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Introduction

This diploma thesis grew out of three seminars about geometric and differential topology the author attended at Göttingen University in the years 2007 and 2008.

The first was concerned with the h–cobordism theorem due to Stephen Smale. As an important application it implies the equivalence of the notions *diffeomorphic* and *h–cobordant* for a certain class of manifolds. This class comprises simply connected closed manifolds of dimension greater than four. In particular, the problem of classifying manifolds homeomorphic to the n–dimensional sphere S^n is restated as a classification problem up to h–cobordism for $n > 4$. Albeit technical, the property h–cobordism has got the advantage of being easier traceable. A possibly even more recognised application of the h–cobordism theorem is a proof of the generalised Poincaré conjecture in dimensions $n > 4$. It states that any homotopy n–sphere is homeomorphic to the standard sphere S^n.

The second seminar was then a work–through of Michel Kervaire's and John Milnor's seminal paper "Groups of homotopy spheres I", [KerMil63]. It shows (at that time excluding the case $n = 3$) that there are only finitely many homotopy n–spheres up to h–cobordism. In other words and provided that $n > 4$, there are only finitely many pairwise non–diffeomorphic manifolds homeomorphic to S^n. In still other words, there are only finitely many distinct exotic spheres of dimension greater than four. Somewhat unsatisfactory, but implied by the very nature of the purpose of the paper, no example of an exotic sphere is constructed in it.

The third seminar filled this gap. It was based on Milnor's book "Singular Points of Complex Hypersurfaces", [Mil68], which carries out the construction of a certain fibre bundle associated with an isolated singularity of a complex algebraic variety. It turns out that the boundary of a small neighbourhood of such a singularity can be an exotic sphere. For the proof, however, which involves computing such invariants as the intersection form or the Arf invariant of a coboundary Milnor refers the reader to the relevant literature.

Now in the first part of this diploma thesis we will finally recall a proof of existence of exotic spheres. We put the task in the more general context of classifying total spaces of S^3–bundles over S^4 up to homeomorphism, PL–homeomorphism and diffeomorphism. This is precisely the class of manifolds Milnor examined in 1956 to give the first examples of exotic spheres. Partial results of the classification appeared sporadically ever since Milnor's discovery proved that there is interest in the theory. The classification was finally completed by D. C. Crowley and C. Escher in 2003, [CrEs03]. We follow their paper, filling in some details a research article must leave out, in an attempt to present some kind of exposition on the subject. As expected the classification shows that also those manifolds among the total spaces which are not homeomorphic to S^7 admit various "exotic" differentiable structures.

The second part comes back to algebraic geometry, the other classical area of mathematics exotic spheres are encountered in. As the preceding classification of sphere bundles suggests, one might expect a similar situation for singularity boundaries of algebraic varieties. They should as well provide examples of homeomorphic and non–diffeomorphic manifolds even if they are not topological spheres. As a starting point, one could reconsider the two criteria Milnor has applied to decide whether a given singularity boundary is a topological sphere. These are determinants of certain integer matrices arising from different contexts. They both take unit values whenever the manifold in question is homeomorphic to the sphere. We will prove that these determinants coincide (up to sign) in general. It is thus natural to assume that they should still be useful when examining singularity boundaries which are not topological spheres.

Throughout this diploma thesis *smooth* will mean differentiable of class C^∞. *Mappings* and *maps* of topological spaces are meant to be continuous. *Manifolds*, with or without boundary, are to be smooth and compact unless otherwise stated. Numbers without brackets refer to sections, theorems or other numbered units. Numbers in round brackets refer to equations. Finally, a token in square brackets refers to the bibliography.

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1 Classification of 3–sphere bundles over the 4–sphere

In this chapter we want to present the classification of S^3–bundles over S^4 up to homeomorphism, PL–homeomorphism and diffeomorphism. We will see that a variety of exotic 7–manifolds occur in this fashion. The outline is as follows. In section 1.1 we will recall a classical result due to Feldbau reducing the classification up to fibre bundle equivalence to the computation of $\pi_3(\mathrm{SO}_4) = \pi_3(\mathrm{O}_4)$. The part of the classification that can directly be deduced from elementary bundle theory will also be included in this section. An overview of the strategy for the general classification of the bundles is given in section 1.2. We will take a closer look on the relevant invariants and their values for our sphere bundles in section 1.3. Finally, section 1.4 concludes with some curiosities that can be found along the way.

1.1 The classifying homotopy group

We want to classify fibre bundles with fibre S^3, base S^4 and structure group SO_4. Our starting point is the following result (see [St44] or [St51], p. 99).

1.1 Theorem. *The equivalence classes of bundles over S^n with structure group G and fibre F are in 1–1 correspondence with equivalence classes of elements of $\pi_{n-1}(G)$ under the operations of $\pi_0(G)$.*

Here are some explanations.

- Two bundles ξ and ξ' are called equivalent if they have the same base space, fibre and group and if there is a bundle map $\xi \to \xi'$ (in the sense of [St51], p. 9) inducing the identity map of the common base space.
- The notation $\pi_0(G) = G/G_e$ with G_e denoting the component of the unit element e in G is understood.
- Any $g \in G$ representing a class in $\pi_0(G)$ operates on G as an inner automorphism thus inducing an automorphism of each homotopy group $\pi_n(G)$. A curve g_t from g_0 to g_1 in G gives a homotopy $(g,t) \mapsto g_t g g_t^{-1}$ leaving e fixed. Hence we have a well–defined action $\pi_0(G) \curvearrowright \pi_n(G)$ for each n.

The construction as it stands is entirely topological. However, setting $n = 4$, $G = \mathrm{SO}_4$ and $F = \mathbb{R}^4$, any resulting four dimensional vector bundle $\zeta : E(\zeta) \to \mathrm{S}^4$ is naturally equipped with a smooth bundle structure which is unique up to smooth bundle equivalence, see e. g. [DeSa68], p. 232. Since ζ has got SO_4 as structure group, we may assume a Euclidean metric is fixed. The smooth bundle charts can then be chosen to be fibrewise isometries. Thus the natural smooth structure on the total space $E(\zeta)$ induces a smooth structure on its submanifold of unit vectors M. The effective action of SO_4 on S^3 is given by the restriction of its action on $\mathbb{R}^4$. Therefore M is precisely the total space of the sphere bundle $\xi\colon E(\xi) = M \to \mathrm{S}^4$ obtained by replacing the fibre $\mathbb{R}^4$ by $F = \mathrm{S}^3$. Now given any two topologically equivalent vector bundles ζ and ζ' we also have a smooth bundle equivalence by approximation when the natural smooth structures are assigned. Moreover, we may assume that it restricts to isometries on fibres. From this follows uniqueness of the smooth structure on the manifold M up to diffeomorphism.

By the same line of arguments the total space of the disc bundle $\eta\colon W \to \mathrm{S}^4$ obtained by replacing the fibre $\mathbb{R}^4$ by the Euclidean 4–disc $F = \mathrm{D}^4$ is a smooth fibre bundle. Its total space W is a smooth manifold with boundary M.

As $\pi_1(\mathrm{S}^4) = 0$, any of our vector bundles ζ is orientable. Both S^4 and $\mathbb{R}^4$ carry a canonical orientation. The choice of SO_4 as structure group thus implies that a vector bundle $\zeta : E(\zeta) \to \mathrm{S}^4$ comes along with a canonically preferred orientation. This induces an orientation on W and thus on the boundary M. The choice of SO_4 as structure group for the bundles ξ and η also ensures that the induced orientations of the fibres are preserved under equivalence. Similarly, the fact that we require bundle equivalences to induce the identity of the base space (and not a degree -1 homeomorphism) guarantees that the induced orientation of the base is preserved under equivalence. We will comment later on the question which bundles merely differ by these orientation issues.

Since $\pi_0(\mathrm{SO}_4)$ is trivial, the bundles are in 1–1 correspondence with elements of $\pi_3(\mathrm{SO}_4)$. It will be convenient to have an explicit description of this group.

Consider $\mathbb{R}^4$ as the quaternionic skew field $\mathbb{H}$. Then S^3 is the subgroup of unit quaternions and an identity element $\mathbf{1} \in \mathrm{S}^3$ is distinguished. We can define Lie group homomorphisms $\rho, \sigma\colon \mathrm{S}^3 \to \mathrm{SO}_4$ via quaternionic multiplication

$$\rho(u)\, v = u\, v\, u^{-1} \quad \text{and} \quad \sigma(u)\, v = u\, v\ .$$

These are clearly linear in v and indeed orthogonal since the norm preserves quaternionic products. They are rotations as there is a curve joining u with $\mathbf{1}$ in S^3.

Embed $\mathbb{R}^3 \overset{i}{\hookrightarrow} \mathbb{H}$ as the hyperplane with zero real part in the obvious way. Then we can define a Lie group homomorphism $\varrho\colon \mathrm{S}^3 \to \mathrm{SO}_3$ by restricting

$$\varrho(q) = i^* \rho(q)\ .$$

This homomorphism actually maps to SO_3, for $\varrho(q)$ is $\mathbb{R}\mathbf{1}$–invariant and $\mathbb{R}\mathbf{1}$ is orthogonal to $\mathbb{R}^3 \subset \mathbb{H}$. Thus it is also $\mathbb{R}^3$–invariant.

1.2 Lemma. *The triple* $\varrho : \mathrm{S}^3 \to \mathrm{SO}_3$ *carries the structure of a smooth (principal) bundle which is isomorphic to the canonical twofold cover* $\mathrm{S}^3 \to \mathbb{RP}^3$ *of real projective three–space.*

Proof. Pure imaginary quaternions only commute with pure real ones. Hence ϱ has kernel $\{\mathbf{1}, -\mathbf{1}\}$ and cosets $\{q, -q\}$. It thus suffices to show that ϱ is surjective. But it is a well–known fact that a rotation of $v \in \mathbb{R}^3$ by the angle φ around the axis spanned by $u \in \mathrm{S}^2$ can be represented as $\varrho(\cos\left(\frac{\varphi}{2}\right)\mathbf{1} + \sin\left(\frac{\varphi}{2}\right)u)\,v$. □

Lifting any $\varphi \in \mathrm{SO}_3$ by ϱ to $\pm q \in \mathrm{S}^3$ yields a well–defined inclusion $\tilde{\rho}\colon \mathrm{SO}_3 \subset \mathrm{SO}_4$ as subgroup by sending φ to $\rho(q)$. In particular, this defines a smooth action $\gamma\colon\ \mathrm{SO}_3 \curvearrowright \mathrm{S}^3$. Finally, the homomorphism σ is mono and provides an embedding $\mathrm{S}^3 \subset \mathrm{SO}_4$ as normal subgroup.

1.3 Theorem. *The four–dimensional rotation group* SO_4 *factorises into a semidirect product* $\mathrm{SO}_3\,{}_\gamma\!\ltimes \mathrm{S}^3$ *of topological groups.*

Proof. For any rotation $\phi \in \mathrm{SO}_4$ there are unit quaternions $q_L, q_R \in \mathrm{S}^3$ such that $\phi(v) = q_L\, v\, q_R$ for each $v \in \mathbb{H}$ and these are unique up to simultaneous change of sign. A proof of this by pure matrix computations has recently been given in [Me05]. We compute

$$\phi(v) = q_L\, v\, q_R = (\pm q_R^{-1})\,(q_R\, q_L\, v)\,(\pm q_R)\ ,$$

i. e. $\phi = \rho(\pm q_R^{-1})\,\sigma(q_R q_L)$ and thus $\tilde{\rho}(\mathrm{SO}_3)\,\sigma(\mathrm{S}^3) = \mathrm{SO}_4$. Since $\rho(q) = \sigma(\pm p)$ if and only if $q\,v q^{-1} = p\,v$ for any v, we may set $v = \mathbf{1}$ to see that $p = \mathbf{1}$ and in turn $q = \pm\mathbf{1}$. This implies $\tilde{\rho}(\mathrm{SO}_3) \cap \sigma(\mathrm{S}^3) = \{\,\mathrm{id}_{\mathbb{H}}\,\}$, completing the algebraic part of the theorem.

Now the isomorphism $(\pm p, q) \mapsto \rho(\pm p)\sigma(q)$ is a continuous bijection of compact Hausdorff spaces. Thus it is a homeomorphism. □

With the help of the following basic result from homotopy theory the structure of the classifying homotopy group will be evident. For simplicity now and in the remainder we do not want to distinguish between mappings from the sphere $\mathrm{S}^n \to X$ and their homotopy classes in $\pi_n(X)$ in our notation.

1.4 Lemma. *Let* $(G, e, \cdot)$ *be a topological group and let* φ, ψ *be any mappings* $\mathrm{S}^n \to G$. *Then the pointwise product* $\varphi \cdot \psi$ *is homotopic to the sum* $\varphi + \psi$ *in* $\pi_n(G)$.

Proof. Denote by $\varepsilon : \mathrm{S}^n \to G$ the constant map $x \mapsto e$. Then clearly $\varphi + \varepsilon \simeq \varphi$ and $\varepsilon + \psi \simeq \psi$. The sum $\varphi + \varepsilon$ maps to e in the second half of the cube $[0,1]^n$ while $\varepsilon + \psi$ maps to e in the first half of the cube. We thus have

$$\varphi \cdot \psi \simeq (\varphi + \varepsilon) \cdot (\varepsilon + \psi) = \varphi + \psi\ .$$

□

A nice consequence for the n–th quaternionic power is the following

1.5 Corollary. *The mapping* $\mathrm{S}^3 \to \mathrm{S}^3,\ u \mapsto u^n$ *has degree* n.

Proof. The common method of summing up local degrees of some finite preimage might be difficult in view of the fact that for instance $u^2 + \mathbf{1} = 0$ has all of $\mathrm{S}^2 = \mathbb{R}^3 \cap \mathrm{S}^3 \subseteq \mathbb{H}$ as solutions. But by the Hurewicz theorem we can compute the degree in homotopy where the assertion is immediate from lemma 1.4. □

1.6 Theorem. *The group $\pi_3(\mathrm{SO}_4)$ is free abelian with generators ρ and σ.*

Proof. We apply the π_3–functor to theorem 1.3 to obtain

$$\pi_3(\mathrm{SO}_4) \cong \pi_3(\mathrm{SO}_3) \oplus \pi_3(\mathrm{S}^3) \ .$$

By Lemma 1.2 ϱ induces an isomorphism $\pi_3(\mathrm{S}^3) \xrightarrow{\sim} \pi_3(\mathrm{SO}_3)$ in the exact homotopy sequence. Thus the generator $\mathrm{id}_{\mathrm{S}^3} \in \pi_3(\mathrm{S}^3)$ is mapped to the generator $\varrho \in \pi_3(\mathrm{SO}_3)$ and the isomorphism of 1.3 carries $m\,\varrho \oplus n\,\mathrm{id}_{\mathrm{S}^3}$ over to $\rho^m\sigma^n = m\rho + n\sigma \in \pi_3(\mathrm{SO}_4)$ by lemma 1.4. □

Now let the pair of integers (m, n) determine the classes

$$\zeta_{m,n} \colon E_{m,n} \to \mathrm{S}^4, \quad \eta_{m,n} \colon W_{m,n} \to \mathrm{S}^4 \quad \text{and} \quad \xi_{m,n} \colon M_{m,n} \to \mathrm{S}^4$$

of $\mathbb{R}^4$, D^4 and S^3–bundles corresponding to $m\rho + n\sigma \in \pi_3(\mathrm{SO}_4)$. As a first step towards the classification of the total spaces $M_{m,n}$, we will discuss how the integers m and n transform when the orientations of fibre and base are reversed.

If we enlarge the structure group SO_4 to O_4, bundle equivalences may reverse the induced orientation of the fibre. Recall that a bundle map consists of fibre automorphisms which can be described as the action of elements of the structure group. An equivalence reverses the orientation of the fibres if and only if these elements lie in the reflection component of O_4. In this case, the classifying homotopy classes in $\pi_3(\mathrm{O}_4) \cong \pi_3(\mathrm{SO}_4)$ are conjugate under the nontrivial element $\alpha \in \pi_0(\mathrm{O}_4) \cong \mathbb{Z}_2$, see theorem 1.1.

1.7 Lemma. *The action of $\alpha \in \pi_0(\mathrm{O}_4)$ on $\pi_3(\mathrm{SO}_4)$ is given by*

$$\alpha(m\rho + n\sigma) = (m+n)\rho - n\sigma \ .$$

Proof. Represent α by quaternionic conjugation $r \in \mathrm{O}_4$ which has determinant -1. Then α acts trivially on $\mathbb{R}^3 \subset \mathbb{H}$ and thus trivially on $\mathrm{SO}_3 \subset \mathrm{SO}_4$. So also on homotopy level we have $\alpha(\rho) = \rho$.

