Branch groups and their trees, and ordered sets

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Definition. A subgroup H of a group G is a precomponent if H commutes with its distinct conjugates.
Then $\langle H^G \rangle = \langle H^g \mid g \in G\}$ is the central product of the conjugates.

Examples: normal subgroups, subgroups of nilpotent groups of class 2, groups H with $H/(H \cap \mathrm{Z}(G))$ non-abelian simple.

They arise often:

- components in finite groups,
  (precomponents H with $H/(H \cap \mathrm{Z}(G))$ simple and H perfect)
- the 'natural' direct summands of base groups of wreath products
- restricted stabilizers for group actions on rooted trees
- restricted stabilizers for actions on other sets, e.g. totally ordered sets

Aim: unified approach to precomponents via first-order group theory.

Let $H \leqslant G$.

$H^x \sim H^y$ if $\exists n, \exists x_0 = x, x_2, \ldots, x_n = y$ with $[H^{x_{i-1}}, H^{x_i}] \neq 1$ for all i.

$P = \langle H^x \mid H^x \sim H\}$ is the unique smallest precomp. containing H.

Notation: $C_G^2(X) = C_G(C_G(X))$.

Let P be a precomponent.

$P \lhd \langle P^x \mid x \in G \rangle \lhd G$. Also $P \lhd C_G^2(P)$:

$x \in C_G^2(P) \Rightarrow x$ centralizes $C_G(P)$
$\Rightarrow x$ normalises all $P^g \neq P \Rightarrow x \in N_G(P)$.

When does the obvious graph have (uniformly) bounded diameter?

First-order sentences/formulae

$(\forall x \forall y \forall z)([x, y, z] = 1)$ G nilp. of class $\leqslant 2$ Yes!

$(\forall x \in G')(\forall z)([x, z] = 1)$ G nilp. of class $\leqslant 2$ No!

$(\forall x_1 \forall x_2 \forall x_3 \forall x_4)(\exists y_1, y_2)([x_1, x_2][x_3, x_4] = [y_1, y_2])$

every element of G' is a commutator

$(\forall x_1 \forall x_2 \exists y)(y \neq x_1 \wedge y \neq x_2)$ $|G| \geqslant 3$

$(\forall x_1 \forall x_2 \forall x_3 \forall x_4)(\bigvee_{1 \leqslant i < j \leqslant 4} x_i = x_j)$ $|G| \leqslant 3$

$(\forall x)(x^6 = 1 \rightarrow x = 1)$ no elements of order $2, 3$

$g^4 = 1 \wedge g^2 \neq 1$ g has order 4

$(\forall k \neq 1)(\forall g)(\exists r \in \mathbb{N})(\exists x_1, \ldots, x_r)(g = k^{x_1} k^{x_2} \ldots k^{x_r})$ No!

Classes of finite groups defined by a sentence

(1) $\{$groups of order $\leqslant n\}$, $\{$groups of order $\geqslant n\}$, $\{$groups with no elements of order $n\}$, nilpotent groups of class $\leqslant 2$.

(2) **Felgner's Theorem (1990).** $\exists$ sentence σ (in the f.-o. language of group theory) such that, for G finite, $G \models \sigma \Leftrightarrow G$ is non-abelian simple.

$\sigma = \sigma_1 \wedge \sigma_2$ with

σ_1: $(\forall x \forall y)(x \neq 1 \wedge \mathrm{C}_G(x,y) \neq \{1\} \to \bigcap_{g \in G} (\mathrm{C}_G(x,y)\mathrm{C}_G(C_G(x,y)))^g = \{1\})$,
σ_2: 'each element is a product of κ_0 commutators' for a fixed $\kappa_0 \in \mathbb{N}$.

(3) Finite soluble groups:

They are characterized (among finite groups) by 'no $g \neq 1$ is a prod. of commutators $[g^h, g^k]$'; that is, ρ_n holds $\forall n$

$$\rho_n: (\forall g \forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_n)(g = 1 \vee g \neq [g^{x_1}, g^{y_1}] \dots [g^{x_n}, g^{y_n}]).$$

Theorem (JSW 2005). Finite G is soluble iff it satisfies ρ_{56}.

