- $\mathrm{Aut}(X^*)$
- $\mathrm{Aut}_p(X^*) = p$-ary automorphisms, where p is a prime, $X = \{0, 1, \ldots, p-1\}$, and all vertex permutations are powers of the standard cycle $(012\ldots p-1)$.
- More generally, $\mathrm{Aut}_{\Sigma'}(X^*) = (\Sigma')^{X^*}$ where Σ' is a subgroup of Σ.
- ?

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Examples of self-similar groups

- $\mathrm{Aut}_f(X^*)$ = finitary automorphisms = automorphisms whose vertex permutations are trivial below some level = automorphisms with only finitely many nontrivial vertex permutations (this is a countable, locally finite group)
- More generally, for $\Sigma'' \leq \Sigma' \leq \Sigma$, automorphisms whose vertex permutations come from Σ', but only finitely many are outside of Σ''.
- $\mathrm{Aut}_{f.r.}(X^*)$ = automorhisms finitary along rays = automorphisms that, along every ray, have only finitely many nontrivial vertex permutations
- More generally, for $\Sigma'' \leq \Sigma' \leq \Sigma$, automorphisms whose vertex permutations come from Σ', but, along every ray, only finitely many are outside of Σ''.
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Note that $\mathrm{Aut}_f(X^*)$ and $\mathrm{Aut}_{f.r}(X^*)$ are dense in $\mathrm{Aut}(X^*)$ and, in the more general cases, the closures are equal to $\mathrm{Aut}_{\Sigma'}(X^*)$.

How to build more examples?

Start with a self-similar set, generate a self-similar group, then close it topologically.

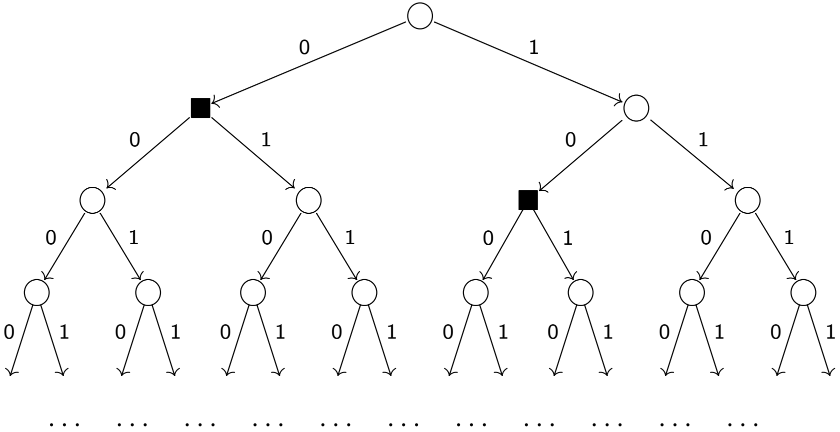
Moreover, in order to obtain “small” examples, start with “small” self-similar sets.

Finite state automorphisms

Definition

$g \in \mathrm{Aut}(X^*)$ is a finite-state automorphism if it has finitely many distinct sections (its orbit under X^* is finite).

The portrait of an automorphism as a Moore automaton



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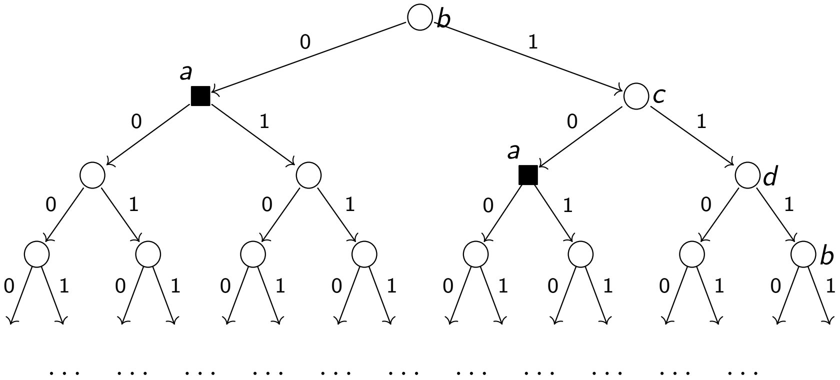

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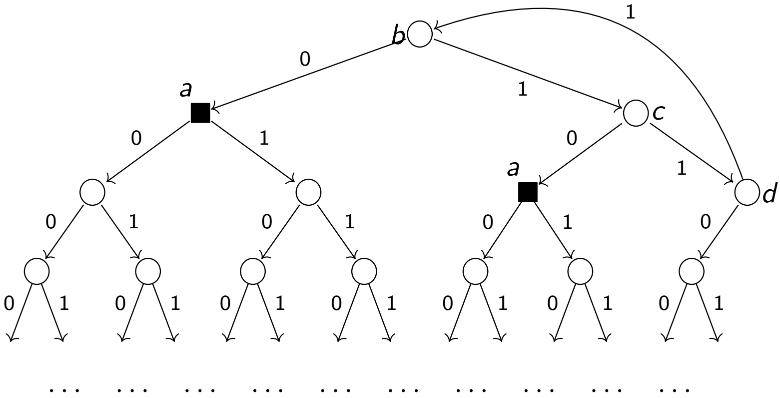