On the other hand, we compute for $p \in \mathrm{S}^3$ and $q \in \mathbb{H}$

$$(r\,\sigma(p)\,r^{-1})\,q = r(p\,r(q)) = qr(p) = q\,p^{-1} = p\,p^{-1}q\,p^{-1} = \rho(p)(p^{-1}q) = \rho(p)\sigma(p)^{-1}q \ ,$$

i. e. $\alpha(\sigma) = \rho\sigma^{-1}$. By lemma 1.4 we may conclude $\alpha(\sigma) = \rho - \sigma$ on homotopy level. □

We have thus proven that for any m, n the manifolds $M_{m,n}$ and $M_{m+n,-n}$ are (orientation reversing) diffeomorphic.

Allowing the fibre transformations to be reflections gave rise to equivalences reversing the orientation of the fibres. Likewise, allowing bundle equivalences to induce base mappings homotopic to reflections will yield equivalences reversing the orientation of the base.

1.8 Definition. Two bundles ξ and ξ' are called *weakly equivalent* if they have the same base space, fibre and group and if there is a bundle map $\xi \to \xi'$ inducing a homeomorphism of the common base space.

1.9 Lemma. *Let ξ, ξ' be two bundles over S^n with fibre F, group G and $\pi_0(G)$–conjugacy classes τ, τ' in $\pi_{n-1}(G)$. Then ξ and ξ' are weakly equivalent if and only if $\tau' = \pm\tau$.*

Sketch of proof. A weak equivalence induces a deg ± 1–mapping of S^n. If the degree is $+1$, the mapping is homotopic to the identity and the corresponding covering homotopy ends in a map $\xi \to \xi'$ inducing the identity on S^n, so in this case $\tau = \tau'$. If the degree is -1, the mapping is homotopic to a reflection which, restricted to the equatorial S^{n-1}, is a reflection of S^{n-1}. The corresponding covering homotopy ends in a map $\xi \to \xi'$ inducing a reflection of S^{n-1}. From this it follows that τ' can be represented by an element of τ with opposite sign.

Conversely, if $\tau = \tau'$ then ξ and ξ' are equivalent. If $\tau = -\tau'$, choose a reflection of S^n which restricted to S^{n-1} is a reflection as well. Then the pullback bundle of ξ with respect to this reflection is weakly equivalent to ξ and has class $-\tau$. □

Again set $n = 4$, $G = \mathrm{SO}_4$ and $F = \mathrm{S}^3$. We have discussed below theorem 1.1 that the total spaces $E(\xi)$ and $E(\xi')$ are diffeomorphic if $\tau = \tau'$. If $\tau = -\tau'$, we pull back ξ via a smooth reflection of S^4 as above. This gives us a bundle whose total space is clearly (orientation reversing) diffeomorphic to $E(\xi)$ and which has class $-\tau$.

We thus have that for any m, n the manifolds $M_{m,n}$ and $M_{-m,-n}$ are (orientation reversing) diffeomorphic. In particular, we may assume $n \geq 0$ in the remainder. Combining the orientation reversions of fibre and base shows that $M_{m,n}$ and $M_{-m-n,n}$ are (orientation preserving) diffeomorphic.

Before proceeding to our general classification pattern, we will take care of the case $n = 0$. We have a well–defined generator $\alpha \in H^4(W_{m,n})$ which restricts to the orientation class $\iota \in H^4(\mathrm{S}^4)$. Let $\beta \in H^4(M_{m,n})$ be given by the restriction $i^*\alpha$ where $i : M_{m,n} \to W_{m,n}$ is the inclusion of the boundary. Since $H^5(W_{m,n}, M_{m,n}) \cong H_3(W_{m,n}) \cong 0$ by Lefschetz duality, the exact cohomology sequence implies that i^* is surjective. So β generates $H^4(M_{m,n})$. Unless otherwise stated, integer coefficients are to be understood.

1.10 Lemma. *The cohomology group $H^4(M_{m,0})$ is infinite cyclic generated by β.*

Proof. The bundles $\xi_{m,0}$ have structure group reducible to SO_3 fixing a zero–sphere in each fibre. Thus any bundle $\xi_{m,0}$ possesses a cross–section and we may conclude $\pi_3(M_{m,0}) = \pi_3(\mathrm{S}^3) \oplus \pi_3(\mathrm{S}^4) \cong \pi_3(\mathrm{S}^3) \cong \mathbb{Z}$ and $\pi_1(M_{m,0}) = \pi_2(M_{m,0}) = 0$. By the Hurewicz theorem $H_3(M_{m,0}) \cong \mathbb{Z}$. By Poincaré duality $H^4(M_{m,0}) \cong \mathbb{Z}$, too. □

1.11 Lemma. *The first Pontrjagin class $p_1(W_{m,n}) \in H^4(W_{m,n})$ is given by*

$$p_1(W_{m,n}) = 2(n + 2m)\,\alpha\ .$$

Proof. The tangent bundle of $W_{m,n}$ splits as a Whitney sum of the subbundle tangent and the subbundle normal to the fibre. The first bundle is the pullback of the vector bundle $\zeta_{m,n}$ under the projection $W_{m,n} \to \mathrm{S}^4$. According to lemma 3 of [Mil56a], the bundle $\zeta_{m,n}$ has first Pontrjagin class $p_1(\zeta_{m,n}) = 2(n + 2m)\,\iota$. The second bundle is the pullback of the tangent bundle of S^4 which has first Pontrjagin class $p_1(\mathrm{S}^4) = 0$. □

For any smooth submanifold X of any smooth manifold Y let $\tau(X)$ and $\nu(X)$ denote the tangent bundle and the normal bundle of X in Y respectively.

1.12 Corollary. *The first Pontrjagin class $p_1(M_{m,0}) \in H^4(M_{m,0})$ is given by*

$$p_1(M_{m,0}) = 4m\,\beta\ .$$

Proof. The collar neighbourhood theorem states precisely that the normal bundle of a boundary is trivial. We thus have

$$p_1(M_{m,0}) = p_1(\tau(M_{m,0}) \oplus \nu(M_{m,0})) = p_1(i^*\tau(W_{m,0})) = i^*p_1(W_{m,0}) = 4m\,\beta\ .$$

□

1.13 Corollary. *The manifolds $M_{m,0}$ and $M_{m',0}$ are homeomorphic or diffeomorphic if and only if $m' = \pm m$.*

Proof. If $m' = \pm m$, then $M_{m,0}$ and $M_{m',0}$ are diffeomorphic by lemma 1.9. On the other hand, the rational Pontrjagin classes are a topological invariant of smooth manifolds as proved by Novikov. So if $m' \neq \pm m$, then $M_{m,0}$ and $M_{m',0}$ are not homeomorphic by corollary 1.12. □

This completes the discussion of the manifolds $M_{m,0}$. We will assume $n > 0$ in what follows.

1.2 Methods of the total space classification

In order to have a rough idea of the homotopy type of the manifolds $M_{m,n}$, we first of all compute their cohomology rings. This will primarily be a matter of the following lemma.

1.14 Lemma. *The vector bundle $\zeta_{m,n}$ has got Euler class $e(\zeta_{m,n}) = n\,\iota \in H^4(\mathrm{S}^4)$.*

Proof. The differentiable structure of the total space $E_{m,n}$ of the bundle $\zeta_{m,n}$ can be made precise by the following construction. Form the disjoint union

$$\mathbb{R}^4 \times \mathbb{R}^4 \coprod \mathbb{R}^4 \times \mathbb{R}^4$$

and write $(u,v)_1$ for elements in the left hand and $(u',v')_2$ for elements in the right hand component. Then identify points $(u,v)_1 \sim (u',v')_2$ in $\mathbb{R}^4 \setminus \{0\} \times \mathbb{R}^4$ if and only if

$$u' = \frac{u}{\|u\|^2} \quad \text{and} \quad v' = \rho^m(\tfrac{u}{\|u\|})\,\sigma^n(\tfrac{u}{\|u\|})\,v\ .$$

Now decompose the base S^4 into the northern hemisphere S^4_+ of points $x \in \mathrm{S}^4$ with fifth coordinate $x_5 \geq 0$ and the southern hemisphere S^4_- of points $x \in \mathrm{S}^4$ with fifth component $x_5 \leq 0$. Then points $x \in \mathrm{S}^4_+$, $x' \in \mathrm{S}^4_-$ can be mapped to points $u, u' \in \mathbb{R}^4$ in the plane via stereographic projection through the south and the north pole

$$u = \tfrac{1}{1+x_5}(x_1, x_2, x_3, x_4) \quad \text{and} \quad u' = \tfrac{1}{1-x_5'}(x_1', x_2', x_3', x_4') \ .$$

We define a section $s\colon \mathrm{S}^4 \longrightarrow E_{m,n}$ via

$$x \longmapsto \begin{cases} (u, \tfrac{1-x_5}{1+x_5}\mathbf{1})_1 & \text{if } x \in \mathrm{S}^4_+ \\ (u', \ u'^{\,n} \ )_2 & \text{if } x \in \mathrm{S}^4_- \end{cases}$$

where again $\mathbf{1} \in \mathbb{H}$ is the quaternion unit and the power $u'^{\,n}$ is given by quaternionic multiplication. Points in the equator $x \in \mathrm{S}^3 = \mathrm{S}^4_+ \cap \mathrm{S}^4_-$ are mapped to

$$s(x) = (u, \mathbf{1})_1 = ((x_1, x_2, x_3, x_4), \mathbf{1})_1 = (u', \rho^m(u)\sigma^n(u)\mathbf{1})_2 = (u', u'^{\,n})_2 = s(x)$$

so the section is well–defined. The factor $\frac{1}{\|u\|^2} = \frac{1-x_5}{1+x_5}$ comes from the transition map of the two charts of S^4 and ensures that the section is smooth. We thus can compute the Euler number of $\zeta_{m,n}$ by summing up the indices of the isolated zeros in the two poles. In a neighbourhood of the north pole the vector field clearly is smoothly isotopic to a nowhere zero one. So the index of this root is zero. In a neighbourhood of the south pole the vector field is just given by taking the n–th quaternionic power $u \mapsto u^n$. This induces a degree n–mapping of S^3 by corollary 1.5. So the index of this root, the overall index and the Euler number of the bundle is n. □

We will now see that the Euler class $e(\zeta_{m,n})$ entirely determines the cohomology ring of the associated manifold $M_{m,n}$.

1.15 Corollary. *The nontrivial cohomology groups of $M_{m,n}$ are*

$$H^0(M_{m,n}) \cong H^7(M_{m,n}) \cong \mathbb{Z} \quad \textit{and} \quad H^4(M_{m,n}) \cong \mathbb{Z}_n \ .$$

Proof. These are the only finitely generated abelian groups that fit into the Gysin sequence

$$\ldots \longrightarrow H^i(\mathrm{S}^4) \overset{\cup e(\zeta_{m,n})}{\longrightarrow} H^{i+4}(\mathrm{S}^4) \longrightarrow H^{i+4}(M_{m,n}) \longrightarrow H^{i+1}(\mathrm{S}^4) \overset{\cup e(\zeta_{m,n})}{\longrightarrow} \ldots$$

since by lemma 1.14 the cup product $\cup\, e(\zeta_{m,n})$ is multiplication by the integer n. □

As an immediate consequence we have that $M_{m,n}$ and $M_{m,n'}$ are not homeomorphic if $n \neq n'$. So the classification problem is reduced to deciding whether $M_{m,n}$ and $M_{m',n}$ are diffeomorphic, PL–homeomorphic or homeomorphic. The integer m is not reflected in the elementary algebraic topology of $M_{m,n}$.

We are thus faced with the problem of classifying a collection of $(s-1)$–connected $(2s+1)$–manifolds. There is a general theory due to C. T. C. Wall (see [Wa67]) for all s except $s = 3,\ 7$. Wall suggested the latter two cases as a problem for his student D. L. Wilkens who essentially solved it in [Wi72]. We will comment later on the ambiguity Wilkens left open. The methods of Wall and Wilkens are taken from the theory of surgery and handlebodies. The invariants they introduce realise a classification of highly connected manifolds up to the following concept.

1.16 Definition. Two n–manifolds M and N are called *almost diffeomorphic* if there is a homeomorphism $f\colon M \longrightarrow N$ and a homotopy sphere Σ^n such that

$$f\colon M \longrightarrow N\#\Sigma^n$$

is a diffeomorphism.

One might hope that there is a convenient category, endowed with "forgetful" functors from the smooth and into the topological category, which realises definition 1.16 as its isomorphism notion. In the dimension we are interested in, this turns out to be true.

1.17 Theorem. *Two closed 7–manifolds are almost diffeomorphic if and only if they are PL–homeomorphic.*

Proof. Any smooth manifold possesses a smooth triangulation or equivalently polyhedral structure which is unique up to combinatorial equivalence (see [Wh40]). The construction is functorial in that the diffeomorphism $f : M \to N\#\Sigma^7$ yields a PL–homeomorphism $f_{PL} : M \to N\#\Sigma^7$. It remains to show that $N = N\#\,\mathrm{S}^7$ and $N\#\Sigma^7$ are PL–homeomorphic. For this purpose we recall a classical lemma from differential topology ([Mil56b], lemma A.3). It says that the manifold obtained by gluing together two given manifolds M_1, M_2 by means of a diffeomorphism of their boundaries carries a natural smooth structure. This structure is compatible with the canonical embeddings of M_1 and M_2 and is unique up to diffeomorphism. The proof carries over to the analogous result in the PL–category (see [Mu61], theorem 10.4). Now choose a smooth orientation preserving embedding $i\colon \mathrm{D}^7 \to N$, set $N_0 = N \setminus i(\mathring{\mathrm{D}}^7)$ and let $j\colon \mathrm{S}^6 \to \mathrm{D}^7$ be an orientation reversing embedding into the boundary. Let $\varphi \in \mathrm{Diff}^+(\mathrm{S}^6)$ be an orientation preserving diffeomorphism that describes $\Sigma^7 \in \Gamma_7$ as a twisted sphere (see [Mil06]). Then the manifolds $N\#\,\mathrm{S}^7$ and $N\#\Sigma^7$ are given by the C^∞–category pushouts

$$\begin{tikzcd} & \mathrm{S}^6 \arrow[dl, "i\circ j"'] \arrow[dr, "j"] & \\ N_0 \arrow[dr] & & \mathrm{D}^7 \arrow[dl] \\ & N\#\,\mathrm{S}^7 & \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{S}^6 \arrow[dl, "i\circ j"'] \arrow[dr, "j\circ\varphi"] & \\ N_0 \arrow[dr] & & \mathrm{D}^7 \arrow[dl] \\ & N\#\Sigma^7 & \end{tikzcd} .$$

We know that $\pi_0(\mathrm{PL}^+(\mathrm{S}^6)) \cong 0$. For a PL–homeomorphism not PL–isotopic to the identity gave rise to a non-standard PL–sphere by the PL twisted sphere construction. But the Hauptvermutung is correct for S^n. So φ is PL–isotopic to the identity (though not necessarily smoothly isotopic) and we have a PL–automorphism ϕ of $\mathrm{S}^6 \times I$ which restricts to the identity on $\mathrm{S}^6 \times\{0\}$ and to φ on $\mathrm{S}^6 \times\{1\}$. Pasting a PL-manifold combinatorially equivalent to the 7-simplex onto the boundary component $\mathrm{S}^6 \times\{0\}$ and defining ϕ to map identically on the added cell we obtain a PL–automorphism ϕ extending φ to all of D^7. We thus have a diagram in the PL–category

$$\begin{tikzcd}
 & \mathrm{S}^6 \arrow[dl, "i\circ j"'] \arrow[d, "j\circ"] \arrow[dr, "j\circ\varphi"] & \\
N_0 \arrow[dr] & \mathrm{D}^7 \arrow[r, "\phi", "\sim"'] \arrow[d] & \mathrm{D}^7 \arrow[dl] \\
 & N\#\mathrm{S}^7 \cong N\#\Sigma^7 &
\end{tikzcd}$$

where the left hand triangle and the outer square clearly give isomorphic pushouts.

For the converse general smoothing theory (compare [HiMa74], part II, theorem 5.3) implies that a smooth structure on a PL-manifold M corresponds to a section in a certain bundle $\mathcal{E}(M)$. This is induced from the universal bundle BO $\to$ BPL with fibre PL/O by the map $M \to$ BPL classifying its PL–structure. The obstruction to extending such a section over the k–skeleton lies in the group

$$H^k(M;\pi_{k-1}(\mathrm{PL/O}))\ .$$

One can show that the bundle of coefficients is trivial. Moreover, it is a story on its own to show that PL/O is 6–connected. The references can be found in [Ru01], IV.4.27(iv). It follows that any closed 7–PL–manifold is smoothable. The differentiable structures are in 1–1 correspondence with elements of $\pi_7(\mathrm{PL/O})$. Now Hirsch and Mazur have proven also in [HiMa74] that $\pi_n(\mathrm{PL/O})$ is isomorphic to the group of smooth structures on a PL–sphere S^n. By [Th58b], Théorème 9, a smooth manifold PL–isomorphic to the sphere can be constructed as a twisted sphere. We thus have that $\pi_7(\mathrm{PL/O}) \cong \Gamma_7$. As a consequence of the h–cobordism theorem (see [Mil65a]) we know that Γ_7 is isomorphic to the group Θ_7 of homotopy 7–spheres. Milnor has computed this group as $\Theta_7 \cong \mathbb{Z}_{28}$ (see [KerMil63]). The μ–invariant we will define in definition 1.35 is additive with respect to connected sum (lemma 1.36) and distinguishes all 28 homotopy 7–spheres (theorem 1.40). This clearly proves the theorem. $\square$

We will now introduce the invariants the classification will rely on. These will essentially be the torsion linking form and the obstruction to stable parallelisability.