Quasisimple groups

G is quasisimple if G perfect and $G/\mathrm{Z}(G)$ simple

Proposition (JSW 2017). Finite G is quasisimple iff Q satisfies $\mathrm{QS}_1 \wedge \mathrm{QS}_2 \wedge \mathrm{QS}_3$:

QS_1: each element is a product of two commutators;
QS_2: $(\forall x)(\forall u)[x, x^u] \in \mathrm{Z}(G) \to x \in \mathrm{Z}(G)$;
QS_3:
$(\forall x \forall y)(x \notin \mathrm{Z}(G) \wedge \mathrm{C}_G(x,y) > \mathrm{Z}(G)) \to \bigcap_{g \in G}(\mathrm{C}_G(x,y)\mathrm{C}_G^2(x,y))^g = \mathrm{Z}(G)$.

Definable sets

... sets of elements $g \in G$ (or in $G^{(n)} = G \times \cdots \times G$) defined by first-order formulae, possibly with parameters from G.

Examples: • $\mathrm{Z}(G)$, defined by $(\forall y)([x, y] = 1)$

• $\mathrm{C}_G(h)$, defined by $[x, h] = 1$

• Centralizers of definable sets are definable:
Say $S = \{s \mid \varphi(s)\}$; then $\mathrm{C}_G(S) = \{t \mid \forall g(\varphi(g) \to [g, t] = 1)\}$

The (*soluble*) *radical* $\mathrm{R}(G)$ of a finite group G is the largest soluble normal subgroup of G.

Theorem (JSW 2008). There's a f.-o. formula $r(x)$ such that if G is finite and $g \in G$ then $g \in \mathrm{R}(G)$ iff $r(g)$ holds in G.

The sets X_h, W_h

- $X_h = \{[h^{-1}, h^g] \mid g \in G\}$, $\quad W_h = \bigcup\{X_{h^g} \mid g \in G, [X_h, X_{h^g}] \neq 1\}$.

$$
\begin{array}{rll}
\varphi_1(h,x): & (\exists y)(x = [h^{-1}, h^y]) & \text{(defines } X_h\text{)} \\
\varphi_2(h,x): & (\exists t \exists y_1 \exists y_2)(\varphi_1(h, y_1) \wedge \varphi_1(h^t, y_2) \wedge \varphi_1(h^t, x) \wedge \ [y_1, y_2] \neq 1) & \\
& & \text{(defines } W_h\text{)} \\
\varphi_3(h,x): & (\forall y)(\varphi_2(h,y) \to [x,y] = 1) & \mathrm{C}_G(W_h) \\
\gamma(h,x): & (\forall y)(\phi_3(h,y) \to [x,y] = 1) & \mathrm{C}_G^2(W_h)
\end{array}
$$

- $\varepsilon_{\leqslant}(x,y)$: $\varepsilon_{\leqslant}(h_1,h_2)$ iff $\mathrm{C}_G^2(W_{h_1}) \leqslant \mathrm{C}_G^2(W_{h_2})$
$\{(h_1,h_2) \mid \varepsilon_{\leqslant}(h_1,h_2)\}$ definable in $G \times G$; leads to a definable equiv. relation
- $\exists\ \beta(x)$: $\beta(h)$ iff $\mathrm{C}_G^2(W_h)$ commutes with its distinct conjugates.

G finite: component = quasisimple subgroup Q that commutes with its distinct G-conjugates ($\Leftrightarrow$ Q subnormal).

Theorem (JSW 2017). $\exists$ f.o. formulae $\pi(h, y)$, $\pi'(h)$, $\pi'_{\mathrm{c}}(h)$, $\pi'_{\mathrm{m}}(h)$ such that for every finite G, the products of components of G are the sets $\{x \mid \pi(h, x)\}$ for the $h \in G$ satisfying $\pi'(h)$.

The components: the sets $\{x \mid \pi(h, x)\}$ for which $\pi'_{\mathrm{c}}(h)$ holds.
The non-ab. min. normal subgps.: $\{x \mid \pi(h, x)\}$ with $\pi'_{\mathrm{m}}(h)$.

