An Ihara-type theorem for buildings

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Ihara's theorem

the classical theorem of Ihara

- Let $\mathbb{K}$ be a non-archimedian local field, i.e., $\mathbb{K}$ is a field with a discrete valuation $v\colon \mathbb{K} \to \mathbb{Z} \cup \{\infty\}$ and $\mathcal{O} \subseteq \mathbb{K}$, where $\mathcal{O} = \{\, x \in \mathbb{K} \setminus v(x) \leq 0 \,\}$, is a complete discrete valuation ring with finite residue field $\mathbb{F}$.
- Let V be a 2-dimensional $\mathbb{K}$-vector space.
- A $\mathcal{O}$-submodule $L \subseteq V$ is said to be a $\mathcal{O}$-**lattice**, and
- two $\mathcal{O}$-lattices $L_1, L_2 \subseteq V$ are said to be **equivalent**, if there exists $\lambda \in \mathbb{K}^\times$ such that $L_2 = \lambda \cdot L_1$, and **adjacent** if $L_1/L_1 \cap L_2 \simeq \mathbb{F}$.
- Let $\mathscr{V}$ denote the set of equivalence classes of $\mathcal{O}$-lattices, and let $\mathcal{E} = \{\, ([L_1], [L_2]) \in \mathscr{V}^2 \mid L_1, L_2 \text{ adjacent} \,\}$.
- Then $\Gamma(V) = (\mathscr{V}, \mathcal{E})$ with the obvious mappings $o\colon \mathcal{E} \to \mathscr{V}$, $t\colon \mathcal{E} \to \mathscr{V}$, $\bar{\ }\colon \mathcal{E} \to \mathcal{E}$ is a combinatorial graph.
- (Y. Ihara) The graph $\Gamma(V)$ is a tree.

Consequences of Ihara's theorem
using Bass-Serre Theory

- Let $([L_1],[L_2])$ be an edge in the graph $\Gamma(V)$.
- Let $G = \mathrm{SL}_2(\mathbb{K})$, $G_i = \mathrm{stab}_G([L_i])$ and $I = \mathrm{stab}_{G_1}([L_2])$. Then

$$G \simeq G_1 \amalg_I G_2.$$

  The group I is called an Iwahori subgroup of G.
- (Stallings) Every torsion free lattice $\Lambda \subseteq \mathrm{SL}_2(\mathbb{K})$ is a free group.
- As a consequence every lattice $\Lambda \subseteq \mathrm{SL}_2(\mathbb{K})$ is a virtually free group.

The graph

associated to a pair of subgroups

- Let G be a group, and let $A, B \subseteq G$ be subgroups of G. Then
  - $\Gamma(G; A, B) = (\mathscr{V}, \mathcal{E})$ given by
  - $\mathscr{V} = G/A \sqcup G/B$,
  - $\mathcal{E} = \{ (gA, gB) \mid g \in G \}$ with the obvious mappings $t, o \colon \mathcal{E} \to \mathscr{V}$, $\bar{\ } \colon \mathcal{E} \to \mathcal{E}$, is a graph in the sense of J-P. Serre.

Proposition (Trees, J-P. Serre)

Let G be a group, and let $A, B \subseteq G$ be two proper non-trivial subgroups of G. Then the following are equivalent:

- *$\Gamma(G; A, B)$ is a tree.*
- *The canonical homomorphism $\pi \colon A \coprod_C B \to G$ is an isomorphism, where $C = A \cap B$.*

Coxeter groups

Definition

A group W together with a subset $S \subset W$ satisfying

- $\forall \sigma \in S : \operatorname{ord}(\sigma) = 2$;
- For $\sigma, \tau \in S$ let

$$m_{\sigma,\tau} = \operatorname{ord}(\sigma \cdot \tau) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}.$$

Then

$$W \simeq \langle S \mid (\sigma \cdot \tau)^{m_{\sigma,\tau}} = 1 \rangle.$$

is called a **Coxeter system**. Groups with a Coxeter system are called **Coxeter groups**

Decomposing Coxeter groups

Definition

Let (W, S) be a Coxeter system, and let $S_\blacktriangle$, $S_\blacktriangledown \subseteq S$ be subsets of S. Then $S_{\blacktriangledown/\blacktriangle}$ is said to be an ∞-decomposition of (W, S), if

- $S_\blacktriangle \cup S_\blacktriangledown = S$;
- For $S_\bullet = S_\blacktriangledown \cap S_\blacktriangle$ and $S_\triangledown = S_\blacktriangledown \setminus S_\bullet$; $S_\vartriangle = S_\blacktriangle \setminus S_\bullet$ one has :

$$m_{\sigma,\tau} = \infty \quad \forall \sigma \in S_\triangledown,\ \tau \in S_\vartriangle.$$

- The ∞-decomposition $S_{\blacktriangledown/\blacktriangle}$ is said to be **spherical**, if $(W_\bullet, S_\bullet)$ is a finite Coxeter group, where $W_\bullet = W_{S_\bullet} = \langle S_\bullet \rangle$ is the **parabolic subgroup** generated by $S_\bullet$.