For any topological space X let $\beta : H^*(X;\mathbb{Q}/\mathbb{Z}) \to H^{*+1}(X;\mathbb{Z})$ be the Bockstein homomorphism associated with the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{j} \mathbb{Q}/\mathbb{Z} \longrightarrow 0\ .$$

For any finitely generated abelian group G denote by $\mathrm{T}G$ its torsion subgroup.

1.18 Definition. Let X be a closed oriented n–manifold. Then for $k = 0, \dots, n-1$ the bilinear pairing

$$\begin{array}{ccc}
\mathrm{T}H^{k+1}(X) \otimes \mathrm{T}H^{n-k}(X) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\
x \otimes y & \longmapsto & \langle x', i_*(y \cap [X])\rangle
\end{array}$$

with $\beta(x') = x$ will be called the k^{th} *linking pairing.* If n is odd, the sole term linking pairing will refer to the $\left(\frac{n+1}{2}\right)^{\text{th}}$ pairing.

Since x is a torsion element, it is in the image of β. If $z \in \ker\beta = \operatorname{im} j^*$, then $z = j^*z'$ and $\langle j^*z', i_*y \cap [X]\rangle = \langle z', j_*i_*(y \cap [X])\rangle = 0$, so the pairing is well–defined.

Let X be a 2–connected closed 7–manifold. Bott has computed $\pi_2(\mathrm{SO}) \cong 0$ and $\pi_3(\mathrm{SO}) \cong \mathbb{Z}$ where SO denotes the direct limit of the natural inclusions $\mathrm{SO}_n \hookrightarrow \mathrm{SO}_{n+1}$. It follows that the stable tangent bundle $\tau^s(X)$ restricted to the 3–skeleton is trivial whereas the obstruction to stable triviality over the 4–skeleton is a well–defined element $\beta = \beta(X)$ of the group

$$H^4(X; \pi_3(\mathrm{SO})) \cong H^4(X) \ .$$

As $\pi_i(\mathrm{SO}) \cong 0$ for $i = 4,\ 5,\ 6$, there is no other obstruction to stable parallelisability.

1.19 Lemma (Kervaire, [Ker59] lemma 1.1)**.** *The first Pontrjagin class $p_1(X)$ is twice the obstruction class β.*

We will see that this multiplication by two occurs universally. The cohomology of the classifying spaces of the stable classical groups is well–known. In particular, both $H^4(\mathrm{BSO})$ and $H^4(\mathrm{BSpin})$ are infinite cyclic.

1.20 Lemma. *The canonical map* $\mathrm{BSpin} \xrightarrow{\pi} \mathrm{BSO}$ *induces a homomorphism*

$$\pi^* \colon H^4(\mathrm{BSO}) \longrightarrow H^4(\mathrm{BSpin})$$

which is multiplication by ± 2.

Proof. The proof given here can be found in the original source [Ths62]. Since there it is spread over the whole paper, we extract it here for the convenience of the reader. As usual denote by $p_1 \in H^4(\mathrm{BSO})$ the canonical generator. For any given space let ρ_n be the natural cohomology homomorphism induced by the projection $\mathbb{Z} \to \mathbb{Z}_n$. By problem 15-A of [MS74] we have $\rho_2(p_1) = w_2^2 \in H^4(\mathrm{BSO}; \mathbb{Z}_2)$ with w_2 denoting the universal stable second Stiefel–Whitney class. From the long exact sequence of homotopy groups for the classifying space fibration we have $\pi_n(\mathrm{BSpin}) \cong \pi_{n-1}(\mathrm{Spin})$. So BSpin is 3–connected because Spin is 2–connected. In particular, $H^2(\mathrm{BSpin}; \mathbb{Z}_2) = 0$. Thus

$$\rho_2(\pi^* p_1) = (\pi^* w_2)^2 = 0 \ .$$

It follows that there is a unique $q_1 \in H^4(\mathrm{BSpin})$ such that $\pi^* p_1 = 2q_1$. We are done when we have shown that $\rho_2(q_1) = \pi^* w_4 \in H^4(\mathrm{BSpin}; \mathbb{Z}_2)$. For then q_1 is odd and since all groups in the Serre cohomology spectral sequence of the fibration

$$K(\mathbb{Z}_2, 1) \longrightarrow \mathrm{BSpin} \longrightarrow \mathrm{BSO}$$

are either zero or sums of $\mathbb{Z}$s and $\mathbb{Z}_2$s, the element q_1 must be a generator.

Now Wu has shown that the Pontrjagin square

$$\mathfrak{P}_2 \colon H^{2j}(\mathrm{BSO}; \mathbb{Z}_2) \longrightarrow H^{4j}(\mathrm{BSO}; \mathbb{Z}_4)$$

(see [Ths57]) evaluated on even Stiefel–Whitney classes takes the form

$$\mathfrak{P}_2(w_{2j}) = \rho_4(p_j) + \theta\left(w_{4j} + \textstyle\sum_{0<i<j} w_{2i}w_{4j-2i}\right)$$

where θ is induced from the nontrivial coefficient homomorphism $\mathbb{Z}_2 \to \mathbb{Z}_4$. The original proof is given in [Wu54] (in Chinese), a new proof can be found in [Ths60], p. 82 (in English). Setting $j = 1$ and using naturality of Pontrjagin squares and of change of coefficient homomorphisms we thus have

$$0 = \mathfrak{P}_2(0) = \mathfrak{P}_2(\pi^* w_2) = \rho_4(\pi^* p_1) + \theta(\pi^* w_4)\ .$$

Because evidently $2\rho_4 = \theta\rho_2$, we may conclude

$$\theta\left(\rho_2(q_1) - \pi^* w_4\right) = 0\ .$$

From this $\rho_2(q_1) = \pi^* w_4$ follows since the group preceding the domain of θ in the associated Bockstein sequence is $H^3(\mathrm{BSpin}; \mathbb{Z}_2) \cong 0$. □

Hence we can define a universal stable *spin characteristic class* as the generator of $H^4(\mathrm{BSpin})$ that has got the same sign under the monomorphism of lemma 1.20 as the first universal stable Pontrjagin class $p_1 \in H^4(\mathrm{BSO})$. We give this stable spin characteristic class the suggestive name $\frac{p_1}{2}$.

Since X is 2–connected, its second Stiefel-Whitney class vanishes trivially and thus X is spin. This enables us to identify the obstruction class $\beta(X)$ from a more abstract point of view with the spin characteristic class $\frac{p_1}{2}(X)$. The coincidence is ensured by lemma 1.19 as long as $H^4(X)$ contains no two–torsion elements. It is stated (without proof) in [CrEs03] and [Cr02] that the definitions are actually equivalent in general.

The following result will be crucial for the homeomorphism classification.

1.21 Theorem. *The spin characteristic class is a topological invariant of spin manifolds.*

Proof. We denote by BTopSpin (BPLSpin) the classifying space for topological spin microbundles (PL vector bundles). For a smooth spin manifold X its topological spin structure factorises as

$$\begin{tikzcd} & \mathrm{BSpin} \arrow[d] \\ X \arrow[ur] \arrow[r] & \mathrm{BTopSpin}\ . \end{tikzcd}$$

So the theorem is clearly restated by claiming that in the corresponding diagram in fourth cohomology

$$\begin{tikzcd} & H^4(\mathrm{BSpin}) \arrow[dl] \\ H^4(X) & H^4(\mathrm{BTopSpin}) \arrow[l] \arrow[u, "\varphi"'] \end{tikzcd}$$

the homomorphism φ is an isomorphism. To prove this consider the commutative prism

$$\begin{tikzcd}
 & \mathrm{BSpin} \arrow[dl] \arrow[dr] \arrow[d] & \\
\mathrm{BPLSpin} \arrow[rr] \arrow[dd] & \mathrm{BO} \arrow[dl] \arrow[dr] & \mathrm{BTopSpin} \arrow[dd] \\
 & & \\
\mathrm{BPL} \arrow[rr] & & \mathrm{BTop}\ .
\end{tikzcd}$$

This is constructed by choosing CW approximations for the lower triangle so that the upper triangle is induced from it by the 4–connected covering construction which is functorial. For the definition of n–connected coverings we adopt the convention of Husemoller, [Hu08], chapter 12, section 6, p. 142, though it implies confusingly that these 4–connected coverings are only 3–connected. The so obtained spaces occur on the fourth level of the associated Whitehead towers or inverse Postnikov towers. Note that uniqueness of Whitehead towers implies uniqueness of the above prism up to weak homotopy equivalence.

Applying the π_4–functor to the prism, we see directly by definition of the 4–connected coverings that all vertical arrows become isomorphisms. Moreover, we have $\pi_4(\mathrm{BO}) \cong \pi_4(\mathrm{BPL})$ because $\mathrm{BPL} \to \mathrm{BO}$ is a fibration with 7–connected fibre BPL/O (compare the proof of theorem 1.17). The map $\mathrm{BPL} \to \mathrm{BTop}$ lies in the Kirby–Siebenmann fibration

$$K(\mathbb{Z}_2, 3) \longrightarrow \mathrm{BPL} \longrightarrow \mathrm{BTop}$$

which reflects that the only obstruction to finding a PL–structure on a given topological manifold lies in its fourth cohomology group. This yields the short exact sequence

$$0 \longrightarrow \pi_4(\mathrm{BPL}) \longrightarrow \pi_4(\mathrm{BTop}) \longrightarrow \mathbb{Z}_2 \longrightarrow 0\ .$$

Substituting the above isomorphisms, this becomes

$$0 \longrightarrow \pi_4(\mathrm{BSpin}) \longrightarrow \pi_4(\mathrm{BTopSpin}) \longrightarrow \mathbb{Z}_2 \longrightarrow 0\ .$$

Since we are now in the upper triangle, the Hurewicz theorem is available and we have

$$0 \longrightarrow H_4(\mathrm{BSpin}) \longrightarrow H_4(\mathrm{BTopSpin}) \longrightarrow \mathbb{Z}_2 \longrightarrow 0\ .$$

According to [KrSt91], lemma 6.5, this sequence splits. Therefore we can apply the $\mathrm{Hom}(-,\mathbb{Z})$–functor and the universal coefficient theorem to obtain an isomorphism

$$H^4(\mathrm{BTopSpin}) \longrightarrow H^4(\mathrm{BSpin})\ .$$

Naturality of the universal coefficient theorem ensures this isomorphism is just φ. □

Combining 1.20 and 1.21, we have

1.22 Corollary. *The first Pontrjagin class is a topological invariant of spin manifolds.*

1.23 Prospects. A proof that 1.22 is correct without the assumption of an existing spin structure has been announced by Kreck in 2004, see [KrLue05], p. 31. The author was unable to locate such a paper.

It is well–known that the first Stiefel–Whitney class w_1 is an obstruction to orientability while the second Stiefel–Whitney class w_2 is an obstruction to the existence of a spin structure. Similarly, the invariant $\frac{p_1}{2}$ is an obstruction to lifting the structure group $\mathrm{Spin}(n)$ to the non–compact topological group $\mathrm{String}(n)$ which is only defined for $n > 4$. As a space this group is given by the 7–connected cover of $\mathrm{Spin}(n)$.

Even though Grove and Ziller (see [GZ00]) showed that all manifolds $M_{m,n}$ admit metrics with non–negative sectional curvature it is not yet known whether any exotic 7–sphere $M_{m,1}$ admits a metric with positive sectional curvature. Since $M_{m,n}$ is spin, its $\widehat{A}$–genus can be computed by means of the Atiyah-Singer index theorem. For this purpose we have to choose a Riemannian metric and compute the index of the Dirac operator on its spinor bundle. The $\widehat{A}$-genus is known to be an obstruction to the existence of a metric with positive scalar curvature.

Now Husemoller (see [Hu08], p. 145) conjectures that similarly, the existence of a string structure gives rise to spinors and Dirac operators on the free loop space of the manifold. The index of the Dirac operator would be the Witten index which then might be an obstruction to the existence of a metric with positive Ricci curvature.

Total spaces of S^7–bundles over S^8, for which similar arguments as those presented here are possible, might provide interesting examples for the theory. For these bundles a complete classification has not yet been carried out.

1.24 Definition. A *Wilkens triple* $W = (G, b, g)$ consists of a finitely generated abelian group G together with a non-singular symmetric bilinear map $b\colon \mathrm{T}G \otimes \mathrm{T}G \longrightarrow \mathbb{Q}/\mathbb{Z}$ and an even element $g \in G$.

Given two Wilkens triples $W_1 = (G_1, b_1, g_1)$ and $W_2 = (G_2, b_2, g_2)$ we have a natural sum $W_1 \oplus W_2 = (G_1 \bigoplus G_2,\ b_1 \oplus b_2,\ g_1 \oplus g_2)$ where the direct sum of bilinear maps is given by

$$b_1 \oplus b_2\ ((u_1 \oplus u_2) \otimes (v_1 \oplus v_2)) = b_1(u_1 \otimes v_1) + b_2(u_2 \otimes v_2)\ .$$

This sum operation is associative and an identity element is provided by the trivial triple $(0, 0, 0)$. The triples W_1 and W_2 are called isomorphic if there is a group isomorphism $\varphi\colon G_1 \to G_2$ such that $b_2 = b_1 \circ (\varphi \otimes \varphi)$ and such that $\varphi(g_1) = g_2$. Clearly the sum operation for triples is well–defined and commutative on the set of isomorphism classes. In summary, the isomorphism classes $\mathcal{W}$ of Wilkens triples form a commutative monoid.

Similarly, the connected sum operation # establishes the structure of a commutative monoid on the set $\mathcal{P}$ of orientation–preserving almost diffeomorphism classes of closed 2–connected 7–manifolds P where the identity element is represented by the standard sphere S^7.

1.25 Definition. The *Wilkens map* $\Phi : \mathcal{P} \longrightarrow \mathcal{W}$ is defined on representatives and assigns to a manifold P the triple

$$(H^4(P),\ \mathrm{lk}(P),\ \tfrac{p_1}{2}(P))$$

built from its fourth cohomology group, its linking pairing 1.18 and its spin characteristic class $\frac{p_1}{2}$.

For the definition of Φ to be meaningful, it remains to verify

1.26 Lemma. *If P is a manifold of class $\mathcal{P}$, its spin characteristic class $\frac{p_1}{2}(P)$ is even.*

Proof. The proof given here can be found in [GKS04], corollary 3.4, p. 413, without references. We show equivalently that the mod 2–reduction of $\frac{p_1}{2}(P)$ in $H^4(P;\mathbb{Z}_2)$ is zero. This mod 2–reduction is given by the fourth Stiefel–Whitney class $w_4(P)$ as has been shown universally by Thomas, [Ths62] 1.6, p. 58. Now recall that Wu classes $v_i \in H^i(P)$ are well–defined by the formula $v_i \cup x = \mathrm{Sq}^i(x) \in H^7(P)$ for each $x \in H^{7-i}(P)$. Here Sq^i denotes the i–th Steenrod square. The total Wu class $v = 1 + v_1 + v_2 + \dots$ is related to the total Stiefel–Whitney class $w = 1 + w_1 + w_2 + \dots$ by the formula $v \cup w = 1$. In particular, all Stiefel–Whitney classes vanish if and only if all Wu classes do. For $i > 3$ we gather $v_i = 0$ directly from the defining formula as the Steenrod squares are zero in this range. Also $v_3 = 0$ using the Adém relation $\mathrm{Sq}^3 = \mathrm{Sq}^1\,\mathrm{Sq}^2$. Finally $v_1 = v_2 = 0$ because P is 2–connected. □

1.27 Theorem. *The Wilkens map $\Phi\colon \mathcal{P} \longrightarrow \mathcal{W}$ is an epimorphism of monoids.*

Proof. The proof that Φ is a homomorphism requires essentially to show that the linking pairing splits under connected sums. This is similar to the proof of lemma 1.36 below where we will show that the intersection pairing of a manifold splits under connected sum. Clearly $\Phi(\mathrm{S}^n) = 0$. The surjectivity statement is a proposition in section 2 of [Wi72]. □

By setting $-W$ equal to $(G, -b, g)$ and $-P$ equal to P with reversed orientation we have the formula $\Phi(-W) = -\Phi(W)$. For if the orientation is reversed, the fundamental class and thus the linking form changes sign while the Pontrjagin class remains unchanged. Since no nontrivial element of $\mathcal{W}$ has an inverse, the notation $-W$ should not lead to confusion.