Define δ_r for $r \geqslant 1$ recursively by $\delta_1(x_1, x_2) = [x_1, x_2]$ and $\delta_r(x_1, \ldots, x_{2^r}) = [\delta_{r-1}(x_1, \ldots, x_{2^{r-1}}), \delta_{r-1}(x_{2^{r-1}+1}, \ldots, x_{2^r})]$ for $r > 1$.

$$
\begin{aligned}
\gamma(h,x)&: \quad (\forall y)(\phi_3(h,y) \to [x,y] = 1) && \mathrm{C}_G^2(W_h) \\
\alpha^1(h,x)&: \quad (\exists y_1 \ldots \exists y_{16})\big(\big(\textstyle\bigwedge_{n=1}^{16} \gamma(h,y_n)\big) \wedge x = \delta_4(y_1, \ldots, y_{16})\big) \\
& && \delta_4\text{-value in } \mathrm{C}_G^2(W_h) \\
\alpha(h,x)&: \quad (\exists y_1 \exists y_2)(\alpha^1(h,y_1) \wedge \alpha^1(h,y_1) \wedge x = y_1 y_2)
\end{aligned}
$$

Let G be finite, Q a component. If $h \in Q \setminus \mathrm{Z}(Q)$ then $Q = \langle W_h \rangle$, so $Q \leqslant \mathrm{C}_G^2(W_h)$.
Show $Q =$ set of prods. of 2 δ_4-values in $\mathrm{C}_G^2(W_h)$, so $Q = \{x \mid \alpha(h,x)\}$.

Automorphism groups of ordered sets

Write $\mathrm{Aut}_O(\Omega)$ for the group of order AMs of a totally ordered set Ω.

Theorem (Andrew Glass and JSW, 2017). Suppose that $\mathrm{Aut}_O(\Omega)$ acts transitively on Ω.
(a) If $\mathrm{Aut}_O(\Omega)$, $\mathrm{Aut}_O(\mathbb{R})$ satisfy the same first-order sentences then $\Omega \cong \mathbb{R}$ (as ordered set).
(b) If $\mathrm{Aut}_O(\Omega)$, $\mathrm{Aut}_O(\mathbb{Q})$ satisfy the same first-order sentences then $\Omega \cong \mathbb{Q}$ or $\Omega \cong \mathbb{R} \setminus \mathbb{Q}$.

Same conclusion by Gurevich and Holland (1981) with the stronger hypothesis that $\mathrm{Aut}_O(\Omega)$ acts transitively on pairs (α, β) with $\alpha < \beta$.

Transitivity is necessary. Let $\Omega = \mathbb{R} \times \{0, 1\}$ with alphabetic order:
$(r_1, \lambda_1) < (r_2, \lambda_2)$ if $r_1 < r_2$ or if $r_1 = r_2$ and $\lambda_1 < \lambda_2$.
Then $\mathrm{Aut}_O(\mathbb{R} \times \{0, 1\}) \cong \mathrm{Aut}_O(\mathbb{R})$.

Let Ω be totally ordered.
For $f, g \in \mathrm{Aut}_O(\Omega)$ define $f \vee g$, $f \wedge g \in \mathrm{Aut}_O(\Omega)$ by

$$\alpha(f \vee g) = \max\{\alpha f, \alpha g\}, \quad \alpha(f \wedge g) = \min\{\alpha f, \alpha g\} \quad \text{for all } \alpha \in \Omega.$$

An *ℓ-permutation group* on Ω is a subgroup $G \leqslant \mathrm{Aut}_O(\Omega)$ closed for $\vee$, $\wedge$.
Let G be a trans. ℓ-perm. group on Ω. A convex set $\Delta \subseteq \Omega$ is an o-*block* if either $\Delta g = \Delta$ or $\Delta g \cap \Delta = \emptyset$ for each $g \in G$.
Stabilizer and rigid stabilizer of o-block Δ are defined by

$$\mathrm{Stab}(\Delta) := \{g \in G \mid \Delta g = \Delta\}, \quad \mathrm{rst}(\Delta) := \{g \in G \mid \mathrm{supp}(g) \subseteq \Delta\},$$

G is o-primitive if $\nexists$ o-blocks apart from Ω and singletons.
G is o-2 transitive if transitive on all $(\alpha_1, \alpha_2) \in \Omega \times \Omega$ with $\alpha_1 < \alpha_2$.
o-2-transitivity $\Longrightarrow$ o-primitivity.