Decomposing Coxeter groups, II

Proposition

Let (W, S) be a Coxeter system, let $S_{\blacktriangledown/\blacktriangle} \subseteq S$ be proper subsets of S, and let $S_{\bullet} = S_{\blacktriangledown} \cap S_{\blacktriangle}$. Then the following are equivalent:

- $W \simeq W_{\blacktriangle} \coprod_{W_{\bullet}} W_{\blacktriangledown}$, *where* $W_{\blacktriangledown} = W_{S_{\blacktriangledown}} = \langle S_{\blacktriangledown} \rangle$, *etc.*
- $\Gamma(W; W_{\blacktriangledown}, W_{\blacktriangle})$ *is a tree.*
- $W_{\blacktriangledown/\blacktriangle}$ *is an* ∞*-decomposition.*

Coxeter groups with more than one end

Ends of Coxeter Groups

It is known after Stallings that a group can have 0, 1, 2 or infinitely many ends. Are there known results on the space of ends of a Coxeter group?

gr.group-theory gt.geometric-topology

edited Sep 10 '14 at 15:05 Stefan Kohl 9,469 • 5 • 40 • 89

asked Sep 10 '14 at 14:46 Nicolas Boerger 268 • 1 • 8

1 You can find some information on ends of Coxeter groups in this paper by Mihalik: sciencedirect.com/science/article/pii/0022404995001174 – Nick Gill Sep 10 '14 at 15:29

5 I think the theorem stating that a finitely generated group can only have 0, 1, 2 or infinitely many ends is rather due to Hopf. – Seirios Sep 10 '14 at 15:38

1 Answer

The following book has a wealth of material on this topic:

> *The geometry and topology of Coxeter groups* by Michael Davis.

By way of example, here is one result from the book:

> **Theorem 8.7.1** Let (W, S) be a Coxeter System:
>
> 1. W is one-ended if and only $H_c^1(\Sigma) = 0$.
> 2. W has two ends if and only $H_c^1(\Sigma) \cong \mathbb{Z}$.
> 3. W has infinitely many ends if and only if $H_c^1(\Sigma)$ has infinite rank.

Here Σ is the cell complex associated to (W, S). I have an e-copy of the book - email me if you want a copy.

answered Sep 10 '14 at 17:22 Nick Gill 7,086 • 17 • 49

1 I would not consider this as a satisfactory answer, since it changes a problem into another one. I would consider as a full answer one describing in terms of the combinatorial data defining the Coxeter group. For instance just describing when the Coxeter group is finite involves a quite lengthy list. Describing when it has 2 ends is just, in the connected case, the case of the infinite dihedral group. (...) – YCor Sep 10 '14 at 17:42

Here are 1 source of multi-ended Coxeter group: when there is a partition $S = A \cup C \cup B$ with A, B non-empty, such that all edges between A and B are labeled ∞, and such that the subgroup generated by C is finite. This is something checkable. Maybe there are other obvious sources of multi-ended Coxeter groups, and it would be natural to wonder if they are the only ones. – YCor Sep 10 '14 at 17:46

@YCor, of course your comment is entirely reasonable. There are plenty more results in the cited text that give more explicit information (see especially Thm. 8.7.3)... but I don't want to write them all out! – Nick Gill Sep 10 '14 at 18:22

Decomposing Coxeter groups, III

Proposition (M. Davis)

Let (W, S) be a Coxeter system. Then $e(W) > 1$ if and only if there exists a spherical ∞-decomposition $S_{\blacktriangledown/\blacktriangle}$ of (W, S). In particular, Y. Cornulier's question has an affirmative answer.

Definition

A Coxeter system (W, S) is said to be completely spherically ∞-decomposable, if there exists a family of subsets $(S_j)_j \in J$ such that

- $S = \bigcup_{j \in J} S_j$;
- (W_{S_j}, S_j) is a finite Coxeter group for all $j \in J$;
- for $j, k \in J$, $j \neq k$, S_j, S_k is a spherical ∞-decomposition of $(W_{S_j \cup S_k}, S_j \cup S_k)$.

Coxeter groups of dimension 1

Proposition

Let (W, S) be a Coxeter system. Then the following are equivalent:

- $\mathrm{vcd}(W) \leq 1$;
- $\mathrm{cd}_{\mathbb{Q}}(Q) \leq 1$
- *(W, S) is completely spherically ∞-decomposable.*

Reduced expressions

and the length function ...