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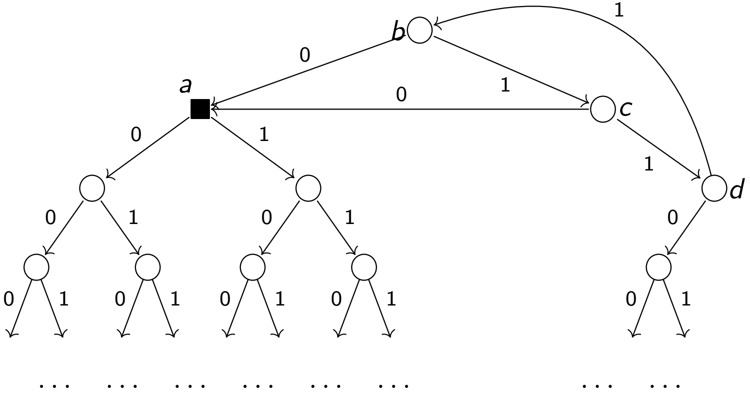
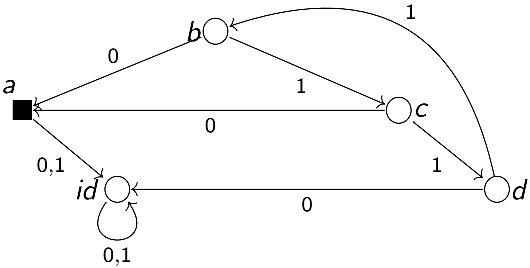


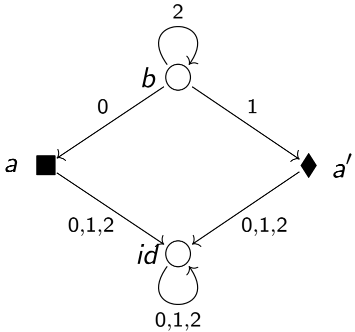
Grigorchuk automaton $\{id, a, b, c, d\}$



An automaton is

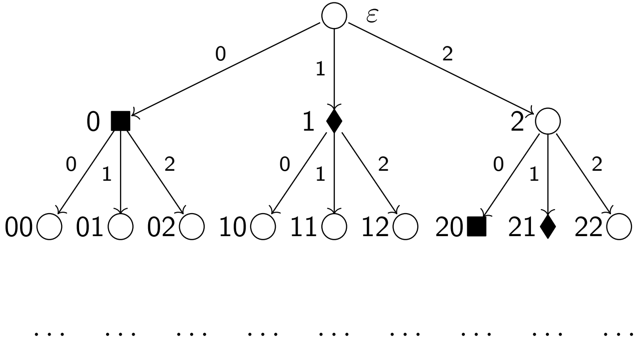
- a finite self-similar set of tree automorphisms.
- a finite set of tree automorphisms closed under taking sections.
- a finite set of finite-state tree automorphisms.
- a finite X^*-invariant set of tree automorphisms
- a finite union of finite orbits of the action $\mathrm{Aut}(X^*) \curvearrowleft X^*$
- a finite graph S with vertices labeled by permutations in $\Sigma(X)$ and,for each $x \in X$ and $s \in S$ one outgoing edge starting at s labeled by x.

(additional words sometimes used: finite, Moore, transducer, ...)

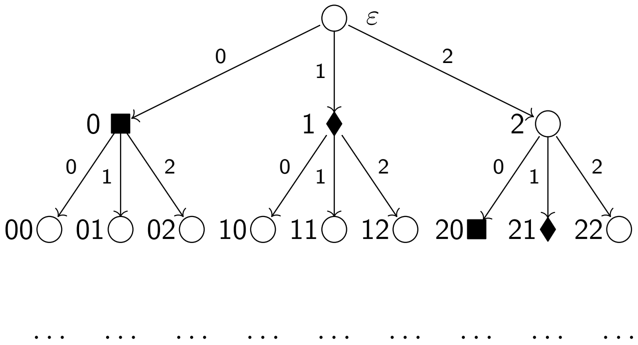


$b|_{(u)}$ = the label on the state reached by reading u starting from b

From an automaton back to a portrait



From an automaton back to a portrait



With the interpretation ○ = (), ■ = (012) and ♦ = (021)
$\langle id, a, a', b\rangle$ = Gupta-Sidki 3-group.

$\mathrm{Aut}_{fs}(X^*)$ = finite state automorphisms

Proposition

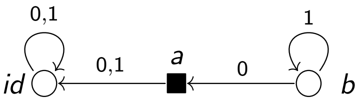
(a) $\mathrm{Aut}_{fs}(X^*) \leq \mathrm{Aut}(X^*)$.
(b) $\mathrm{Aut}_{fs}(X^*)$ *is self-similar.*
(c) $\mathrm{Aut}_{fs}(X^*)$ *is dense in* $\mathrm{Aut}(X^*)$.

Proof.