'McCleary's Trichotomy'. Transitive f.d. o-primitive ℓ-permutation groups are o-2 transitive or right regular representations of subgroups of $\mathbb{R}$.

Technicalities

Lemma. Let G be o-2 transitive on Ω and $g, h \in G$ with $\mathrm{supp}(h) \cap \mathrm{supp}(h^g) = \emptyset$ and $h \neq 1$. Then $\exists\ f, k \in G$ such that

$$[h^{-1}, h^f][h^{-g}, h^{gk}] \neq [h^{-g}, h^{gk}][h^{-1}, h^f].$$

For $g \in G$ and each union Λ of convex g-invariant subsets of Ω, let $\mathrm{dep}(g, \Lambda)$ be the element of $\mathrm{Aut}_O(\Omega)$ that agrees with g on Λ and with the identity elsewhere. Say G *fully depressible* (*f.d.*) on Ω if $\mathrm{dep}(g, \Lambda) \in G$ for all $g \in G$ and all such $\Lambda \subseteq \Omega$.

$\mathrm{Aut}_O(\Omega)$ is fully depressible.

Let G be a f.d. transitive ℓ-perm. group on Ω.
Write $B(\alpha, \beta)$ for the smallest o-block containing both $\alpha, \beta \in \Omega$.
Let $T = \{B(\alpha, \beta) \mid \alpha \neq \beta\}$.
Assume Stab(Δ) acts on Δ as a non-abelian group for all $\Delta \in T$.

Recall that

$$X_h := \{[h^{-1}, h^g] \mid g \in G\} \quad \text{and} \quad W_h = \bigcup \{X_{h^g} \mid g \in G,\ [X_h, X_{h^g}] \neq 1\}.$$

For $\Delta \in T$, let

$$Q_\Delta = \{h \in \mathrm{rst}(\Delta) \mid (\exists \alpha \in \Omega)(B(\alpha h, \alpha)) = \Delta\}.$$

As G transitive and f.d., $Q_\Delta \neq \emptyset$. Since $(\mathrm{rst}(\Delta))^g = \mathrm{rst}(\Delta g)$ commutes with $\mathrm{rst}(\Delta)$ for $g \notin \mathrm{Stab}(\Delta)$, we have

$$X_h \subseteq \mathrm{rst}(\Delta) \quad \text{and} \quad W_h \subseteq \mathrm{rst}(\Delta) \qquad \text{for all } \Delta \in T \text{ and } h \in Q_\Delta.$$

Proposition 1. Let $\Delta \in T$ and $h \in Q_\Delta$.

(a) $W_h = \bigcup\{X_{h^g} \mid g \in \mathrm{Stab}(\Delta)\}$.

(b) $\mathrm{C}_G(W_h)$ is the pointwise stabilizer of Δ.

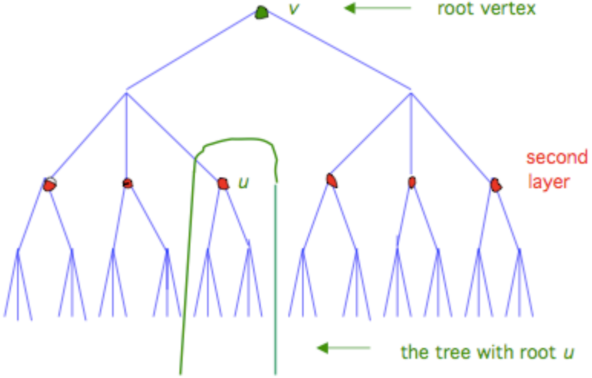
(c) $\mathrm{C}^2_G(W_h) = \mathrm{rst}(\Delta)$. In particular, $\mathrm{C}^2_G(W_h)$ is independent of $h \in Q_\Delta$:

Corollary. G is o-primitive on Ω iff $\mathrm{C}^2_G(W_g) = G$ for all $g \in G \setminus \{1\}$.
So if (G_1, Ω_1), (G_2, Ω_2) are transitive f.d. ℓ-groups that satisfy the same f.-o. sentences, and G_1 is o-primitive, then so is G_2.