Buildings

Definition

A building is a simplicial complex $\mathscr{B}$ which is the union of a family of simplicial subcomplexes $\mathscr{A}$ which elements are called appartments with the following properties:

- B_1 Every appartment $\Sigma \in \mathscr{A}$ is a Coxeter complex;
- B_2 For any two simplices $A, B \in \mathscr{B}$ there is an appartment Σ containing both of them;
- B_3 For any two appartments $\Sigma, \Sigma' \in \mathscr{A}$ containing A and B there is an isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise.

Remark

In the above definition "empty sets" are allowed.

Buildings, II

Remark

All appartments are isomorphic to a Coxeter complex of a fixed Coxeter group (W, S). Moreover, the type function $|\_|\colon C(W,S) \to \mathcal{P}^\sharp(S)$ can be extended to a type function

$$|\_|\colon \mathscr{B} \to \mathcal{P}^\sharp(S) \quad (1)$$

($\mathcal{P}^\sharp$ = proper subsets.)

Decompositions and Graphs

- Let $S_{\blacktriangledown/\blacktriangle} \subseteq S$ be a decomposition of the Coxeter system (W, S), i.e., $S = S_{\blacktriangle} \cup S_{\blacktriangledown}$, and put $S_{\bullet} = S_{\blacktriangle} \cap S_{\blacktriangledown}$.
- Let $\mathscr{B}_{\bullet}$ denote all elements in $\mathscr{B}$ of co-type $S_{\bullet}$.
- For $B \in \mathscr{B}_{\bullet}$ let $B^{\blacktriangledown}$ denote the face of B of cotype $S_{\blacktriangledown}$, and similar $B^{\blacktriangle}$ the face of B of cotype $S_{\blacktriangle}$.
- Let $\Gamma_{\blacktriangledown/\blacktriangle}$ denote the graph (in the sense of J-P. Serre) given by

$$\mathscr{V} = \mathscr{B}_{\blacktriangledown} \cup \mathscr{B}_{\blacktriangle},$$
$$\mathcal{E} = \mathscr{B}_{\bullet} \sqcup \bar{\mathscr{B}}_{\bullet},$$
$$o|_{\mathscr{B}} = \_^{\blacktriangledown}, \quad t|_{\mathscr{B}} = \_^{\blacktriangle}.$$

Then $\Gamma_{\blacktriangledown/\blacktriangle}$ is a combinatorial graph. Moreover one has the following version of Ihara's theorem.

Theorem

Let $S_{\blacktriangledown/\blacktriangle}$ be a decomposition of the Coxeter system (W, S) associated to the building $\mathscr{B}$. Then the following are equivalent:

- $S_{\blacktriangledown/\blacktriangle}$ *is an ∞-decomposition of (W, S);*
- $\Gamma_{\blacktriangledown/\blacktriangle}$ *is a tree.*

Topological Kac-Moody groups

of Rémy-Ronan type ...

- Let $\mathbb{F}$ be a finite field, and let $G(\mathbb{F})$ denote the $\mathbb{F}$-rational points of an almost split Kac-Moody group defined over $\mathbb{F}$ with infinite Weyl group (W, S).
- Let Ξ denote the twin building associated to $G(\mathbb{F})$, and Ξ^+ the positive part of Ξ.
- Let $\bar{G}(\mathbb{F})$ denote the completion of $G(\mathbb{F})$ with respect to its discrete action of the locally finite simplicial complex $|\Xi^+|$, where $|\Xi^+|$ is the Davis-realization of Ξ^+. Then $\bar{G}(\mathbb{F})$ is a totally disconnected locally compact group, and the same is true for the set of type-preserving elements $\bar{G}^\circ(\mathbb{F})$
- If $S_{\blacktriangledown/\blacktriangle}$ is an ∞-decomposition of (W, S), then - by Bass-Serre theory -
$$\bar{G}^\circ(\mathbb{F}) \simeq \bar{G}^\circ_\blacktriangle(\mathbb{F}) \coprod_{\bar{G}^\circ_\bullet(\mathbb{F})} \bar{G}^\circ_\blacktriangledown(\mathbb{F}).$$
- If W is of virtual cohomological dimension 1, then $\bar{G}^\circ(\mathbb{F})$ is isomorphic to the fundamental group of a finite graph of "profinte groups" which edge maps are open embeddings.

The proof

or why is it true!!

- The principal idea of the proof the generalized Ihara theorem is to define maps between the set of minimal galleries $\mathfrak{mgal}^{\pm}(\mathscr{B})$ with signature of the building $\mathscr{B}$ and the set $\mathcal{P}_r(\Gamma_{\blacktriangledown/\blacktriangle})$ of reduced paths in the graph $\Gamma_{\blacktriangledown/\blacktriangle}$, and to study their properties in order to be able to show surjectivity (by induction).