1.28 Definition. A non–singular symmetric bilinear map $\mathrm{T}G \otimes \mathrm{T}G \to \mathbb{Q}/\mathbb{Z}$ is called *irreducible* if it is not the proper sum of two maps. A Wilkens triple $W = (G, b, g)$ is called *indecomposable* if $G \cong \mathbb{Z}$ or G is finite and b is irreducible. A class $P \in \mathcal{P}$ is called indecomposable if $\Phi(P)$ is indecomposable. Denote by $\mathcal{W}^{\mathrm{ind}}$ and $\mathcal{P}^{\mathrm{ind}}$ the set of classes of indecomposable triples and manifolds respectively.

1.29 Theorem (Wilkens, theorem 1 in [Wi72])**.** *The monoid $\mathcal{P}$ is generated by $\mathcal{P}^{ind}$.*

Thus the classification problem is reduced to indecomposable manifolds.

1.30 Theorem (Wall, [Wa63])**.** *If $b\colon G \otimes G \longrightarrow \mathbb{Q}/\mathbb{Z}$ is a finite irreducible form, then $G \cong \mathbb{Z}_{p^k}$ for p prime or $G \cong \mathbb{Z}_{2^k} \oplus \mathbb{Z}_{2^k}$.*

Recall that we have computed $H^4(M_{m,n}) \cong \mathbb{Z}_n \cong \bigoplus_{i=1}^k \mathbb{Z}_{p_i^{\epsilon_i}}$ with the prime decomposition $n = \prod_{i=1}^k p_i^{\epsilon_i}$. As there are no nontrivial homomorphisms of cyclic groups of coprime order, any automorphism of $H^4(M_{m,n})$ must preserve the above decomposition. So we are in the convenient situation that $\Phi(M_{m,n}) = \Phi(M_{m',n})$ if and only if the pairs of corresponding indecomposable Wilkens triples are isomorphic. Thus, if the indecomposable triples classify, so does $\Phi(M_{m,n})$.

We write $\mathcal{P}^{\mathrm{ind}}$ as the disjoint union $\mathcal{P}_1^{\mathrm{ind}} \cup \mathcal{P}_2^{\mathrm{ind}}$ of classes of manifolds with fourth cohomology group isomorphic to

1. $\mathbb{Z}$ or $\mathbb{Z}_{p^k}$, for p an odd prime and
2. $\mathbb{Z}_{2^k}$ or $\mathbb{Z}_{2^k} \oplus \mathbb{Z}_{2^k}$.

Finally, we can state the first part of the Wilkens classification.

1.31 Theorem (Wilkens, theorem 2 (i) in [Wi72])**.** *The restriction of the Wilkens map Φ to the set $\mathcal{P}_1^{ind}$ is a bijection.*

As has already been pointed out, the methods of proving the theorem are from surgery theory. A classical reference is [Wa99]. The manifolds $\mathcal{P}_2^{\mathrm{ind}}$ are not classified by the Wilkens triple. It is a little awkward to state what can be said about these. We need the further refinement $\mathcal{P}_2^{\mathrm{ind}} = \mathcal{P}_{2a}^{\mathrm{ind}} \cup \mathcal{P}_{2b}^{\mathrm{ind}} \cup \mathcal{P}_{2c}^{\mathrm{ind}} \cup \mathcal{P}_{2d}^{\mathrm{ind}}$ to classes of manifolds with fourth cohomology group of order

a) at most four and $\frac{p_1}{2}$ divisible by four,

b) at most four and $\frac{p_1}{2}$ not divisible by four,

c) greater than four and $\frac{p_1}{2}$ divisible by four and

d) greater than four and $\frac{p_1}{2}$ not divisible by four.

1.32 Theorem (Wilkens, theorem 2 (ii) in [Wi72])**.** *The restriction of the Wilkens map Φ to the set $\mathcal{P}_{2b}^{ind} \cup \mathcal{P}_{2c}^{ind}$ is a bijection. The restriction to $\mathcal{P}_{2a}^{ind} \cup \mathcal{P}_{2d}^{ind}$ is two–to–one.*

1.33 Remark. There are classification results analogous to those stated from 1.29 to 1.32 for 6–connected 15–manifolds (see [Wi72]).

In general surgery theory, ambiguities as those in cases a) and d) call for a quadratic refinement of the linking form. But in dimensions 7 and 15 Wall's definition of quadratic refinement of the linking form does not work. This in turn is a consequence of the existence of bundles over S^4 and S^8 with Hopf invariant 1. However, a method to get rid of these $\mathbb{Z}_2$–ambiguities has recently been given by Crowley as the main result of his Ph.D.–thesis [Cr02]. In its complete generality the technique is rather lengthy. But for the sphere bundles we are interested in, it reduces to the computation of one more invariant which we will present later.

Now as the tools for the almost diffeomorphism classification are developed, the question is how to descend to the pure homeomorphism classification and how to refine our invariants to obtain the diffeomorphism case. We have seen in the proof of theorem 1.17 that each class $P \in \mathcal{P}$ contains exactly 28 different diffeomorphism types which can be pairwisely interchanged by connected sum with an appropriate homotopy sphere $\Sigma^7 \in \Theta_7$. Thus all that remains to be done for the diffeomorphism classification is to find an invariant which firstly is additive with respect to connected sum and secondly distinguishes all 28 homotopy 7–spheres. Such an invariant is given by the μ–invariant of Eels and Kuiper (see [EeKui62]). We now recall its definition and verify the two needed properties.

1.34 Definition. A closed oriented $(4k-1)$–manifold M satisfies the *μ–condition* if it bounds a compact oriented spin manifold W such that the natural homomorphisms

$$\begin{aligned} j^*\colon H^{2k}(W,M;\mathbb{Q}) &\longrightarrow H^{2k}(W;\mathbb{Q})\ , \\ j^*\colon H^{4i}(W,M;\mathbb{Q}) &\longrightarrow H^{4i}(W;\mathbb{Q})\ , \quad (0<i<k) \end{aligned}$$

are isomorphisms and the restriction

$$i^*\colon H^1(W;\mathbb{Z}_2) \longrightarrow H^1(M;\mathbb{Z}_2)$$

is an epimorphism.

Observe that our pairs $(W_{m,n}, M_{m,n})$ satisfy the μ–condition. For simplicity we only give the precise definition of the μ–invariant in the dimension we need it.

1.35 Definition. Let M be a 7–manifold satisfying the μ–condition. Then its μ–invariant is given by

$$\mu(M) = \tfrac{1}{896}\Big(\langle {j^*}^{-1}{p_1}^2(W), [W,M]\rangle - 4\,\sigma(W)\Big) \mod 1\ ,$$

where $p_1(W) \in H^4(W;\mathbb{Q})$ is the first rational Pontrjagin class and where $\sigma(W)$ denotes the signature of W, i. e. the signature of its intersection form.

In the general case the invariant μ is also defined to be of the form

$$r_1\,(N(W) - r_2\,\sigma(W))$$

where $r_1, r_2 \in \mathbb{Q}$ and $N(W)$ is a certain rational linear combination of the pulled back rational Pontrjagin numbers of W. The definition is independent of the choice of W ([EeKui62], p. 97). Thus μ is a diffeomorphism invariant of M.

1.36 Lemma. *Let M_1 and M_2 be $(4k-1)$–manifolds satisfying the μ–condition. Then*

$$\mu(M_1 \# M_2) = \mu(M_1) + \mu(M_2) \mod 1\ .$$

The proof will be geometrical in flavour. We will at first explain how to connect two manifolds along the boundary and how to construct inclusions of the manifolds the sum is made from.

In the following, H^n denotes the closed half disc in $\mathbb{R}^n_{\geq 0}$ which has D^{n-1} as its subset of points with $x_n = 0$. Let W_1 and W_2 be compact oriented n–manifolds with connected boundaries M_1 and M_2. We will define a manifold W with boundary $M = M_1 \# M_2$ as follows and write $(W,M) = (W_1,M_1)\#(W_2,M_2)$.

1.37 Definition. Choose two smooth embeddings $f_1 : (\mathrm{H}^n, \mathrm{D}^{n-1}) \to (W_1, M_1)$ and $f_2 : (\mathrm{H}^n, \mathrm{D}^{n-1}) \to (W_2, M_2)$ such that $f_2 \circ f_1^{-1}$ is orientation reversing. Then the *connected sum along the boundary* W is given by

$$(W_1 \setminus f_1(0)) \coprod (W_2 \setminus f_2(0))$$

where $f_1(tu)$ is to be identified with $f_2((1-t)u)$ for each $0 < t < 1$, $u \in \mathrm{S}^{n-1} \cap \mathrm{H}^n$.

Similarly to the gluing boundaries construction as mentioned in the proof of theorem 1.17, W possesses a natural differentiable structure. It has got the homotopy type of the one-point union $W_1 \vee W_2$. Note that $f_1(\frac{1}{2}\,\mathrm{S}^{n-1} \cap \mathrm{H}^n) = f_2(\frac{1}{2}\,\mathrm{S}^{n-1} \cap \mathrm{H}^n)$ as a subset A of W, cf. figure 1.1. This leads us to the following construction.

Consider the homotopy

$$g_i \colon W_i \times I \longrightarrow W_i$$

that is given by the identity on the complement of $f_i(\frac{1}{2}\,\mathrm{H}^n)$ for any t and by

$$(f_i(x), t) \mapsto f_i\left(x + t\left(\left(x_1, \ldots, x_{n-1}, \sqrt{\tfrac{1}{4} - x_1^2 \ldots - x_{n-1}^2}\right) - x\right)\right)$$

for each $x \in \frac{1}{2}\,\mathrm{H}^n$. After "straightening the angle" at the subset $f_i(\partial(\frac{1}{2}\,\mathrm{D}^{n-1}))$, compare the appendix of [Mil59], the manifold $g_i(W_i, 1)$ is clearly diffeomorphic to W_i. Thus we obtain smooth inclusions $W_i \subseteq W$ by regarding $f_i(g_i(W_i, 1))$ as the subset W_i of W.

Proof (of lemma 1.36). Choose coboundaries W_1 and W_2 of M_1 and M_2 as in 1.34 and form the connected sum (W, M) along the boundary as in 1.37. We call two triples (or pairs, or spaces) *equivalent* if they induce the same singular homology and cohomology groups and write "$\cong$" for this equivalence relation. Consider the diagram

$$\begin{tikzcd}
 & 0 & & \\
 & H^*(W_1, M_1) \oplus H^*(W_2, M_2) \arrow[u] & & \\
H^{*-1}(M \cup A, M) \arrow[r] & H^*(W, M \cup A) \arrow[u, "h"', "\sim"] \arrow[r, "j"] & H^*(W, M) \arrow[r] & H^*(M \cup A, M) \\
 & 0 \arrow[u] & &
\end{tikzcd}$$

.

Here and in the remainder of the proof rational coefficients are to be understood. The horizontal line is the exact cohomology sequence of the triple $(W, M \cup A, M)$. This triple can be seen to be equivalent to the excisive triple

$$(W, (M \times [0,1)) \cup B, M \times [0,1))$$

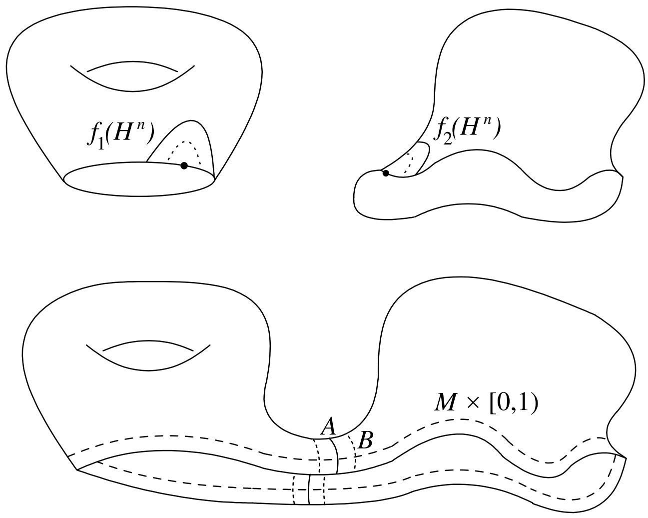


Figure 1.1: The connected sum along the boundary $(W_1, M_1)\#(W_2, M_2)$.

where $M \times [0, 1)$ is a collar neighbourhood of M in W while $B = A \times (-1, 1)$ is a bicollar neighbourhood of A in W.

The vertical line is the Mayer-Vietoris sequence of the decomposition $(W, M \cup A) = (W_1 \cup W_2, M_1 \cup M_2)$ which is equivalent to the excisive pair

$$(W, (M \times [0, 1)) \cup B) = ((W_1 \cup B) \cup (W_2 \cup B), ((M_1 \times [0, 1)) \cup B) \cup ((M_2 \times [0, 1)) \cup B)$$

and hence is excisive, too. Note that

$$(M \cup A, M) \cong (A, f_i(\partial(\tfrac{1}{2}\,\mathrm{D}^{n-1}))) \cong (\mathrm{D}^{n-1}, \partial\,\mathrm{D}^{n-1})$$

thus j is an isomorphism in orders less than $n-1$. This is the corresponding diagram in n–th homology

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
 & H_n(W_1, M_1) \oplus H_n(W_2, M_2) \arrow[d, "h", "\sim"'] & & \\
\mathbb{Q} & H_n(W, M \cup A) \arrow[l] \arrow[d] & H_n(W, M) \arrow[l, "j"'] & 0 \arrow[l] \\
 & 0 & &
\end{tikzcd}$$

.

The embeddings in 1.37 have been so chosen that

$$h^{-1}j([W, M]) = [W_1, M_1] \oplus [W_2, M_2]\ .$$

Now if $\alpha = jh^{-1}(\alpha_1 \oplus \alpha_2) \in H^{2k}(W, M)$ we compute

$$\begin{aligned}\langle \alpha^2 \mid [W,M]\rangle &= \langle jh^{-1}(\alpha_1^2 \oplus \alpha_2^2) \mid [W,M]\rangle = \langle \alpha_1^2 \oplus \alpha_2^2 \mid h^{-1}j[W,M]\rangle = \\ &= \langle \alpha_1^2 \oplus \alpha_2^2 \mid [W_1,M_1] \oplus [W_2,M_2]\rangle = \langle \alpha_1^2 \mid [W_1,M_1]\rangle + \langle \alpha_2^2 \mid [W_2,M_2]\rangle\end{aligned}$$

by usage of the fact that the Kronecker pairing is natural with respect to chain maps such as h taken from the Mayer-Vietoris sequence and j induced by inclusion. From this it is evident that the index σ is additive with respect to connected sum along the boundary, i. e.

$$\sigma(W) = \sigma(W_1) + \sigma(W_2)\ .$$

Recall that by τ_M we denote the tangent bundle of the manifold M. We have smooth projections $\pi_r \colon W \to W_r$ with $r = 1, 2$ that restrict to the identity on $W_r \setminus f_r(0)$ and collapse the complement to the point $f_r(0)$. Denote the trivial n–bundle over W by ε^n. Then the i–th Pontrjagin class of W is given by

$$\begin{aligned}p_i(\tau_W) &= p_i(\tau_W \oplus \varepsilon^n) \\ &= p_i(\pi_1^*\tau_{W_1} \oplus \pi_2^*\tau_{W_2}) && \text{(the pullback-bundle of a point is trivial)} \\ &= \sum_{s+t=i} p_s(\pi_1^*\tau_{W_1})\, p_t(\pi_2^*\tau_{W_2}) && \text{(coefficients in } \mathbb{Q} \text{ forbid mod 2 torsion)} \\ &= \sum_{s+t=i} \pi_1^*p_s(\tau_{W_1})\, \pi_2^*p_t(\tau_{W_2}) && (\pi_i \text{ is covered by the bundle map } \mathrm{D}\,\pi_i) \\ &= \pi_1^*p_i(\tau_{W_1}) + \pi_2^*p_i(\tau_{W_2})\ .\end{aligned}$$

The last step is easy but nontrivial. The idea is that a cycle pushed forward to a single point is of necessity a boundary. It is the same argument which shows that a point has no (co–)homology of order greater than zero.