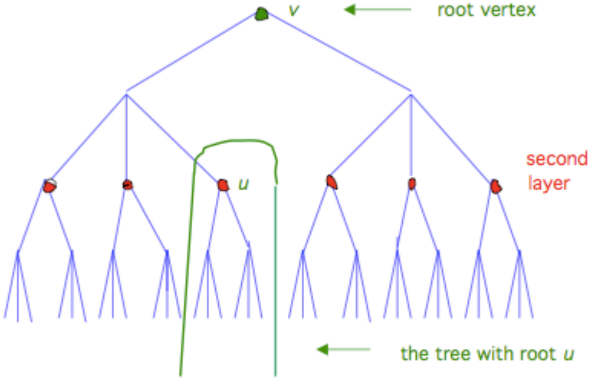
Proof of the Theorem. Let $\Lambda = \mathbb{R}$ or $\Lambda = \mathbb{Q}$, let $\mathrm{Aut}_O(\Omega)$, $\mathrm{Aut}_O(\Lambda)$ satisfy same f.-o. sentences. Enough to prove $\mathrm{Aut}_O(\Omega)$ o-2-transitive.
$\mathrm{Aut}_O(\Lambda)$ is o-2-transitive on Λ, so o-primitive, non-abelian. So $\mathrm{Aut}_O(\Omega)$ is non-abelian and o-primitive by Corollary. Now use McCleary's trichotomy.

(Proof shows that if $G \leqslant \mathrm{Aut}_O(\Omega)$ transitive and f.d. then $\Omega \cong \Lambda$.)

The rooted tree of type $(2, 3, 2, 3, \dots)$



The rooted tree of type $(2, 3, 2, 3, \ldots)$



Let G act faithfully on T fixing v.

Second layer L_2 is a union of G-orbits

$\mathrm{rst}_G(u)$ – elements moving only vertices in T_u

$\mathrm{rst}_G(2) = \langle \mathrm{rst}_G(w) \mid w \in$ 2nd layer$\rangle$, the dir. product.

Fix $(m_n)_{n\geqslant 0}$, a sequence of integers $m_n \geqslant 2$.

The *rooted tree* T of type (m_n) has a root vertex v_0 of valency m_0. Each vertex of distance $n \geqslant 1$ from v_0 has valency $m_n + 1$.
nth layer L_n: all vertices u at distance n from v_0.
So m_n edges descend from each $u \in L_n$.
For a vertex u, the subtree with root u is T_u.

Let G act faithfully on T.
$\mathrm{rst}_G(u) = \{g \mid g \text{ fixes each vertex outside } T_u\}$.
$\mathrm{rst}_G(n) = \langle \mathrm{rst}_G(u) \mid u \in L_n \rangle$.

G acts as a *branch group* on T if for each n,

- G acts transitively on L_n,
- $\mathrm{rst}_G(n)$ has finite index in G.

Definition. G is Boolean if $G \neq 1$ and

- G/K is vA (virtually abelian) whenever $1 < K \leqslant G$;
- G has no non-trivial vA normal subgroups.

Branch groups are Boolean.

Structure lattice

Assume G is Boolean.

$\mathbf{L}(G) = \{H \leqslant G \mid |G : \mathrm{N}_G(H)| \text{ finite}\}$ – a lattice of subgroups of G.