(a) Because sections of a product (inverse) are products (inverses) of sections.
(b) Because sections of sections are sections.
(c) Because it contains $\mathrm{Aut}_f(X^*)$. □

Examples are now easy to generate, but ...

How does the closure of the group generated by



look like? How to describe/characterize/recognize its elements?

How to describe a set of portraits?

How to describe a set of portraits?

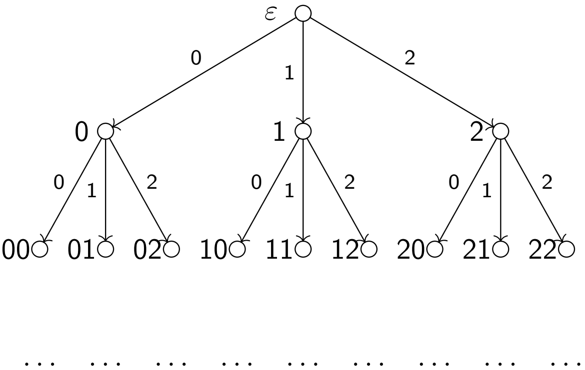
In fact, let us simplify the question and look at the rooted tree of arity 1.

level 0 $= X^0$

level 1 $= X$

level 2 $= X^2$

. . .



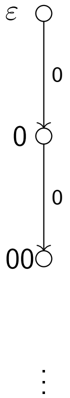
How to describe a set of portraits?

In fact, let us simplify the question and look at the rooted tree of arity 1.

level 0 $= X^0$

level 1 $= X = \{0\}$

level 2 $= X^2$



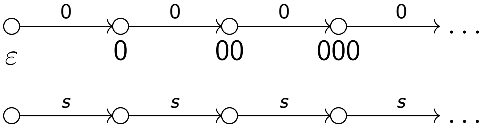
Aut(X^*) is trivial, but the rest of the "story" is not

A mighty giant's standing.
He hath a sword, and moves not,

Heinrich Heine

$X^* =$

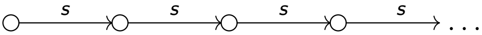
$\mathbb{N} =$



$$\Sigma^{X^*} = \Sigma^{\mathbb{N}} = \text{sequences over the finite alphabet } \Sigma$$

$\mathrm{Aut}(X^*)$ is trivial, but the rest of the "story" is not

$\mathbb{N} =$



$\Sigma^{X^*} = \Sigma^{\mathbb{N}} =$ sequences over the finite alphabet Σ

$g|_{(u)} =$ the term of the sequence g at u

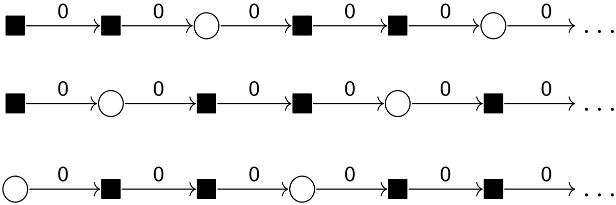
Two sequences are "close" if they agree on a "long" initial segment

$d(g,h) = \inf\{\ d_n \mid |u| = n,\ g|_{(u)} = h|_{(u)}\ \}$

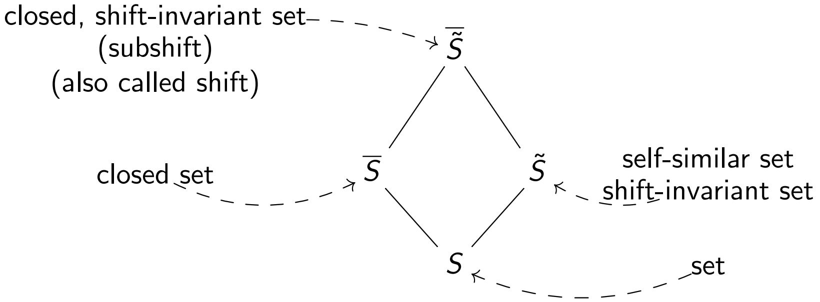
The section map σ_0 is just the shift map $\sigma : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$

$$(g^\sigma)|_{(u)} = g|_{(0u)}$$

$g =$, $g^\sigma =$, $g^{\sigma^2} =$



Closures w.r.t. the topological and self-similarity structures



S = set of sequences (configurations, points in the configuration space)

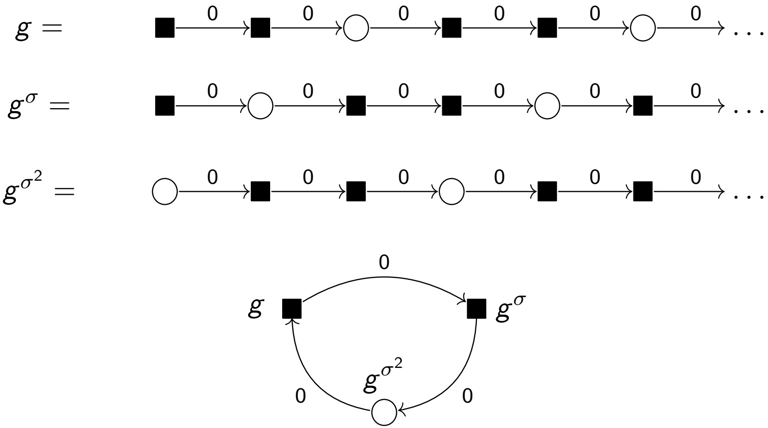
$\overline{S}$ = the smallest closed set containing S