By construction of π_r we have $\pi_1^* \operatorname{pr}_1 + \pi_2^* \operatorname{pr}_2 = h^{-1}$, thus

$$p_i(\tau_W) = h^{-1}(p_i(\tau_{W_1}) \oplus p_i(\tau_{W_2}))\ .$$

Recall that the μ–conditions on M_1 and M_2 imply that the inclusions $i_r \colon (W_r, \emptyset) \to (W_r, M_r)$ where $r = 1, 2$, and thereby also the inclusion $i \colon (W, \emptyset) \to (W, M)$, induce isomorphisms on cohomology in degrees $4u$ where $0 < u < k$. These fit into a diagram

$$\begin{tikzcd}
H^*(W_1) \oplus H^*(W_2) & & H^*(W_1, M_1) \oplus H^*(W_2, M_2) \arrow[ll, "i_1 \oplus i_2"'] \\
H^*(W) \arrow[u, "h"] & & H^*(W, M \cup A) \arrow[ll] \arrow[u, "h"'] \arrow[dl, "j"] \\
& H^*(W, M) \arrow[ul, "i"] &
\end{tikzcd}$$

and we compute

$$\begin{aligned} i^{-1}p_u(\tau_W) &= i^{-1}h^{-1}(p_u(\tau_{W_1}) \oplus p_u(\tau_{W_2})) \\ &= jh^{-1}(i_1^{-1}p_u(\tau_{W_1}) \oplus i_2^{-1}p_u(\tau_{W_2})) \ . \end{aligned}$$

So given any weighted partition $4j_1 + 8j_2 + \ldots + (4k-4)j_{k-1} = n$, we have

$$\begin{aligned} &\langle (i^{-1}p_1(\tau_W))^{j_1} \ldots (i^{-1}p_{k-1}(\tau_W))^{j_{k-1}} \mid [W,M] \rangle \\ =&\langle (i_1^{-1}p_1(\tau_{W_1}) \oplus i_2^{-1}p_1(\tau_{W_2}))^{j_1} \ldots (i_1^{-1}p_{k-1}(\tau_{W_1}) \oplus i_2^{-1}p_{k-1}(\tau_{W_2}))^{j_{k-1}} \mid h^{-1}j[W,M] \rangle \\ =&\langle (i_1^{-1}p_1(\tau_{W_1}))^{j_1} \ldots (i_1^{-1}p_{k-1}(\tau_{W_1}))^{j_{k-1}} \oplus (i_2^{-1}p_1(\tau_{W_2}))^{j_1} \ldots (i_2^{-1}p_{k-1}(\tau_{W_2}))^{j_{k-1}} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \mid [W_1,M_1] \oplus [W_2,M_2] \rangle \\ =& \quad \langle (i_1^{-1}p_1(\tau_{W_1}))^{j_1} \ldots (i_1^{-1}p_{k-1}(\tau_{W_1}))^{j_{k-1}} | [W_1,M_1] \rangle \\ &+ \langle (i_2^{-1}p_1(\tau_{W_2}))^{j_1} \ldots (i_2^{-1}p_{k-1}(\tau_{W_2}))^{j_{k-1}} | [W_2,M_2] \rangle \ , \end{aligned}$$

showing that the rational Pontrjagin numbers are additive with respect to connected sum along the boundary. This clearly proves lemma 1.36. □

1.38 Remark. With no change this also proves the additivity of Milnor's invariant λ (see [Mil56a], footnote p. 400).

1.39 Corollary. *The invariant μ defines a homomorphism of monoids*

$$\mu \colon \mathcal{P} \longrightarrow \mathbb{Q}/\mathbb{Z} \ .$$

Furthermore, we have $\mu(-P) = -\mu(P)$.

Proof. For the homomorphism statement we still have to check that $\mu(\mathrm{S}^n) = 0$. But this is immediate by choosing the contractible disc $W = \mathrm{D}^{n+1}$ as coboundary. The second statement follows since both the Pontrjagin number and the signature in the definition of μ change sign if the orientation – thus the fundamental class – is reversed. □

We can embed $\mathbb{Z}_{28} \leq \mathbb{Q}/\mathbb{Z}$ as the cyclic subgroup generated by $\frac{1}{28} \in \mathbb{Q}/\mathbb{Z}$.

1.40 Theorem. *The invariant μ defines an isomorphism of groups*

$$\mu \colon \Theta_7 \xrightarrow{\sim} \mathbb{Z}_{28} \ .$$

Proof. We have already mentioned that $\Theta_7 \cong \mathbb{Z}_{28}$. So in view of corollary 1.39 everything is proven when we find a homotopy sphere $\Sigma^7 \in \Theta_7$ such that $\mu(\Sigma^7) = \frac{1}{28}$. This will be accomplished as a consequence of lemma 1.46. □

This completes the diffeomorphism classification of the manifolds in $\mathcal{P}$ up to the Wilkens–$\mathbb{Z}_2$–ambiguity. For our sphere bundles the promised invariant that will rule it out comes for free.

1.41 Theorem. *The homomorphism of monoids* $\bar{s}_1 = 28\mu$

$$\bar{s}_1 : \mathcal{P} \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is a topological invariant of manifolds.

Among other things the proof requires knowledge about the spin cobordism ring, Thom spectra and the Atiyah–Hirzebruch spectral sequence. It is given in [KrSt91], proposition 2.5. We will see in lemma 1.48 that the invariant $\bar{s}_1$ distinguishes the ambiguous cases a) and d) at least for the sphere bundles $M_{m,n}$.

The homeomorphism classification is readily obtained because all invariants realising the PL–classification are in fact topological invariants. So the PL– and the topological classification are the same. This completes the discussion of the classification methods. For the convenience of the reader we give a summary of our results.

1.42 Theorem. Let m, m', n, n' be any integers. Abbreviate “orientation preserving” by o. p. and “orientation reversing” by o. r.

1 There is a commutative tetrahedron of diffeomorphisms

$$\begin{tikzcd}
 & M_{m,n} \arrow[dl, squiggly] \arrow[dd, squiggly] \arrow[dr] & \\
M_{-m,-n} \arrow[rr, squiggly] \arrow[dr] & & M_{-m-n,n} \\
 & M_{m+n,-n} \arrow[ur, squiggly] &
\end{tikzcd}$$

where the straight arrows are o. p. and the waved arrows are o. r.

2 The manifolds $M_{m,0}$ and $M_{m',0}$ are diffeomorphic, PL–homeomorphic or homeomorphic if and only if $m = \pm m'$. By 1 any $M_{m,0}$ has an o. r. self–diffeomorphism.

Let $n, n' > 0$.

3 If $n \neq n'$, the manifolds $M_{m,n}$ and $M_{m',n'}$ are not homotopy equivalent.

Let $n = n'$.

4 The manifolds $M_{m,n}$ and $M_{m',n}$ are o. p. [o. r.] PL–homeomorphic or o. p. [o. r.] homeomorphic if and only if

$$\Phi(M_{m,n}) = [-]\, \Phi(M_{m',n}) \quad \text{and} \quad \bar{s}_1(M_{m,n}) = [-]\, \bar{s}_1(M_{m',n}) \ .$$

5 The manifolds $M_{m,n}$ and $M_{m'n}$ are o. p. [o. r.] diffeomorphic if and only if

$$\Phi(M_{m,n}) = [-]\, \Phi(M_{m',n}) \quad \text{and} \quad \mu(M_{m,n}) = [-]\, \mu(M_{m',n}) \ .$$

1.3 Computation of invariants

The task list theorem 1.42 assigns to us is to compute for any $M_{m,n}$ the invariants lk, $\frac{p_1}{2}$, μ and $\bar{s}_1$. Afterwards we will verify that as promised the invariant $\bar{s}_1$ distinguishes the ambiguous cases. The classification of the occurring Wilkens triples will be easy. We will conclude this section by stating the results as simplest possible congruences the integers m, n and m',n' will satisfy if and only if $M_{m,n}$ and $M_{m',n'}$ are homeomorphic, PL–homeomorphic or diffeomorphic respectively. Still let $n > 0$.

1.43 Lemma. *The linking form* $\mathrm{lk}\colon H^4(M_{m,n}) \otimes H^4(M_{m,n}) \to \mathbb{Q}/\mathbb{Z}$ *is isomorphic to the standard form*

$$\begin{array}{rcl} \mathrm{lk}_{\mathrm{std}}\colon \mathbb{Z}_n \otimes \mathbb{Z}_n & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ (r,s) & \longmapsto & \frac{rs}{n} \ . \end{array}$$

So this invariant does not contribute to the classification.

Proof. By bilinearity of the Kronecker pairing in definition 1.18, it suffices to show $\mathrm{lk}(\beta,\beta) = \frac{1}{n}$ where $\beta = i^*\alpha \in H^4(M_{m,n})$ is the generator as in section 1.1 on page 10. We will use the classical technique developed in [SeiThr34], §77, computing the self linking number of the dual homology class $\beta \cap [M_{m,n}] \in H_3(M_{m,n})$. Omitting the indices m and n, Poincaré–Lefschetz duality reads

$$\begin{tikzcd} 0 \arrow[r] & H^4(W,M) \arrow[r, "j^*"] \arrow[d, "\cap\,[W,M]"', "\sim"] & H^4(W) \arrow[r, "i^*"] \arrow[d, "\cap\,[W,M]"', "\sim"] & H^4(M) \arrow[r] \arrow[d, "\cap\,[M]"', "\sim"] & 0 \\ 0 \arrow[r] & H_4(W) \arrow[r, "j_*"] & H_4(W,M) \arrow[r, "\partial"] & H_3(M) \arrow[r] & 0 \ . \end{tikzcd}$$

In particular, we have $\partial(\alpha \cap [W_{m,n}, M_{m,n}]) = \beta \cap [M_{m,n}]$. Since $H_3(M_{m,n}) \cong \mathbb{Z}_n$, there is $\gamma \in H_4(W_{m,n})$ such that $j_*\gamma = n\,\alpha \cap [W_{m,n}, M_{m,n}]$. So the denominator of $\mathrm{lk}(\beta,\beta)$ is n. The numerator is given by the intersection number of the classes γ and $\alpha \cap [W_{m,n}, M_{m,n}]$. Let $D(\,\cdot\,) = \cdot \cap [W_{m,n}, M_{m,n}]$ denote the duality isomorphism. Then expressed in terms of the cohomology cup product this intersection number is given by

$$\langle \alpha \cup D^{-1}(\gamma),\ [W_{m,n}, M_{m,n}]\rangle = \langle \alpha,\ D^{-1}(\gamma) \cap [W_{m,n}, M_{m,n}]\rangle = \langle \alpha,\ \gamma\rangle = 1 \ .$$

$\square$

Recall the setting previous to lemma 1.10 on page 10. Combining lemma 1.11 and lemma 1.20, we have

1.44 Lemma. *The spin characteristic class* $\frac{p_1}{2}(W_{m,n}) \in H^4(W_{m,n})$ *is given by*

$$\tfrac{p_1}{2}(W_{m,n}) = (n+2m)\,\alpha \ .$$

1.45 Corollary. *The spin characteristic class* $\frac{p_1}{2}(M_{m,n}) \in H^4(M_{m,n})$ *is given by*

$$\tfrac{p_1}{2}(M_{m,n}) = 2m\,\beta \ .$$

Proof. Similarly to the proof of corollary 1.12 we compute

$$\begin{aligned}\tfrac{p_1}{2}(M_{m,n}) &= \tfrac{p_1}{2}(\tau(M_{m,n}) \oplus \nu(M_{m,n})) = \\ &= \tfrac{p_1}{2}(i^*\tau(W_{m,n})) = i^*\tfrac{p_1}{2}(W_{m,n}) = (n+2m)\,\beta = 2m\,\beta\end{aligned}$$

by stability and naturality of the spin characteristic class. □

In particular, we see that the ambiguous case a) translates to

a') $\quad n = 2 \mod 4, \quad m$ arbitrary $\qquad$ or $\qquad n = 4 \mod 8, \quad m$ even

while the ambiguous case d) carries over to

d') $\quad n = 0 \mod 8, \quad m$ odd.

1.46 Lemma. *The μ–invariant is given by*

$$\mu(M_{m,n}) \equiv \frac{(n+2m)^2 - n}{224 \cdot n} \mod 1 \ .$$

Proof. Of course we choose $W_{m,n}$ as spin coboundary. Again let $E = E_{m,n}$ denote the total space of the vector bundle $\zeta_{m,n} : E \xrightarrow{\pi} \mathrm{S}^4$. Denote by E_0 the total space E with the zero section removed and let j' be the inclusion $(E, \emptyset) \to (E, E_0)$. By the five lemma we have natural isomorphisms $H^*(E, E_0) \cong H^*(W_{m,n}, M_{m,n})$. The quadratic form associated to the intersection pairing

$$s\colon H_4(W_{m,n}) \otimes H_4(W_{m,n}) \longrightarrow \mathbb{Z}$$

is thus given by

$$\begin{aligned} H^4(E, E_0) &\longrightarrow \mathbb{Z} \\ v &\longmapsto \langle v \cup v, [E, E_0]\rangle \end{aligned}$$

with the fundamental class $[E, E_0] \in H_8(E, E_0)$ determined by the preferred orientation. The preferred orientation also gives a well–defined generator $u \in H^4(E, E_0)$ called the Thom class of the vector bundle $\zeta_{m,n}$. Using lemma 1.14 we compute

$$\langle u \cup u, [E, E_0]\rangle = \langle {j'}^* u \cup u, [E, E_0]\rangle = \langle \pi^* e(\zeta_{m,n}) \cup u, [E, E_0]\rangle = n\langle \iota \cup u, [E, E_0]\rangle = n \ .$$

The last step is justified by Thom's theorem that $\cdot \cup u$ is an isomorphism. Since we assume $n > 0$, we have $\sigma(W_{m,n}) = +1$.

Let $\alpha \in H^4(W_{m,n}; \mathbb{Q})$ and $\eta \in H^4(W_{m,n}, M_{m,n}; \mathbb{Q})$ be generators such that $j^*\eta = n\alpha$. Using lemma 1.11 which is the same for rational coefficients we compute

$$\begin{aligned} &\langle j^{*-1} p_1^2(W_{m,n}), [W_{m,n}, M_{m,n}]\rangle \\ = \ &\langle \tfrac{1}{n} 4(n+2m)^2 \eta \cup \eta, [W_{m,n}, M_{m,n}]\rangle \\ = \ &\tfrac{4(n+2m)^2}{n} \ . \end{aligned}$$

This proves the lemma. □

Any $M_{m,1}$ is a simply connected homology sphere and thus homeomorphic to S^7 as verified by Smale (see [Mil65a]). For $M_{1,1}$ we have $\mu(M_{1,1}) = \frac{1}{28} \mod 1$ finishing the proof of theorem 1.40.

1.47 Corollary. *The invariant $\bar{s}_1$ is given by*

$$\bar{s}_1(M_{m,n}) = 28\mu(M_{m,n}) = \tfrac{(n+2m)^2-n}{8\cdot n} \mod 1 \; = \tfrac{m^2}{2n} + \tfrac{m}{2} + \tfrac{n-1}{8} \mod 1 \, .$$

Only for the moment, let m and n be subject to either of the ambiguous cases a') or d'). For another integer m' we have $\frac{p_1}{2}(M_{m,n}) = \frac{p_1}{2}(M_{m',n})$ if and only if $m' = m \mod \frac{n}{2}$. Since Tamura has constructed explicit orientation preserving homeomorphisms $M_{m,n} \cong M_{m',n}$ for $m' = m \mod n$ in [Ta58], theorem 3.1, it remains to us to take care of the case $m - m' = \frac{n}{2}$ which is assumed in the following lemma.

1.48 Lemma. *We have $\bar{s}_1(M_{m,n}) \neq \bar{s}_1(M_{m',n})$.*

Proof. Computing

$$\bar{s}_1(M_{m,n}) - \bar{s}_1(M_{m',n}) = \tfrac{m-m'}{2} + \tfrac{m^2-m'^2}{2n} \mod 1 = \tfrac{m}{2} + \tfrac{n}{8} \mod 1$$

it is easily checked that in none of the three ambiguous combinations of m and n this term is zero mod 1. □

As we have just seen, the occurring Wilkens triples are of the form $(\mathbb{Z}_n, \mathrm{lk}_{\mathrm{std}}, 2m)$. Any homomorphism preserving the standard linking form $\mathrm{lk}_{\mathrm{std}}$ is given by multiplication with an element $\alpha \in \mathbb{Z}_n$ such that $\alpha^2 = 1$. But multiplication by any such α clearly defines an automorphism of $\mathbb{Z}_n$. So adopting the notation of [CrEs03], we define the set $A^+(n) = \{\alpha \in \mathbb{Z}_n \mid \alpha^2 = 1\}$ and conclude that $\Phi(M_{m,n}) = \Phi(M_{m',n})$ if and only if

$$2m = \alpha\, 2m' \mod n$$

for some $\alpha \in A^+(n)$. By the remarks below theorem 1.27 we have $\Phi(M_{m,n}) = -\Phi(M_{m',n})$ if and only if the above condition holds for an element in $A^-(n) = \{\alpha \in \mathbb{Z}_n \mid \alpha^2 = -1\}$.

So for counting different (PL)–homeomorphism types represented by bundles $M_{m,n}$ the numbers $\#A^\pm(n)$ are of interest.

1.49 Lemma.

1. *Let r be the number of distinct prime divisors of n and let u be 0, 1 or 2 according to $n \neq 0 \mod 4$, $n = 4 \mod 8$ or $n = 0 \mod 8$. Then we have $\#A^+(n) = 2^{r+u}$.*
2. *The set $A^-(n)$ is nonempty if and only if $n = \varepsilon\, p_1^{i_1} \dots p_k^{i_k}$, $p_i = 1 \mod 4$ and p_i prime for all $i = 1, \dots, k$ and $\varepsilon = 0$ or 1. In this case we have $\#A^-(n) = 2^k$.*

The proof is a matter of elementary number theory as can be found in [Lev77], theorem 5.2. It follows, for example, that there are no orientation reversing homeomorphisms $M_{m,3} \cong M_{m',3}$. Finally, we can give the main theorems of the classification.