(A) Let $H, K \in \mathbf{L}(G)$. Then $H \cap K = 1$ iff $[H, K] = 1$.
(B) If $H \in \mathbf{L}(G)$ then $\exists U \leqslant_f G$ with

$$\langle H, \mathrm{C}_G(H) \rangle = H \times \mathrm{C}_G(H) \geqslant U'.$$

Write $H_1 \sim H_2$ iff $\mathrm{C}_G(H_1) = \mathrm{C}_G(H_2)$. An equiv. reln. on $\mathbf{L}(G)$.

(C) The lattice operations in $\mathbf{L}(G)$ induce well-defined join and meet operations $\vee$, $\wedge$ in

$$\mathcal{L} = \mathcal{L}(G) = \mathbf{L}(G)/\sim.$$

$\mathcal{L}$ is the structure lattice of G; greatest and least elements $[G]$ and $[1] = \{1\}$. It's a Boolean lattice:

$a \vee (b_1 \wedge b_2) = (a \vee b_1) \wedge (a \vee b_2), \ldots,$ has complements

Characterization of branch groups

Assume G Boolean. Let
$\Gamma(G) = \{[B] \in \mathcal{L} \mid B \text{ a precomp. } \}$.

Conjugation induces an action of G on $\Gamma(G)$; faithful if

(Branch1) $\bigcap(\mathrm{N}_G(B) \,|\, B$ a precomponent in $\mathcal{L}) = 1$.

Now assume also

(Branch2) For each precomponent $A \neq 1$ in $\mathcal{L}$ the normal closure of
$\bigcap(\mathrm{N}_G(B) \mid B$ non-triv. precomp. in $\mathcal{L}$, $A \cap B = 1)$
has fin. index in G.

Then $\Gamma(G)$ has subtrees on which G acts as a branch group.
Conversely, branch groups on T satisfy these conditions and T embeds G-equivariantly in $\Gamma(G)$.

Interpretations: an example

K a field witn $|K| > 2$, let T the mult. group $K \setminus \{0\}$.

$$G = \left\{ \begin{pmatrix} 1 & x \\ 0 & t \end{pmatrix} \mid x \in K, t \in T \right\}.$$

Write (x, t) for above matrix,
$A = \{(x, 1) \mid x \in K\} \cong K_+$ and $H = \{(0, t) \mid t \in T\} \cong T$.
So $A \triangleleft G$, $G = A \rtimes H$. Fix $e = (1, 1) \in A$ and $f = (0, \lambda) \in H \setminus \{1\}$.
$A = \{k \mid (\forall g)\,[k, k^g] = 1\}$ definable in G,
$H = \{g \mid g^f = g\} = C_G(f)$ definable (with parameters e, f).
For a, b in A define
$a + b = ab$,

$$a * b = \begin{cases} 1 & \text{if } a \text{ or } b = 1 \\ a^g & \text{if not, where } b = e^g \text{ with } g \in G. \end{cases}$$

A becomes a field isomorphic to K.
The set A and the operations on A are definable in G. An interpretation (with parameter e) of the field K in the group G.

Structure graph

Branch groups G can have branch actions on essentially different maximal trees. These actions are encoded in the structure graph $\Gamma(G)$:

- vertices the classes $[B] \in \mathcal{L}(G)$ containing a precomp. B;
- join $[B_1]$, $[B_2]$ by an edge if $[B_1] \leqslant [B_2]$ or $[B_2] \leqslant [B_1]$, and $\nexists$ intermediate classes in the ordering inherited from $\mathcal{L}(G)$.

Conjugation in G induces an action on $\mathcal{L}(G)$ and $\Gamma(G)$.

The tree on which G acts embeds equivariantly in the structure graph; often the embedding is an equivariant IM of trees.

In this case G 'knows' its tree: we can find the tree 'within' G.

New description of structure graph

$C_G^2(Y)$ for $C_G(C_G(Y))$, etc. So $Y \subseteq C_G^2(Y)$, $C_G^3(Y) = C_G(Y)$.
$H \in \mathbf{L}(G)$ is C^2-closed if $H = C_G^2(H)$.

(E) Let G be a branch group.

(a) If H_1, $H_2 \in \mathbf{L}(G)$ have same centralizer then $C_G^2(H_1) = C_G^2(H_2)$.