$\tilde{S}$ = the smallest shift-invariant set containing S

$\overline{\tilde{S}}$ = the smallest closed and shift-invariant set containing S

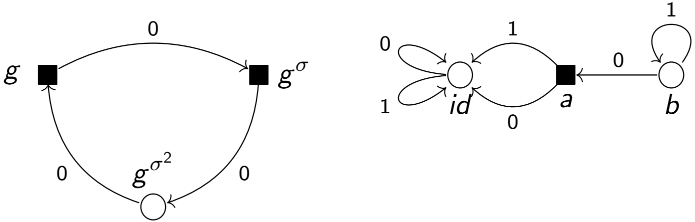
Finite state is same as periodic

finite state = finite orbit under shift(s) = periodic



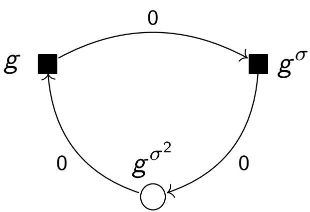
Finite self-similar (shift-invariant) sets as automata

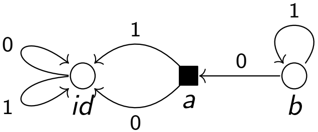
Automaton = finite graph S with vertices labeled by symbols in Σ and,for each $x \in X$ and $s \in S$ one outgoing edge starting at s labeled by x.

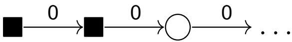


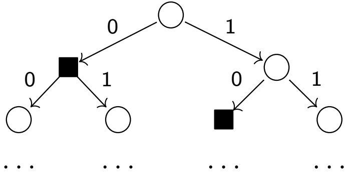
Finite self-similar sets as (deterministic) automata

Every state describes/defines exactly one element.









Enough pictures, now some serious math:

How to describe/define subsets of $\Sigma^{\mathbb{N}}$

Enough pictures, now some serious math:

How to describe/define subsets of $\Sigma^{\mathbb{N}}$

Use S1S

monadic second order formalism of the one successor structure X^*

S1S

first order varaiables: $x, y, z, \ldots$ (range over vertices in X^*)
second order varaibles: $X, Y, Z, \ldots$ (range over subsets of X^*)
successor: s ($s(u)$ is interpreted as the successor of u in X^*)

---

equality symbol: $=$
membership symbol: $\in$
quantifiers: $\forall, \exists$
logical connectives: $\neg, \wedge, \vee, \implies$
parentheses: (and)
basic terms: x
terms: for every term t, the expression $s(t)$ is also a term.
atomic formulas: $t_1 = t_2$, $X = Y$, $t \in X$.
formulas: for all formulas Φ and Ψ the following are also formulas

$$(\neg\Phi),\ (\Phi \wedge \Psi),\ (\Phi \vee \Psi),\ (\Phi \implies \Psi),$$
$$(\exists x\ \Phi),\ (\exists X\ \Phi),\ (\forall x\ \Phi),\ (\forall X\ \Phi).$$

Every formula defines a subset of Σ^{X^*}

$$\Phi: \quad X \text{ is a subset of } Y$$

$$\Phi: \quad \forall x \, (x \in X \implies x \in Y)$$

There are 2 free variables, X and Y. Thus, we set

$$\Sigma = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

A sequence over Σ defines a pair of subsets of X^*,
the top labels describe the (characteristic function of the) set X,
the bottom labels describe the set Y.

The set of sequences in Σ^{X^*} defined by Φ is the set of sequences that do not use the symbol $\binom{1}{0}$ (this symbol is "forbidden").

Every formula defines a subset of Σ^{X^*}

$$\Phi : \quad X \text{ is an inductive set}$$

$$\Phi : \quad \forall x \; (x \in X \implies s(x) \in X)$$

There is one free variable, X. Thus, we set

$$\Sigma = \{0, 1\}$$

A sequence over Σ defines a subset of X^* (its characteristic function).

The set of sequences in Σ^{X^*} defined by Φ is the set of consisting of the sequence of all 0s together with sequences with finite initial sequence of 0s followed by an infinite tail of 1s. Thus the appearance of the "pattern" 10 is "forbidden".