1.50 Theorem (Homeomorphism and PL–homeomorphism classification)**.**

1 The manifolds $M_{m,n}$ and $M_{m',n}$ are o. p. homeomorphic or o. p. PL–homeomorphic if and only if

- *for n odd:*
  $m' = \alpha m \mod n$ *with* $\alpha^2 = 1 \mod n$.
- *for $n = 4 \mod 8$ with m odd, or $n = 0 \mod 8$ with m odd:*
  $m' = \alpha m \mod \frac{n}{2}$ *with* $\alpha^2 = 1 \mod n$.
- *for $n = 2^a q$ with q odd, $a = 1$ or 2 with m even, or $a > 2$ with m odd:*
  $m' = \alpha m \mod n$ *with* $\alpha = \pm 1 \mod 2^a$ *and* $\alpha^2 = 1 \mod n$.

2 The manifolds $M_{m,n}$ and $M_{m',n}$ are o. r. homeomorphic or o. r. PL–homeomorphic if and only if there are prime powers $p_l^{i_l}$ with $l = 1, \ldots, k$ such that $p_l = 1 \mod 4$ and $n = \varepsilon\, p_1^{i_1} \ldots p_k^{i_k}$ with $\varepsilon = 1$ or 2 and

- *for $\varepsilon = 1$:*
  $m' = \alpha m \mod n$ *with* $\alpha^2 = -1 \mod n$.
- *for $\varepsilon = 2$:*
  $m' = \alpha(m + \frac{n}{2}) \mod n$ *with* $\alpha^2 = -1 \mod n$.

1.51 Theorem (Diffeomorphism classification)**.**

1 The manifolds $M_{m,n}$ and $M_{m',n}$ are o. p. diffeomorphic if and only if

$m'(n + m') = m(n + m) \mod 56n$ *and*
$2m' = 2\alpha m \mod n$ *with* $\alpha^2 = 1 \mod n$.

2 The manifolds $M_{m,n}$ and $M_{m',n}$ are o. r. diffeomorphic if and only if

$4m'(n + m') + n(n - 1) = -4m(n + m) - n(n - 1) \mod 224n$ *and*
$2m' = 2\alpha m \mod n$ *with* $\alpha^2 = -1 \mod n$.

Proof. Theorem 1.51 is the direct translation of 1.42, part 5. The first two cases of part 1 and the first case of part 2 in theorem 1.50 are also clear since in these cases the invariant $\frac{p_1}{2}$ determines the (PL)–homeomorphism type uniquely. For the remaining cases we need to verify that the stated conditions are equivalent to those given in part 4 of 1.42. This is again a problem of elementary number theory. The reader is referred to pp. 10 and 11 of [CrEs03]. □

Note that part 2 of theorem 1.51 is slightly mistaken in [CrEs03], corollary 1.6, p. 366.

1.52 Remark (Homotopy classification)**.** By its very nature the classification up to homotopy equivalence requires different tools from those used here. For completeness we give the result which is also due to Crowley and Escher, [CrEs03], theorem 1.1, p. 364.

1 The manifolds $M_{m',n}$ and $M_{m,n}$ are o. p. homotopy equivalent if and only if $m' = \alpha m \mod \gcd(n, 12)$ with $\alpha^2 = 1 \mod \gcd(n, 12)$.

2 The manifolds $M_{m',n}$ and $M_{m,n}$ are o. r. homotopy equivalent if and only if there exist prime powers $p_l^{i_l}$ with $l = 1, \ldots, k$ such that $p_l = 1 \mod 4$ and $n = \varepsilon\, p_1^{i_1} \ldots p_k^{i_k}$ with $\varepsilon = 1$ or 2 and

- $\varepsilon = 1$ then there is only one homotopy type which admits an orientation reversing self homotopy equivalence.
- $\varepsilon = 2$ and the integers m and m' are neither both even nor both odd.

1.4 Miscellaneous results

Having completed the hard classification work, we reward ourselves by proving some probably meaningless problems the classification suggests. A manifold is called *almost parallelisable* if it is parallelisable after removing finitely many points.

1.53 Lemma. *The manifolds $W_{m,1}$ are not (stably) parallelisable, not even almost.*

For the signature $\sigma(W_{m,1}) = 1$ is not divisible by eight though $W_{m,1}$ is bounded by a homology sphere (see [Mil59], lemma 3.2). On the other hand, Milnor (see [KerMil63]) has proven that $\Theta_7 = \mathrm{bP}_8$, which says every homotopy 7–sphere and thus any $M_{m,1}$ bounds a parallelisable manifold. So choose a parallelisable coboundary $\widetilde{W}_{m,1}$ for $M_{m,1}$. Performing surgery below the middle dimension, we may assume that $\widetilde{W}_{m,1}$ is 2–connected and thus spin. The μ–condition is automatic by exact cohomology sequence. Since every Pontrjagin number of $\widetilde{W}_{m,1}$ vanishes, we see

$$\mu(M_{m,1}) = -\tfrac{\sigma(\widetilde{W}_{m,1})}{224} \ .$$

Setting $m = 10$ we thus have $\sigma(\widetilde{W}_{10,1}) = 8 \mod 224$. Now by the corollary in [KerMil58], p. 457, there is a closed almost parallelisable 8–manifold with index 224. Removing an open 8-cell from it, we obtain a parallelisable 8–manifold W^{224} with index 224 and boundary S^7. By forming a finite times the connected sum along the boundary of $\widetilde{W}_{10,1}$ and $\pm W^{224}$ we obtain

1.54 Theorem. *There is a parallelisable 8–manifold which is bounded by $M_{10,1}$ and has index eight.*

More generally, for any $k > 1$ Milnor constructed parallelisable $4k$–manifolds W_0 of index 8 which are bounded by homotopy spheres (see section 4 of [Mil59]). The construction follows the very direct approach of realising the matrix

$$\begin{pmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 2 & 1 & 0 & -1 & 0 & 0 & 0\\
0 & 1 & 2 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 2 & 1 & 0 & 0 & 0\\
0 & -1 & 0 & 1 & 2 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 2 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2
\end{pmatrix}$$

as the intersection form of W_0. For $k = 2$ theorem 1.54 together with theorem 1.40 imply

1.55 Theorem. *The boundary of Milnor's 8–manifold W_0 is diffeomorphic to $M_{10,1}$.*

Precisely 16 of the 28 homotopy 7–spheres occur as S^3–bundles over S^4, realising the μ–values 0, 1, 3, 6, 7, 8, 10, 13, 14, 15, 17, 20, 21, 22, 24, 27. To any two of these numbers which add up to 28, there thus is a pair of orientation reversing diffeomorphic exotic 7–spheres. Representing them by bundles with lowest possible m, these are

$$(M_{1,1}, M_{10,1}), (M_{3,1}, M_{12,1}), (M_{6,1}, M_{13,1}), (M_{8,1}, M_{24,1}) \text{ and } (M_{5,1}, M_{17,1}) \ .$$

Interestingly, also the μ–value 14 appears. Hence the only other homotopy 7–sphere apart from S^7 that admits an orientation reversing self–diffeomorphism can be given the bundle structure of, for instance, $M_{20,1}$. These diffeomorphisms are not fibre preserving (in which case they were weak equivalences). The author was unable to construct any of these diffeomorphism explicitly.

Another class of bundles worthwhile examining are those with Euler number 10. For it has been asked in [GZ00] whether the Berger manifold $B = \mathrm{SO}_5\,/\,\mathrm{SO}_3$ is diffeomorphic to a sphere bundle $M_{m,10}$ for some m. This is of special interest since it is known that B admits a homogeneous metric of positive sectional curvature. That B is PL–homeomorphic to such an $M_{m,10}$ has been shown in [KiSh01]. So in view of our classification scheme all that remains to be done is to compute $\mu(B)$. The problem in doing so is that no spin coboundary for B has been found. Its existence, however, is ensured by triviality of the cobordism group Ω_7. But there is an analytic formula due to Donnelly, Kreck and Stolz, based on the Atiyah-Patodi-Singer index theorem, which expresses $\mu(B)$ in terms of η–invariants. By this approach Goette, Kitchloo and Shankar showed in [GKS04]

1.56 Theorem. *The Berger manifold B is diffeomorphic to $M_{\pm1,\mp10}$.*

They also conclude that the only homogeneous S^3–bundles over S^4 are the trivial bundle $M_{0,0}$, the Hopf bundle $M_{0,\pm1}$, the unit tangent bundle of the 4–sphere $M_{\mp1,\pm2}$ and the Berger space $M_{\pm1,\mp10}$.

A completely different source of interesting, possibly exotic 2–connected 7–manifolds K can be found in the field of algebraic geometry. These appear as boundaries of singularities of certain complex hypersurfaces, so called Brieskorn varieties, defined by complex polynomials. The procedure will be made precise in the following chapter. A natural task then might be to compute $\mu(K)$. For we can determine by convenient criteria if K is a topological sphere in which case $\Phi(K)$ is zero. If so, we can look for the computed μ–value in the above list to see whether the singularity boundary K is diffeomorphic to some $M_{m,1}$ and thus admits the structure of an S^3–bundle over S^4. This, admittedly, was the original idea for my diploma thesis, overlooking that the spin coboundary $\overline{F}_\theta$ which K is naturally endorsed with, is parallelisable. So all Pontrjagin classes of $\overline{F}_\theta$ are trivial and $\mu(K)$ reduces again to $-\frac{\sigma(\overline{F}_\theta)}{224} \mod 1$ or equivalently $\sigma(\overline{F}_\theta) \mod 224$ which is precisely the invariant used in [KerMil63], theorem 7.5, p. 529, to distinguish all homotopy spheres in bP_{4k-1}. This invariant has in turn been computed by Brieskorn, Hirzebruch and Pham, see [Bri66], lemma 8, p. 13, or for an exposition [HirMa68], Satz 14.7, p. 109. In particular, they show that all homotopy 7–spheres occur as singularity boundaries. At least, we can still relate this and the following chapter by enumerating 16 example polynomials, realising the 16 pairwise non–diffeomorphic sphere bundles of class $M_{m,1}$. This will be done in example 2.18.

2 Singularities in Brieskorn varieties

This chapter is concerned with the topology of complex algebraic varieties. More precisely, we will be interested in closed neighbourhoods of their singularities. As pointed out, Brieskorn, Hirzebruch and Pham have shown that the boundaries of these neighbourhoods can be diffeomorphic to exotic spheres.

The outline is as follows. Section 2.1 gives a short report on the Milnor fibre bundle of an isolated singularity as introduced in [Mil68]. Milnor further associates two integer matrices with a singularity which arise from rather different contexts. He then shows that their determinants are ± 1 if and only if the singularity boundary is a topological sphere. This will be presented in some detail in section 2.2 and section 2.3. In section 2.4 we will show that these determinants are equal up to sign in general. An example will conclude the presentation.

2.1 The fibre bundle associated with an isolated singularity

Let $f \in \mathbb{C}[z_1, \ldots, z_{n+1}]$ be a complex polynomial in $n+1$ variables such that $f(0) = 0$ and such that the origin is an isolated critical point. In this context "critical" means all complex partial derivatives vanish. Denote by $V \subseteq \mathbb{C}^{n+1}$ the set of roots of f. Let $K = V \cap \mathrm{S}_\varepsilon$ be the intersection of V with a small sphere S_ε centred at the origin. Then K is a $(2n-1)$–dimensional smooth manifold. The proof specifically involves Whitney's finiteness theorem for algebraic sets (again see [Mil68]). Moreover, it can be shown that the complement $\mathrm{S}_\varepsilon \setminus K$ carries the structure of a smooth fibre bundle over S^1 such that K can be regarded as the common boundary of the closure of all fibres of this bundle. The core of the construction is the following technical lemma.

2.1 Lemma (Curve Selection Lemma)**.** *Let $V, U \subseteq \mathbb{R}^m$ such that V is real algebraic, $U = \{x \in \mathbb{R}^m |\ g_1(x) > 0, \ldots, g_l(x) > 0\}$ for finitely many polynomials $g_i \in \mathbb{R}[x_1, \ldots, x_m]$ and such that $0 \in \overline{U \cap V}$. Then there is a real analytic curve*

$$p : [0, 1) \longrightarrow \mathbb{R}^m$$

with $p(0) = 0$ and $p(t) \in U \cap V$ for each $t > 0$.

This is proven by reducing the general case to varieties of dimension one. In this case, a convenient description of $W \cap V$ for a neighbourhood W of the origin is possible. It is the union of "branches" homeomorphic to $[0, 1)$ by power series. Then one "selects" one of these branches as the curve p.

2.2 Theorem (Fibration Theorem)**.** *In the above setting let* $\phi(z) = \frac{f(z)}{|f(z)|}$ *for each* $z \in \mathrm{S}_\varepsilon \setminus K$ *and write* $F_\theta = \phi^{-1}(e^{i\theta})$ *where* $\theta \in [0, 2\pi]$*. Then*

$$F_0 \hookrightarrow \mathrm{S}_\varepsilon \setminus K \xrightarrow{\phi} \mathrm{S}^1$$

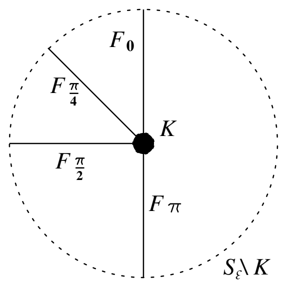
possesses the structure of a smooth fibre bundle.

This is proven in two steps. At first one realises that the real critical points of the map ϕ turn out to be precisely those $z \in \mathrm{S}_\varepsilon \setminus K$ for which the vector $i \operatorname{grad} \log f(z)$ is a real multiple of z. Here the gradient $\operatorname{grad} g$ of a holomorphic function g is to be understood as the vector whose entries is given by the complex conjugates of the partial derivatives of g. The curve selection lemma then rules out the possibility that there are points $z \in \mathbb{C}^{n+1}$ arbitrarily close to the origin so that z and $i \operatorname{grad} \log f(z)$ are linearly dependent over $\mathbb{R}$. This shows that $e^{i\theta}$ is a regular value of ϕ and hence each F_θ is a smooth manifold.

The second part of the proof providing the local triviality property is done by methods of differential topology. A suitable smooth tangent vector field on $\mathrm{S}_\varepsilon \setminus K$ can be constructed locally using the hermitian inner product of $\mathbb{C}^{n+1}$. By a partition of unity we have a global vector field whose integral curves project under ϕ to paths winding around the circle with unit velocity. Since these curves smoothly depend both of time and initial conditions, we obtain the desired bundle charts.

2.3 Lemma. *Decreasing* ε *if necessary the closure* $\overline{F}_\theta$ *of each fibre is a smooth* $2n$*–dimensional manifold with boundary such that* $\partial \overline{F}_\theta = K$ *and* $\mathring{\overline{F}}_\theta = F_\theta$*.*

Again the proof is an application of the curve selection lemma. It can be applied to show that for sufficiently small ε the restriction $f_{\restriction \mathrm{S}_\varepsilon}$ has no critical points on K. This permits the choice of local coordinates at points of K at least for $\overline{F}_0$. The argument for the other fibres works similarly. A schematic image of the fibre bundle $\mathrm{S}_\varepsilon \setminus K$ is presented on the right. The fibres F_θ wind around the common boundary K. We will see that one positive integer determines the fibre homotopy type.



2.4 Definition. Let $g \colon \mathbb{C}^m \to \mathbb{C}^m$ be a holomorphic mapping with an isolated zero in the origin. Then define its *multiplicity* μ by the degree of the mapping

$$\mathrm{S}_\varepsilon \longrightarrow \mathrm{S}^{2m-1}, \quad z \longmapsto \frac{g(z)}{\|g(z)\|}$$

from a small sphere centred at the origin to the standard sphere.

2.5 Theorem. *There is a homotopy equivalence*

$$\mathrm{S}^n \vee \ldots \vee \mathrm{S}^n \longrightarrow F_\theta$$

from the μ*–fold wedge of* n*–dimensional spheres to the fibre* F_θ *where* μ *is the multiplicity of the collection* $\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_{n+1}}\right)$ *of partial derivatives of* f*.*

This middle Betti number μ of the fibre is always positive and has become known as the *Milnor number* of the isolated singularity.