(b) B a precomp. $\Rightarrow C_G^2(B)$ a precomp.

(c) $B_1 < B_2$ precomps., C^2-closed $\Rightarrow N_G(B_1) < N_G(B_2)$.

The graph $\mathcal{B}(G)$ has

- vertices the non-trivial C^2-closed precomps.,
- edge between vertices if one a maximal proper C^2-closed precomp. in the other.

G acts on $\mathcal{B}(G)$ by conjugation.

(F) G branch, on tree T, and v a vertex. Then
$\mathrm{C}_G^2(\mathrm{rst}_G(v)) = \mathrm{rst}_G(v)$, so $\mathrm{rst}(v) \in \mathcal{B}(G)$.

Proof. Clearly $\mathrm{rst}_G(v) \leqslant \mathrm{C}_G^2(\mathrm{rst}_G(v))$. Let $h \in \mathrm{C}_G^2(\mathrm{rst}(v))$. Must prove h fixes every vertex $\notin T_v$, follows if h fixes every such u of level $\geqslant$ level of v. We have $\mathrm{rst}_G(u) \leqslant \mathrm{C}(\mathrm{rst}_G(v))$, so h centralizes $\mathrm{rst}_G(u)$. Thus $\mathrm{rst}_G(u) = (\mathrm{rst}_G(u))^h = \mathrm{rst}_G(uh)$, and so $uh = u$.

Theorem (JSW, 2015). G branch, acting on T.

(a) $B \mapsto [B]$ is a G-equivariant IM $\mathcal{B}(G) \to \Gamma(G)$.

(b) $v \mapsto \mathrm{rst}_G(v)$ is a G-equivt. order-preserving injective map $T \to \mathcal{B}(G)$.

Properties of $\mathcal{B} = \mathcal{B}(G)$ for branch G:

- G is the only vertex fixed in the G-action on $\mathcal{B}$
- the orbit $O(B)$ of each vertex B is finite
- each vertex B is connected to vertex G by a finite path; all simple such paths have length $\leqslant \log_2(|O(B)|)$
- $\forall\ B \in \mathcal{B}\ \ \exists$ branch action for which B is the restricted stabilizer of a vertex
- if $\mathcal{B}$ is a tree then G acts on it as a branch group.

Questions about $\mathcal{B} = \mathcal{B}(G)$

• finite valency?

• can there be exactly $\aleph_0$ maximal trees?

Now G is a branch group.

Recall
$X_h = \{[h^{-1}, h^g] \mid g \in G\}$,
$W_h = \bigcup\{X_{h^g} \mid g \in G, [X_h, X_{h^g}] \neq 1\}$.
$\beta(x)$: $\beta(h)$ iff $C^2_G(W_h)$ commutes with its distinct conjugates

Key Proposition. $\forall\, B \in \mathcal{B}(G)$, $\exists\, h \in G$ with $B = C^2_G(W_h)$.

Proof uses (among other things) the result of Hardy, Abért: branch groups satisfy no group laws. In particular, if $u \in T$ then $\exists\, x, y \in \mathrm{rst}_G(u)$ with $(xy)^2 \neq y^2x^2$.

Interpretation in branch groups

Theorem (JSW, 2015). There are first-order formulae τ, $\beta(x)$, $\varepsilon(x, y)$ s.t. the following holds for each branch group G:

(a) G has a branch action on a unique maximal tree up to G-equivariant IM iff $G \models \tau$;

(b) $S = \{x \mid \beta(x)\}$ is a union of conj. classes, so G acts on it by conjugation;

(c) the relation on S defined by $\varepsilon(x, y)$ is a G-invariant preorder. So $Q = S/\sim$, where $\sim$ is the equiv. relation defined by $\delta(x, y) \wedge \delta(y, x)$, is a poset on which G acts;

(d) Q is G-equivariantly isom. as poset to structure graph $\mathcal{L}(G)$.

When G has a branch action on a unique maximal tree T, this represents T as quotient of a definable subset of G modulo a definable equivalence relation. A parameter-free interpretation for T, and for the action on T.