We can express a lot in S1S

$$\Phi : \qquad x \leq y$$

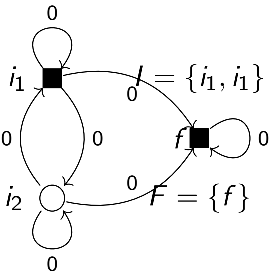
$$\Phi : \qquad \forall X \ (\ (X \text{ is inductive} \wedge x \in X) \implies y \in X \ )$$

Recognizing subsets of Σ^{X^*} by Büchi automata

Φ : X contains a nonempty inductive set

Φ : $\exists Y\ (\ \exists y(y \in Y) \wedge Y \text{ is inductive } \wedge X \subseteq Y\ )$

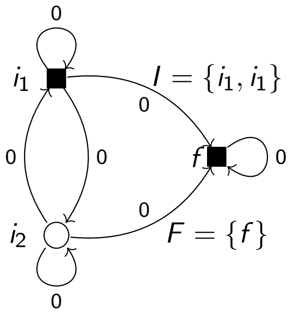
The sequences defined by Φ are precisely those with a tail of 1s. We can recognize the sequences defined by Φ by the following Büchi automaton (in which $\bigcirc = 0$ and $\blacksquare = 1$)

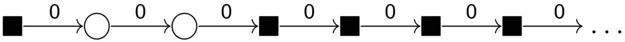


Start at i_1 or i_2, but make sure to visit f.

Büchi automaton

Büchi automaton = finite graph S
- with vertices labeled by symbols in Σ,
- a set of initial states I, and
- a set of final states F.





A sequence over Σ is accepted (recognized) if there exists a graph homomorphism from the sequence to the automaton such that
- the labels are preserved
- the root of the sequence goes to an initial state
- at least one of the final states is visited infinitely many times

Büchi Theorem(s)

Theorem (Büchi 1962)

The S1S theory of the one successor structure X^* is decidable.

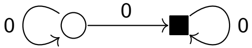
Theorem (Büchi 1962)

A set of sequences is definable in S1S if and only if it can be recognized by a Büchi automaton.

Note that it is much easier to stare at an automaton and decide if there exists a sequence that is accepted than to stare at the formula and decide the same thing. In the automaton, we just need to check if there is a path from an initial state to a cycle containing a final state.

Where do subshifts fit?

The inductive sets (their characteristic functions) form a subshift. They are accepted by



where both states are initial and both are final.

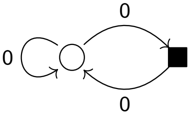
They can also be described as the sequences that do not contain the pattern ■○ (the pattern 10)

The golden shift

$$\forall x \; (\; x \in X \implies \neg(s(x) \in X) \;)$$

These are sequences without consecutive 1s (forbidden pattern ■■)

They are accepted by



where both states are initial and both are final.

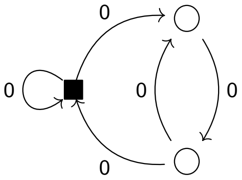
The even shift

$\{\, g \in \Sigma^{X^*} \mid$ there are even number of 0s between any two 1s in $g \,\}$

These are sequences that do not contain any pattern of the form

$$\blacksquare \underbrace{\bigcirc \bigcirc \dots \bigcirc \bigcirc}_{2k+1} \blacksquare$$

They are accepted by



where all states are initial and all are final.

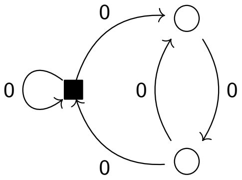
The even shift

$\{\, g \in \Sigma^{X^*} \mid$ there are even number of 0s between any two 1s in $g \,\}$

These are sequences that do not contain any pattern of the form

$$\blacksquare \underbrace{\bigcirc\,\bigcirc \ldots \bigcirc\,\bigcirc}_{2k+1} \blacksquare$$

They are accepted by



where all states are initial and all are final.

Thus, it can be defined in S1S! (Homework)

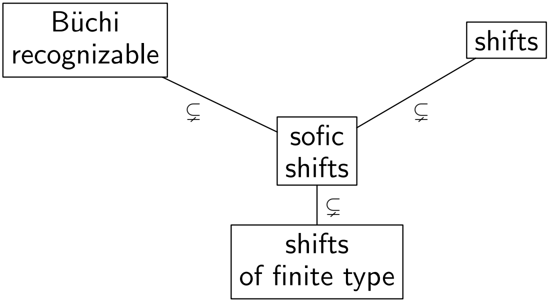
Sofic shifts and shifts of finite type

defined by frbddn ptrns	$\overset{thm}{\Longleftrightarrow}$	Subshift (closed and shift-invariant set)		
		$\supsetneq$		
		Sofic shift	$\overset{def}{\Longleftrightarrow}$	a set recognizable by a Büchi autom. no restrictions
		$\supsetneq$		
defined by finitely many frbddn pttrns	$\overset{def}{\Longleftrightarrow}$	Shift of finite type	$\overset{thm}{\Longrightarrow}$	a set recognizable by a Büchi autom. no restrictions

Sofic shifts and shifts of finite type

defined by frbddn ptrns	$\overset{thm}{\Longleftrightarrow}$	Subshift (closed and shift-invariant set)		
		$\supsetneq$		
quotient of a SFT	$\overset{thm}{\Longleftrightarrow}$	Sofic shift	$\overset{def}{\Longleftrightarrow}$	recognizable by a Büchi autom. no restrictions
		$\supsetneq$		
defined by finitely many frbddn pttrns	$\overset{def}{\Longleftrightarrow}$	Shift of finite type	$\overset{thm}{\Longrightarrow}$	recognizable by a Büchi autom. no restrictions

Sofic = Büchi rcgnzble ∩ shifts = S1S dfnble ∩ shifts



The shift-invariance removes the need for initial states and the topological closure property removes the need for final states.