The proof can roughly be outlined as follows. By the construction of a suitable Morse function one can show that any fibre F_θ has the homotopy type of a finite CW-complex of dimension n. Since

$$\mathrm{S}_\varepsilon \setminus \overline{F}_\theta \overset{\phi_{\restriction \mathrm{S}_\varepsilon \setminus \bar{F}_\theta}}{\longrightarrow} \mathrm{S}^1 \setminus \{e^{i\theta}\}$$

is trivial as a bundle over a contractible base, the total space has any other fibre as deformation retract. So we have $H_*(\overline{F}_\theta) \cong H_*(\mathrm{S}_\varepsilon \setminus \overline{F}_\theta)$. Hence by Alexander duality F_θ has the homology of a point in dimensions less than n. So if $n \geq 2$, by the Hurewicz theorem we know that F_θ is $(n-1)$–connected as soon as we can show that it is simply connected. This is again a Morse theory argument. The fibre F_θ can be built up from a $2n$–dimensional disc by adjoining handles of index $\leq n$ which all lie in the ambient space S_ε. But the complement of the disc is certainly simply connected and the adjunction of handles of index $\leq 2n-2$ cannot alter the fundamental group of the complementary set. By induction $\mathrm{S}_\varepsilon \setminus \overline{F}_\theta$ and hence F_θ is simply connected. We conclude that $\pi_n(F_\theta) \cong H_n(F_\theta)$ is free abelian since torsion subgroups would prevent $H^{n+1}(F_\theta)$ from being trivial. This justifies choosing a finite number of maps $(\mathrm{S}^n, \mathrm{pt}) \to (F_\theta, \mathrm{pt})$ representing a basis which one can unify to a map

$$\mathrm{S}^n \vee \ldots \vee \mathrm{S}^n \longrightarrow F_\theta$$

inducing isomorphisms on homology. By Whitehead's theorem this map is a homotopy equivalence.

To see that the number of these spheres is the multiplicity μ is once again a matter of differential topology. Consider the smooth mapping

$$\begin{aligned} v\colon \mathrm{S}_\varepsilon &\longrightarrow \mathrm{S}_\varepsilon \\ z &\longmapsto \varepsilon \tfrac{\operatorname{grad} f(z)}{\|\operatorname{grad} f(z)\|} \end{aligned}$$

and let $M = \overline{\phi^{-1}([-\frac{\pi}{2}, \frac{\pi}{2}])}$. Then one can show as an application of the Lefschetz fixed point theorem that the Euler characteristic $\chi(M)$ and the degree d of v are related by the formula

$$\chi(M) = 1 - d\ .$$

If one takes into account that the gradient as mentioned above is defined by means of the complex conjugates of the partial derivatives, one has $d = (-1)^{n+1}\mu$. For complex conjugation reflects $(n+1)$ real coordinates. Clearly M deformation retracts onto $\overline{F}_0$. We conclude $\operatorname{rank} H_n(F_0) = \mu$.

2.6 Lemma. *The manifold $\overline{F}_\theta$ is parallelisable.*

For $\overline{F}_\theta$ is embedded in S_ε with trivial normal bundle. So $\overline{F}_\theta$ is s–parallelisable and as a manifold with non–vacuous boundary parallelisable. It follows that $\overline{F}_\theta$ has first Stiefel–Whitney class zero and is orientable. From now on an orientation is meant to be chosen, i. e. we fix a fundamental class $[\overline{F}_\theta, K]$. The following remark might be helpful for intuition.

2.7 Remark. The coboundaries $W_{m,n}$ of the manifolds $M_{m,n}$ in chapter 1 and the coboundaries $\overline{F}_\theta$ of the manifolds K in chapter 2 are contrary in the following sense. The manifolds $W_{m,n}$ have middle Betti number one and thus a one–entry intersection form. The manifolds $\overline{F}_\theta$ have middle Betti number μ which will be quite large in general and allows highly nontrivial intersection forms. On the other hand, the manifolds $W_{m,n}$ have a nontrivial tangent bundle as detected by the first Pontrjagin class. The manifolds $\overline{F}_\theta$ have trivial tangent bundle. These observations translate into the intuitive statements that the manifolds $W_{m,n}$ have an easy homotopy type but a complicated differentiable structure and the manifolds $\overline{F}_\theta$ vice versa. Both a complicated homotopy type and a complicated differentiable structure of the coboundary can induce complicated differentiable structures of the boundary. This is precisely what the two summands of the invariant μ ensure.

The last result in this summary says that K belongs to the class of highly connected manifolds so that the machinery of Wall is applicable in principal.

2.8 Theorem. *The closed manifold K is $(n-2)$–connected.*

Another Morse function reveals that the whole sphere S_ε can be reconstructed from a neighbourhood of K that retracts onto K by adjoining handles of index at least n. But these adjunctions do not affect the homotopy groups in dimensions $0 \leq i \leq n-2$. Hence $\pi_i(K) \cong \pi_i(\mathrm{S}_\varepsilon) \cong 0$.

2.2 The first criterion

The key to identifying K as a topological sphere is the first reduced homology group $\tilde{H}_{n-1}(K)$ that might not vanish.

2.9 Lemma. *Let $n \neq 2$. The manifold K is homeomorphic to S^{2n-1} if and only if the reduced homology group $\tilde{H}_{n-1}(K)$ is trivial.*

Proof. The case $n = 1$ is trivial so let $n \geq 3$. By 2.8 we have $H_0(K) \cong \mathbb{Z}$ and $H_1(K) \cong \ldots \cong H_{n-2}(K) \cong 0$. Moreover $H_{n-1}(K) \cong 0$ by assumption. For $k \geq 1$

$$H_{n+k}(K) \cong H_{n+k+1}(\overline{F}_\theta, K) \cong (H_{n-k-1}(\overline{F}_\theta, K)/T_{n-k-1}) \oplus T_{n-k-2} \cong \begin{cases} \mathbb{Z} & \text{if } k = n-1 \\ 0 & \text{otherwise} \end{cases}$$

due to exact pair sequence, theorem 2.5, Poincaré duality and universal coefficient theorem. Here $T_{n-k-1} \subseteq H_{n-k-1}(\overline{F}_\theta, K)$ and $T_{n-k-2} \subseteq H_{n-k-2}(\overline{F}_\theta, K)$ denote the torsion subgroups. In case $k = 0$ the exact sequence is

$$0 \longrightarrow H_n(K) \longrightarrow \mathbb{Z}^\mu \xrightarrow{\sim} \mathbb{Z}^\mu \longrightarrow 0$$

and hence $H_n(K) \cong 0$.

So we have verified that K is a simply connected homology sphere of dimension ≥ 5. By the generalised Poincaré conjecture (see [Mil65a]) it is homeomorphic to S^{2n-1}. $\square$

2.10 Remark. For $n = 2$ this is wrong. The polynomial $f(z_1, z_2, z_3) = z_1^2 + z_2^3 + z_3^5$ yields the Poincaré sphere as a counterexample.

2.11 Theorem (1st criterion)**.** *Let $n \neq 2$. The manifold K is homeomorphic to S^{2n-1} if and only if the intersection pairing*

$$s : H_n(F_\theta) \otimes H_n(F_\theta) \longrightarrow \mathbb{Z}$$

has determinant ± 1.

2.12 Remark. Recall that the graded commutativity of the cohomology cup product effects that S is symmetric if n is even and skew-symmetric if n is odd. In the latter case we have $\det S = 0$ so any topological sphere K of this kind is necessarily due to a polynomial of an odd number of variables.

Proof (of theorem 2.11). Consider the homology exact sequence

$$H_n(\overline{F}_\theta) \xrightarrow{j_*} H_n(\overline{F}_\theta, K) \longrightarrow \tilde{H}_{n-1}(K) \longrightarrow 0 \ .$$

The group $H_n(\overline{F}_\theta)$ as well as $H_n(\overline{F}_\theta, K)$ by Poincaré duality are free abelian of rank μ as observed in theorem 2.5. So $\tilde{H}_{n-1}(K) \cong 0$ if and only if j_* is an isomorphism. There is another intersection pairing

$$s' : H_n(\overline{F}_\theta, K) \otimes H_n(\overline{F}_\theta) \longrightarrow \mathbb{Z}$$

defined as either composition in the diagram

$$\begin{tikzcd}
H_n(\overline{F}_\theta, K) \otimes H_n(\overline{F}_\theta) \arrow[r, "\mathrm{id} \otimes h \circ D^{-1}"] \arrow[d, "D^{-1} \otimes D^{-1}"'] & H_n(\overline{F}_\theta, K) \otimes \mathrm{Hom}_{\mathbb{Z}}(H_n(\overline{F}_\theta, K), \mathbb{Z}) \arrow[d, "\text{evaluate}"] \\
H^n(\overline{F}_\theta) \otimes H^n(\overline{F}_\theta, K) \arrow[r, "h(\,\cdot\,\cup\,\cdot\,)({[\overline{F}_\theta, K]})"'] & \mathbb{Z} \ .
\end{tikzcd}$$

Here $D : H^n(\overline{F}_\theta) \longrightarrow H_n(\overline{F}_\theta, K)$ and $D : H^n(\overline{F}_\theta, K) \longrightarrow H_n(\overline{F}_\theta)$ are the duality isomorphisms given by

$$\alpha \longmapsto [\overline{F}_\theta, K] \cap \alpha$$

and h is the isomorphism

$$0 \longrightarrow \underbrace{\mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(\overline{F}_\theta, K), \mathbb{Z})}_{\cong\ 0} \longrightarrow H^n(\overline{F}_\theta, K) \xrightarrow{h} \mathrm{Hom}_{\mathbb{Z}}(H_n(\overline{F}_\theta, K), \mathbb{Z}) \longrightarrow 0$$

appearing in the universal coefficient theorem induced by evaluation on chains. Let $\alpha \otimes \beta \in H^n(\overline{F}_\theta) \otimes H^n(\overline{F}_\theta, K)$. We have

$$\begin{aligned}
h(\alpha \cup \beta)([\overline{F}_\theta, K]) &= h \circ (\alpha \cup (\cdot))(\beta)([\overline{F}_\theta, K]) \\
&= ((\cdot) \cap \alpha)^* \circ h(\beta)([\overline{F}_\theta, K]) \\
&= h(\beta)([\overline{F}_\theta, K] \cap \alpha)
\end{aligned}$$

so the diagram commutes. Now for a basis (a minimal generating set) $\{\alpha_i\}_{i=1}^{\mu}$ in $H_n(\overline{F}_\theta, K)$ let $\{\alpha^j\}_{j=1}^{\mu}$ be the dual basis in $\mathrm{Hom}(H_n(\overline{F}_\theta, K), \mathbb{Z})$. Then setting $\beta_j = D \circ h^{-1}(\alpha^j)$ we have a naturally distinguished basis $\{\beta_j\}_{j=1}^{\mu}$ in $H_n(\overline{F}_\theta)$. Since

$$s'(\alpha_i, \beta_j) = h \circ D^{-1}(\beta_j)(\alpha_i) = h \circ D^{-1}(D \circ h^{-1}(\alpha^j))(\alpha_i) = \alpha^j(\alpha_i) = \delta_{ij} \ , \qquad (2.1)$$

the intersection pairing s' is represented as the unit matrix of size μ in these bases. So $\det s' = 1$. The two intersection pairings are related by the critical homomorphism j_* in that for all $\alpha, \beta \in H_n(\overline{F}_\theta)$

$$\begin{aligned} s(\alpha, \beta) &= h(j^* D^{-1}(\alpha) \cup D^{-1}(\beta))([\overline{F}_\theta, K]) \\ &= h(D^{-1}(j_*\alpha) \cup D^{-1}(\beta))([\overline{F}_\theta, K]) \\ &= s'(j_*\alpha, \beta) \ . \end{aligned} \qquad (2.2)$$

Let S_{ij} and J_{ij} with $1 \leq i, j \leq \mu$ be the entries of the intersection matrix of s and of the transformation matrix of j_* in each case defined by the chosen bases. Then by (2.1) and (2.2) we have

$$S_{ij} = s(\beta_i, \beta_j) = s'(j_*\beta_i, \beta_j) = \sum_{k=1}^{\mu} J_{ki}\, s'(\alpha_k, \beta_j) = J_{ji} \ . \qquad (2.3)$$

So the transformation matrix J is the transpose of the intersection matrix S. The assertion follows. □

Before we proceed to the second criterion, let us clarify how the matrices S and J behave when passing to a different basis $\{\tilde{\alpha}_i\}_{i=1}^{\mu} \subseteq H_n(\overline{F}_\theta, K)$. The multiple indices as well as the interplay of transposed and inverted matrices that arise when dealing with changes of coordinates, are a reliable source of confusion. Therefore we want to do this in some detail.

Let $\tilde{C}$ be the corresponding change of basis matrix, i. e. the i–th column of $\tilde{C}$ consists of the coordinates of $\tilde{\alpha}_i$ with respect to the basis $\{\alpha_i\}_{i=1}^{\mu}$. Set $C = \tilde{C}^{-1}$. As specified above we also have a canonical new basis $\{\tilde{\beta}_i\}_{i=1}^{\mu} \subseteq H_n(\overline{F}_\theta)$ obtained from $\{\tilde{\alpha}_i\}_{i=1}^{\mu}$ via $D \circ h^{-1}$.

2.13 Lemma. *A change of basis from $\{\alpha_i\}_{i=1}^{\mu}$ to $\{\tilde{\alpha}_i\}_{i=1}^{\mu}$ in $H_n(\overline{F}_\theta, K)$ results in the transformations*

$$\tilde{S} = CSC^\top \quad \textit{and} \quad \tilde{J} = CJC^\top \ .$$

Note the reversed order compared to how one usually defines congruence of matrices ($\tilde{J} = C^\top JC$).

Proof. By definition we have $\tilde{\alpha}_i = \sum_{k=1}^{\mu} \tilde{C}_{ki}\alpha_k$ and thus $\sum_i \lambda_i\tilde{\alpha}_i = \sum_i \lambda_i \sum_k \tilde{C}_{ki}\alpha_k = \sum_k \left(\sum_i \tilde{C}_{ki}\lambda_i\right) \alpha_k = \sum_i \left(\sum_k \tilde{C}_{ik}\lambda_k\right) \alpha_i$, so $\tilde{C}$ transforms the coordinates λ_i with respect to $\{\tilde{\alpha}_i\}_{i=1}^{\mu}$ to the corresponding coordinates $\sum_k \tilde{C}_{ik}\lambda_k$ with respect to $\{\alpha_i\}_{i=1}^{\mu}$.

The question is how to describe the change of coordinates in $H_n(\overline{F}_\theta)$ when passing from $\{\beta_i\}_{i=1}^{\mu}$ to $\{\tilde{\beta}_i\}_{i=1}^{\mu}$.

Here we observe

$$\delta_{ij} = \tilde{\alpha}^i(\tilde{\alpha}_j) = \tilde{\alpha}^i\left(\textstyle\sum_k \tilde{C}_{kj}\alpha_k\right) = \textstyle\sum_k \tilde{\alpha}^i(\alpha_k)\tilde{C}_{kj}$$

showing that

$$\tilde{\alpha}^i(\alpha_k) = ((\tilde{C})^{-1})_{ik} = C_{ik}$$

for each $1 \le k \le \mu$ and consequently

$$\tilde{\alpha}^i = \textstyle\sum_k C_{ik}\alpha^k \ .$$

This yields

$$\tilde{\beta}_i = D \circ h^{-1}(\tilde{\alpha}^i) = D \circ h^{-1}\left(\textstyle\sum_k C_{ik}\alpha^k\right) = \textstyle\sum_k C_{ik}\beta_k = \textstyle\sum_k (C^\top)_{ki}\beta_k \ .$$

Arguing as above $C^\top$ transforms coordinates with respect to $\{\tilde{\beta}_i\}_{i=1}^{\mu}$ to coordinates with respect to $\{\beta_i\}_{i=1}^{\mu}$. We deduce

$$\tilde{S} = (C^\top)^\top S C^\top = CSC^\top \quad \text{and} \quad \tilde{J} = \tilde{C}^{-1} J C^\top = CJC^\top \ .$$

□

This result is of course consistent with (2.3).

Setting up the second criterion will be easy as soon as we have constructed a certain exact sequence which we will introduce in the general context of any fibre bundle

$$F_0 \hookrightarrow E \longrightarrow \mathrm{S}^1$$

over the circle. For convenience set $I_{2\pi} = [0, 2\pi]$ and think of $F_0 \times I_{2\pi}$ as embedded in $F_0 \times \mathbb{R}$ justifying the notation $\partial(F_0 \times I_{2\pi}) = F_0 \times \{0\} \cup F_0 \times \{2\pi\}$. From the diagram

$$\begin{tikzcd} F_0 \arrow[r] \arrow[d] & E \arrow[d, "\phi"] \\ F_0 \times I_{2\pi} \arrow[r] \arrow[ur, dashed, "h"] & \mathrm{S}^1 \end{tikzcd}$$

we have a lift

$$h : (F_0 \times I_{2\pi}, \partial(F_0 \times I_{2\pi})) \longrightarrow (E, F_0)$$

which is a continuous one–parameter family of homeomorphisms, unique up to homotopy. We want to refer to its restriction

$$h_{2\pi} : F_0 \longrightarrow F_0$$

as the *characteristic homeomorphism* of the fibre F_0.