2 is more complicated than 1

We can repeat, *mutatis mutandis*, most of the story on the binary tree (or any tree of higher (finite) arity). It is definitely more complicated, but it is also the same.

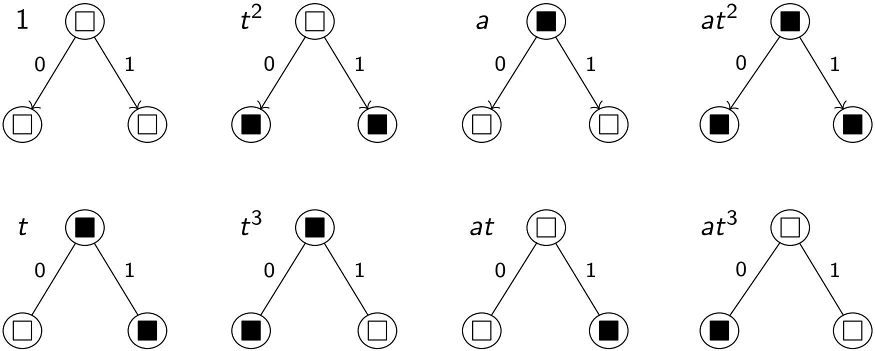
And stared, as if doubting my meaning,
And said: For the sake of heaven explain

Heinrich Heine

Tree patterns

Definition

An X-tree **pattern** of size s over Σ is a map in $\Sigma^{X^{(s)}}$, where $X^{(s)} = \bigcup_{i=0}^{s-1} X^i$.



Two tree shifts of finite type

If we forbid the tree-patterns in the bottom row

$$\mathcal{G}(\mathcal{B}) = \{ \ g \in \Sigma^{X^*} \mid g_{(u0)} = g_{(u1)}, \ \text{for all } u \ \}$$

If we forbid the tree-patterns in the right half

$$\mathcal{G}(\mathcal{R}) = \{ \ g \in \Sigma^{X^*} \mid g_{(u0)} + g_{(u1)} = g_{(u)}, \ \text{for all } u \ \}$$

that was no real paradise,
with a tree forbidden in it.

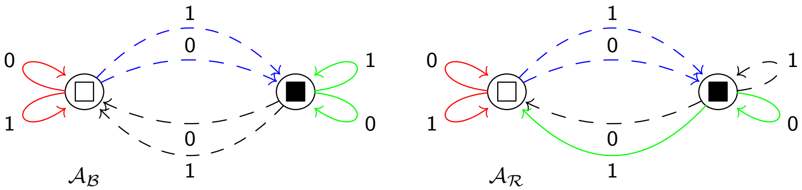
Heinrich Heine

Definition

An X-tree shift over Σ defined by finitely many forbidden patterns is called a **tree shift of finite type**.

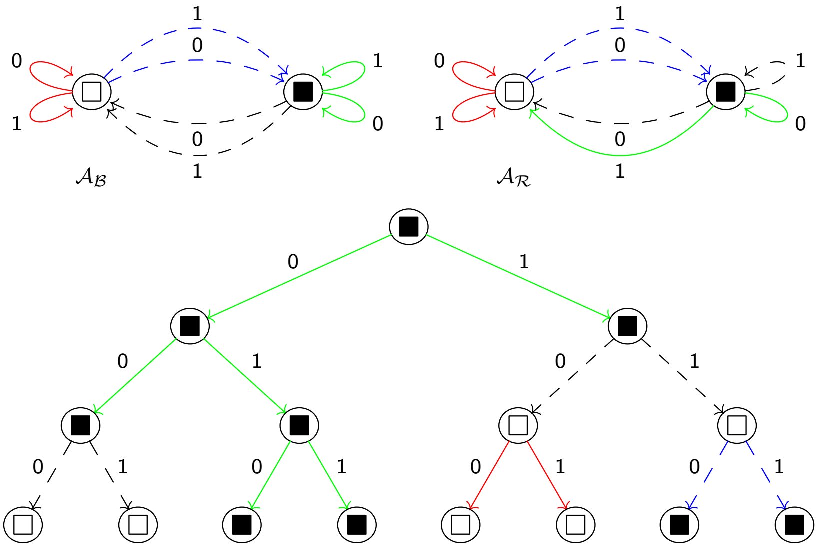
Graphical (automaton) representation

$$\mathcal{G}(\mathcal{B}) = \{\ g \in \Sigma^{X^*} \mid g|_{(u0)} = g|_{(u1)},\ \text{for all } u\ \}$$



$$\mathcal{G}(\mathcal{R}) = \{\ g \in \Sigma^{X^*} \mid g|_{(u0)} + g|_{(u1)} + g|_{(u)} \text{ is even, for all } u\ \}$$

Acceptance of a portrait



Unrestrictive Rabin automaton (Rabin-Moore automaton)

An unrestrictive Rabin X-tree graph over Σ is a 5-tuple $A = (S, X, \Sigma, \tau, \lambda)$, where

- S is a nonempty set, called the set of states (or vertices),
- X is a finite alphabet, called transition (or edge, or tree) alphabet,
- Σ is a finite alphabet, called decoration (or labeling, or state, or vertex) alphabet,
- $\tau \subseteq S \times S^X$ is a relation, called the transition relation, whose elements are called transition bundles (or edge bundles), and
- $\lambda : S \to \Sigma$ is a decoration map (or labeling) map, assigning label to each state.

An unrestrictive Rabin X-tree automaton over Σ is an unrestrictive Rabin X-tree graph over Σ in which the set S of states is finite.

Acceptance by unrestrictive Rabin automata

Let

$$g = (S_g, X, \Sigma, \tau_g, \lambda_g) \qquad \text{and} \qquad h = (S_h, X, \Sigma, \tau_h, \lambda_h)$$

be two unrestrictive X-tree Rabin graphs over Σ.
A homomorphism $\alpha : g \to h$ is a map $\alpha : S_g \to S_h$ on the vertices that is compatible with the labeling and with the transition functions, i.e., a map such that

- the labels on the vertices in g agrees with the labels of their images in h. More precisely, for every vertex s in S_g,

$$\lambda_g(s) = \lambda_h(\alpha(s)),$$

  and

- the edge bundles are preserved. More precisely, for all $s, s_0, \ldots, s_{k-1}$ in S_g,

$$(s, s_0, \ldots, s_{k-1}) \in \tau_g \implies (\alpha(s), \alpha(s_0), \ldots, \alpha(s_{k-1})) \in \tau_h.$$

Acceptance by unrestrictive Rabin automata

Let $f : X^* \to \Sigma$ be an X-tree over Σ. We can think of it as the unrestrictive Rabin X-tree graph over Σ

$$f = (X^*, X, \Sigma, \tau_X, f)$$

where

$$\tau_X = \{(w, (wx)_{x \in X}) \mid w \in X^*\}$$

Let $g = (S, X, \Sigma, \tau, \lambda)$ be an unrestrictive Rabin X-tree automaton over Σ. The automaton g accepts f if there exists a homomorphism $\alpha : f \to g$.

The language $\mathcal{L}(g)$ of g is the set of all X-trees f over Σ accepted by g, i.e.,

$$\mathcal{L}(g) = \{f \in \Sigma^{X^*} \mid \text{there exists a homomorphism } \alpha : f \to g\}.$$

first order varaiables: $x, y, z, \ldots$ (range over vertices in X^*)
second order varaibles: $X, Y, Z, \ldots$ (range over subsets of X^*)
2 successors: s_0 and s_1 ($s_0(u)$ and $s_1(u)$ interpreted as the successors of u in X^*)

---

equality symbol: $=$
membership symbol: $\in$
quantifiers: $\forall, \exists$
logical connectives: $\neg, \wedge, \vee, \implies$
parentheses: (and)
basic terms: x
terms: for every term t, the expressions $s_0(t)$ and $s_1(t)$ are terms.
atomic formulas: $t_1 = t_2$, $X = Y$, $t \in X$.
formulas: for all formulas Φ and Ψ the following are also formulas

$$(\neg\Phi),\ (\Phi \wedge \Psi),\ (\Phi \vee \Psi),\ (\Phi \implies \Psi),$$
$$(\exists x\ \Phi),\ (\exists X\ \Phi),\ (\forall x\ \Phi),\ (\forall X\ \Phi).$$

Some simple formulas in S2S

Φ : z is the root

Φ : $\forall x\ (z \neq s_0(x) \wedge z \neq s_1(x))$

Ψ : X is a ray

Ψ : the root is in $X \wedge$

$\forall x\ (x \in X \implies (s_0(x) \in X \vee s_1(x) \in X,$ but not both$)\wedge$

$\forall x \forall y\ (x \in X \wedge (x = s_0(y) \vee x = s_1(y)) \implies y \in X)\wedge$

Rabin Theorem(s)

Theorem (Rabin 1969)

The S2S theory of the two successor structure X^* is decidable.

Theorem (Rabin 1969)

A set of labeled trees is definable in S2S if and only if it can be recognized by a Rabin automaton.

Rabin Automaton?

Well, same as unrestricted Rabin automaton, but with restrictions!

O stop, or youll drive me quite crazy!

Heinrcih Heine

Rabin Automaton?

Well, same as unrestricted Rabin automaton, but with restrictions!

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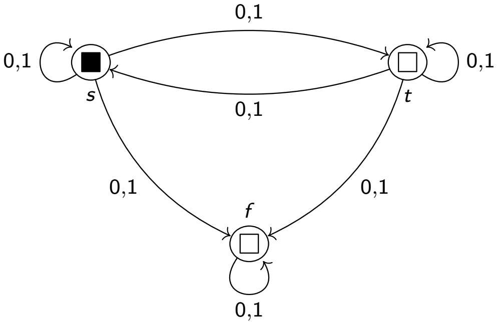
Heinrcih Heine

- A set I of initial states is given
- A family F of sets of states is given (final sets of states)
- Acceptance means: (1) start in an initial state and (2) for every ray, the set of states visited infinitely many times by that ray is a set in the family F,

Finitary automorphisms can be recognized by Rabin

Three states (and the compactness of the boundary of X^*) suffice.



$$I = \{s, t, f\}$$

$$F = \{\{f\}\}$$

defined by frbddn ptrns	$\overset{thm}{\Longleftrightarrow}$	Subshift (closed and shift-invariant set)		
		$\cup\!\!\!\!\!\!\not\,$		
quotient of a SFT	$\overset{thm}{\Longleftrightarrow}$	Sofic shift	$\overset{def}{\Longleftrightarrow}$	recognizable by a Rabin autom. no restrictions
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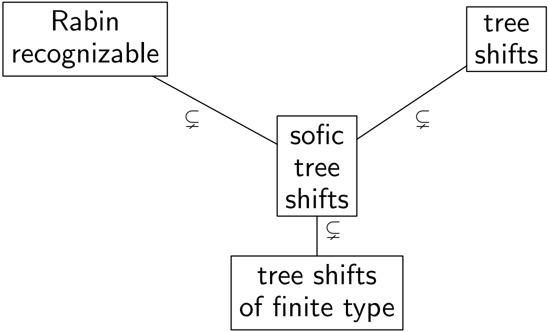
Sofic = Rabin rcgnzble ∩ shifts = S2S dfnble ∩ shifts

C-S,C,F,Š, 2013



The shift-invariance (self-similarity) removes the need for initial states and the topological closure property removes the need for final states.

Recall that there was a group structure!

'Tis now full time that my folly I drop,
And return to sober reason;
This comedy now 'twere better to stop
That weve played for so long a season.

Heinrich Heine

A simple (?) observation

Proposition

Every group tree shift of finite type (in fact, every sofic group tree shift) is a closure of a group of finite-state automorphisms.

A characterization of group tree shifts of finite type

Theorem (Grigorchuk 2006 (i) implies (ii), Š. 2007 converse)

Let G be a subgroup of $\mathrm{Aut}(X^*)$. The following are equivalent.
(i) *G is defined by forbidden tree patterns of size s*
(ii) *G is the closure of a regular branch group H, branching over its stabilizer H_{s-1} of level $s-1$.*

Three examples

Example

The first Grigorchuk group (1980) branches over its stabilizer of level 3, so its closure is a group tree shift of finite type defined by patterns of size 4
(total of $2^{12} = 4096$ allowed patterns; thus $2^{15} - 2^{12}$ forbidden).

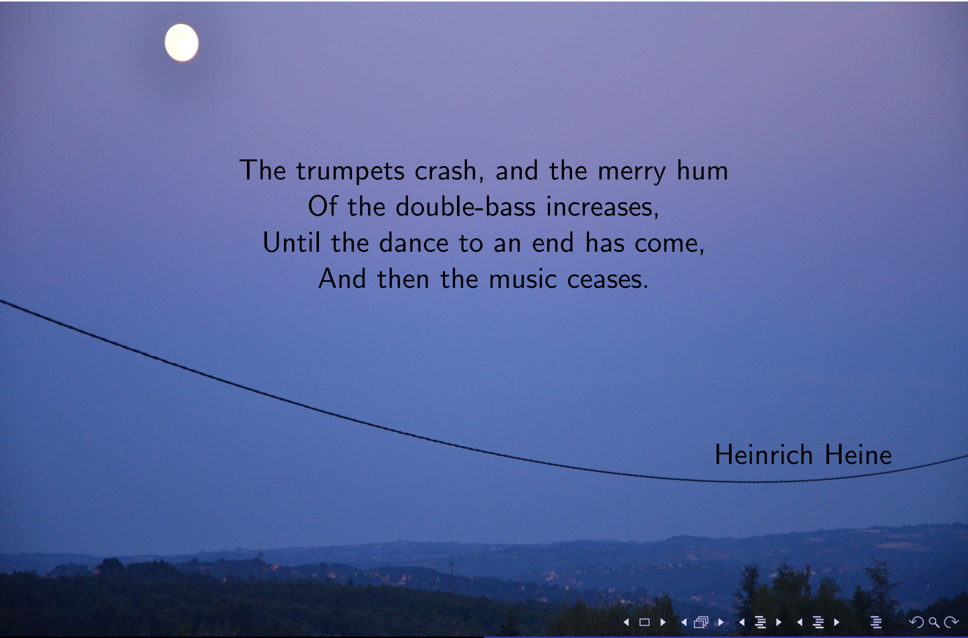
Example

The Gupta-Sidki 3-group (1983) branches over its stabilizer of level 2, so its closure is a group tree shift of finite type defined by patterns of size 3
(total of $3^8 = 6561$ allowed patterns; thus $3^{13} - 3^8$ forbidden).

Example

The closure of the Hanoi Towers group (3 pegs) is a group tree shift of finite type defined by patterns of size 2 (even product)
(total of $3 \cdot 6^3$ allowed patterns; thus $6^4 - 3 \cdot 6^3$ forbidden).

from "Knave of Bergen"



The trumpets crash, and the merry hum
Of the double-bass increases,
Until the dance to an end has come,
And then the music ceases.

Heinrich Heine