2.14 Lemma (Wang sequence I)**.** *We have an associated long exact sequence*

$$\ldots \longrightarrow H_{j+1}(E) \longrightarrow H_j(F_0) \overset{h_{2\pi *}-\mathrm{id}_*}{\longrightarrow} H_j(F_0) \longrightarrow H_j(E) \longrightarrow \ldots \ .$$

Proof. This sequence emanates from the long exact homology sequence of the pair (E, F_0) by substituting the groups $H_{*+1}(E, F_0)$ by $H_*(F_0)$ as indicated in the following diagram

$$\begin{tikzcd}
H_{j+1}(E) \arrow[r] \arrow[ddr, bend right] & H_{j+1}(E,F_0) \arrow[r, "\partial"] & H_j(F_0) \arrow[r] & H_j(E) \ . \\
& H_{j+1}(F_0 \times I_{2\pi}, \partial(F_0 \times I_{2\pi})) \arrow[u, "h_*"', "\sim"] & & \\
& H_j(F_0) \arrow[u, "\sim"] \arrow[uur, bend right, "h_{2\pi *}-\mathrm{id}_*"'] & &
\end{tikzcd}$$

There are at least three different possible view points for the lower vertical isomorphism. It is the homological suspension isomorphism $\sigma_\pm$ as e. g. defined in [tDie00], Beispiel 8.7, p. 139. It is the homology cross product $\times [I_{2\pi}, \partial I_{2\pi}]$ with one of the two generators of $H_1(I_{2\pi}, \partial I_{2\pi})$ as e. g. defined in [tDie00], Satz 4.2, p. 165. And it is the isomorphism P_* induced by the prism operator P of chain complexes as e. g. defined in [Hat02], p. 112. In our case, P is associated with the obvious homotopy $F_0 \times I \to F_0 \times I_{2\pi}$ of the two edge inclusions i^0 and $i^{2\pi}$ with 0 and 2π interchangeable. In each case the two possible choices differ by the factor -1.

That h_* is an isomorphism can be seen from the diagram

$$\begin{tikzcd}
H_*(F_0 \times [0,2\pi], F_0 \times \{0\} \cup F_0 \times \{2\pi\}) \arrow[r, "h_*"] \arrow[d, "\sim"'] & H_*(E, F_0) \arrow[d, "\sim"] \\
H_*(F_0 \times [\delta, 2\pi-\delta], F_0 \times \{\delta\} \cup F_0 \times \{2\pi - \delta\}) \arrow[r, "\tilde{h}_*"] \arrow[dr, "\sim", "\tilde{h} = h_{\restriction F_0 \times [\delta, 2\pi-\delta]\,*}"'] & H_*(E, F_{[-\delta,\delta]}) \\
& H_*(E_{[\delta,2\pi-\delta]}, F_{-\delta} \cup F_\delta) \arrow[u, "\mathrm{exc.}", "\sim"']
\end{tikzcd}$$

where $\delta > 0$ is suitably small and the notations

$$E_{[\delta,2\pi-\delta]} = \phi^{-1}(\{e^{i\theta} \mid \theta \in [\delta, 2\pi-\delta]\}) \quad \text{and} \quad F_{[-\delta,\delta]} = \phi^{-1}(\{e^{i\theta} \mid \theta \in [-\delta, \delta]\})$$

are to be understood.

It remains to prove the identity $\partial h_* P_* = h_{2\pi *} - \mathrm{id}_*$. Since all homology maps involved are induced from chain maps, we can verify this on the underlying singular chain complex. There we compute

$$\partial h_\# P(\alpha) = h_\# \partial P(\alpha) = h_\#(i^{2\pi}_\# \alpha - i^0_\# \alpha) = (h_{2\pi \#} - \mathrm{id}_\#)(\alpha)$$

for each cycle $\alpha \in Z_*(F_0)$. $\square$

2.15 Remark. In the above reference ([tDie00], Beispiel 8.7, p. 139) it is also shown that the relative version

$$\times [0, 2\pi]\colon\ H_n(\overline{F}_0, K) \longrightarrow H_{n+1}(\overline{F}_0 \times I_{2\pi}, \overline{F}_0 \times \partial I_{2\pi} \cup K \times I_{2\pi})$$

of the homology cross product is an isomorphism as well. It coincides with the isomorphism P_* induced by the corresponding relative prism operator P as defined in [Hat02], p. 118.

2.3 The second criterion

From the assertion of theorem 2.11 it is not quite obvious that the fibre bundle structure of the space $\mathrm{S}_\varepsilon \setminus K$ surrounding F_θ is involved. This is different from the second criterion which is concerned with the transformation of homology classes a fibre transport effects.

2.16 Theorem (2^{nd} criterion)**.** *Let $n \neq 2$. The manifold K is homeomorphic to S^{2n-1} if and only if the endomorphism*

$$h_{2\pi *} - \mathrm{id}_* : H_n(F_0) \longrightarrow H_n(F_0)$$

has determinant ± 1.

Proof. This is immediate from the Wang sequence

$$H_n(F_0) \overset{h_{2\pi *} - \mathrm{id}_*}{\longrightarrow} H_n(F_0) \longrightarrow H_n(\mathrm{S}_\varepsilon \setminus K) \longrightarrow 0$$

since $H_n(\mathrm{S}_\varepsilon \setminus K) \cong H^n(K) \cong \tilde{H}_{n-1}(K)$ by Alexander and Poincaré duality. □

2.17 Remark. The determinant in 2.16 of course occurs as $(-1)^n$ times the evaluation at 1 of the characteristic polynomial

$$\Delta(t) = \det\,(t\,\mathrm{id}_* - h_{2\pi *})$$

of the characteristic endomorphism $h_{2\pi *}$ in degree n. There is an algorithm to compute these characteristic polynomials, compare [KWa78].

2.18 Example. For $k > 0$ let f_k be the complex polynomial given by

$$f_k(z_1, z_2, z_3, z_4, z_5) = z_1^3 + z_2^{6k-1} + z_3^2 + z_4^2 + z_5^2\ .$$

It follows from the formulas of Brieskorn and Pham (see [Mil68], theorem 9.1, p. 71) that the associated characteristic polynomial Δ is given by

$$\Delta(t) = \textstyle\prod_{l=1}^{6k-2} (t^2 + t\zeta^l + \zeta^{2l})$$

where ζ is a primitive $(6k-1)^{\text{th}}$ root of unity. With some algebra we have that $\Delta(1) = \pm 1$ for all k. Thus K as defined by f_k is a topological sphere. Moreover, Satz 14.7 of [HirMa68] says that $\sigma(\overline{F}_\theta) = 8k$. With the remarks at the end of section 1.4 we have

$$\mu(K) = \tfrac{27k}{28} \mod 1\ .$$

So letting k vary from 1 to 28, we obtain all of the 28 homotopy 7–spheres. In particular, the following table lists the 16 possible μ–values in $\mathbb{Z}_{28}$ for 3–sphere bundles over the 4–sphere homeomorphic to S^7 together with smallest values of m and k such that $\mu = \mu(M_{m,1}) = \mu(K)$, i.e. such that K can be given the sphere bundle structure of $M_{m,1}$.

μ	0	1	3	6	7	8	10	13	14	15	17	20	21	22	24	27
m	0	1	2	3	13	8	11	17	20	5	9	24	14	12	16	10
k	28	27	25	22	21	20	18	15	14	13	11	8	7	6	4	1

2.19 Remark. If H denotes the transformation matrix of $h_{2\pi*} - \mathrm{id}_*$ in degree n with respect to the basis $\{\beta_i\}_{i=1}^{\mu}$, the change of basis from lemma 2.13 obviously effects that $h_{2\pi*} - \mathrm{id}_*$ is represented by

$$\tilde{H} = (C^\top)^{-1} H C^\top$$

with respect to the basis $\{\tilde{\beta}_i\}_{i=1}^{\mu}$.

One question might arise at this point. Is there a relation between the two criteria 2.11 and 2.16 which goes beyond the fact that both determinants take unit values whenever K is a topological sphere? In particular, it was desirable that there be a formula somehow relating the two matrices S and H the criteria are concerned with.

2.4 The criteria relation formula

A first guess for the relation of the matrices S and H might be they are equal. But this is overly optimistic since it contradicts the transformation behaviour observed in 2.13 and 2.19. Nevertheless, we will arrive at the following point.

2.20 Theorem. *There is an isomorphism* $\Psi : H_n(\overline{F}_0, K) \longrightarrow H_n(\overline{F}_0)$ *such that*

$$h_{2\pi*} - \mathrm{id}_* = \Psi \circ j_* \quad .$$

If a basis is chosen and recalling that $S^\top = (-1)^n S$, the theorem takes the form

2.21 Theorem. *There is* $P \in \mathrm{GL}_n(\mathbb{Z})$ *such that* $H = PS$.

2.22 Corollary. *The matrices* S *and* H *share up to sign the same determinant.*

2.23 Corollary. *The radical of the intersection pairing*

$$s \colon H_n(F_0) \otimes H_n(F_0) \longrightarrow \mathbb{Z}$$

is equal to the kernel of the endomorphism

$$h_{2\pi*} - \mathrm{id}_* \colon H_n(F_0) \longrightarrow H_n(F_0) \ .$$

These conclusions should motivate proving theorem 2.20. As a preparation we will first of all verify that the homotopy h which induces the isomorphism

$$h_*\colon H_{n+1}(F_0 \times I_{2\pi}, \partial(F_0 \times I_{2\pi})) \longrightarrow H_{n+1}(\mathrm{S}_\varepsilon \setminus K, F_0)$$

also induces an isomorphism

$$\overline{h}_*\colon H_{n+1}(\overline{F}_0 \times I_{2\pi}, \overline{F}_0 \times \partial I_{2\pi} \cup K \times I_{2\pi}) \longrightarrow H_{n+1}(\mathrm{S}_\varepsilon, \overline{F}_0)\ .$$

The construction is as follows. Remove a collar neighbourhood of K in $\overline{F}_0$ from $\overline{F}_0$ to obtain a manifold $\widetilde{F}_0$ with boundary $\widetilde{K}$. Clearly $(\overline{F}_0, K)$ and $(\widetilde{F}_0, \widetilde{K})$ are homeomorphic. Let (U, V) be a pair of open neighbourhoods of $(\overline{F}_0, K)$ in S_ε which deformation retract onto $(\overline{F}_0, K)$. (Since $\overline{F}_0$ and K are smoothly imbedded in S_ε, one can e. g. construct them as tubular neighbourhoods.) Choose $\delta > 0$ such that h maps $\widetilde{F}_0 \times \{\delta, 2\pi - \delta\}$ to U. Moreover, we may assume that V was so chosen that h maps $\widetilde{K} \times I_{2\pi}$ to V. Finally, let $\widetilde{h}$ be the restriction of h to $\widetilde{F}_0 \times I_{2\pi}$. As in the proof of lemma 2.14 we have a diagram

$$\begin{tikzcd}
H_*(\widetilde{F}_0 \times I_{2\pi}, \widetilde{F}_0 \times \partial I_{2\pi} \cup \widetilde{K} \times I_{2\pi}) \arrow[r, "\widetilde{h}_*"] \arrow[d, "\sim"'] & H_*(\mathrm{S}_\varepsilon, \overline{F}_0 \cup V) \arrow[d, "\sim"] \\
H_*(\widetilde{F}_0 \times [\delta, 2\pi - \delta], \widetilde{F}_0 \times \{\delta, 2\pi - \delta\} \cup \widetilde{K} \times [\delta, 2\pi - \delta]) \arrow[r] \arrow[dr, "\sim"'] & H_*(\mathrm{S}_\varepsilon, U) \\
& H_*(\mathrm{S}_\varepsilon \setminus \overline{F}_0, U \setminus \overline{F}_0)\ . \arrow[u, "\text{exc.}"', "\sim"]
\end{tikzcd}$$

Now the isomorphism $\overline{h}_*$ is given by $\widetilde{h}_*$ together with the obvious isomorphisms

$$H_{n+1}(\overline{F}_0 \times I_{2\pi}, \overline{F}_0 \times \partial I_{2\pi} \cup K \times I_{2\pi}) \cong H_{n+1}(\widetilde{F}_0 \times I_{2\pi}, \widetilde{F}_0 \times \partial I_{2\pi} \cup \widetilde{K} \times I_{2\pi})$$

and

$$H_{n+1}(\mathrm{S}_\varepsilon, \overline{F}_0 \cup V) \cong H_{n+1}(\mathrm{S}_\varepsilon, \overline{F}_0)\ .$$

As side result we obtain a relative version of the Wang sequence for our special setting.

2.24 Lemma (Wang sequence II)**.** *We have a long exact sequence*

$$\ldots \longrightarrow H_{j+1}(\mathrm{S}_\varepsilon, K) \longrightarrow H_j(\overline{F}_0, K) \xrightarrow{\widetilde{h}_{2\pi *} - \mathrm{id}_*} H_j(\overline{F}_0, K) \longrightarrow H_j(\mathrm{S}_\varepsilon, K) \longrightarrow \ldots\ .$$

Proof. This time we use the long exact homology sequence of the triple $(\mathrm{S}_\varepsilon, \overline{F}_0, K)$. The substitution by isomorphisms is

$$H_*(\overline{F}_0, K) \xrightarrow{P_*} H_{*+1}(\overline{F}_0 \times I_{2\pi}, \overline{F}_0 \times \partial I_{2\pi} \cup K \times I_{2\pi}) \xrightarrow{\overline{h}_*} H_{*+1}(\mathrm{S}_\varepsilon, \overline{F}_0)\ .$$

□

Now theorem 2.20 is included in the following lemma. For a clear arrangement we will use abbreviations $FI = F_0 \times I_{2\pi}$ and the like.

2.25 Lemma. *We have a commutative diagram of free abelian groups*

$$\begin{tikzcd}
H_n(F_0) \arrow[rr, "j_*"] \arrow[d, "P_*", "\sim"'] & & H_n(\overline{F}_0, K) \arrow[d, "P_*", "\sim"'] \\
H_{n+1}(FI, F\partial I) \arrow[rr, "j_*"] \arrow[d, "h_*", "\sim"'] & & H_{n+1}(\overline{F}I, \overline{F}\partial I \cup KI) \arrow[d, "\overline{h}_*", "\sim"'] \\
H_{n+1}(\mathrm{S}_\varepsilon \setminus K, F_0) \arrow[rr, "i_*"] \arrow[dr, "\partial"'] & & H_{n+1}(\mathrm{S}_\varepsilon, \overline{F}_0) \arrow[dl, "\sim"', "\partial"] \\
& H_n(F_0) &
\end{tikzcd}$$

whose "$\sim$"–labelled arrows are isomorphisms.

It shall be understood that the canonical isomorphisms

$$H_n(F_0) \cong H_n(\overline{F}_0), \quad H_{n+1}(FI, F\partial I) \cong H_{n+1}(\overline{F}I, \overline{F}\partial I) \quad \text{and}$$
$$H_{n+1}(\mathrm{S}_\varepsilon, \overline{F}_0) \cong H_{n+1}(\mathrm{S}_\varepsilon, F_0)$$

are to be inserted if applicable.

Proof. The upper square commutes by naturality of suspension (e. g. [tDie00], Satz 9.1, p. 142). The lower square commutes since the underlying diagram of pairs of spaces

$$\begin{tikzcd}
(\widetilde{F}_0 \times I_{2\pi}, \widetilde{F}_0 \times \partial I_{2\pi}) \arrow[r] \arrow[d, "\widetilde{h}"'] & (\widetilde{F}_0 \times I_{2\pi}, \widetilde{F}_0 \times \partial I_{2\pi} \cup \widetilde{K} \times I_{2\pi}) \arrow[d, "\widetilde{h}"'] \\
(\mathrm{S}_\varepsilon \setminus K, F_0) \arrow[r] & (\mathrm{S}_\varepsilon, \overline{F}_0 \cup V)
\end{tikzcd}$$

obviously commutes. The triangle commutes as part of the braid diagram of the triple $(\mathrm{S}_\varepsilon, \mathrm{S}_\varepsilon \setminus K, F_0)$. It also shows that the lower right boundary map is an isomorphism.

$$\begin{tikzcd}[column sep=small]
H_{n+1}(\mathrm{S}_\varepsilon \setminus K, F_0) \arrow[dr] \arrow[rr, bend left] & \otimes & H_n(F_0) \arrow[dr] \arrow[rr, bend left] & & H_n(\mathrm{S}_\varepsilon) \arrow[dr] \arrow[rr, bend left] & & {} \\
& H_{n+1}(\mathrm{S}_\varepsilon, F_0) \arrow[ur] \arrow[dr] & & H_n(\mathrm{S}_\varepsilon \setminus K) \arrow[ur] \arrow[dr] & & H_n(\mathrm{S}_\varepsilon, F_0) \arrow[ur] \arrow[dr] & \\
H_{n+1}(\mathrm{S}_\varepsilon) \arrow[ur] \arrow[rr, bend right] & & H_{n+1}(\mathrm{S}_\varepsilon, \mathrm{S}_\varepsilon \setminus K) \arrow[ur] \arrow[rr, bend right] & & H_n(\mathrm{S}_\varepsilon \setminus K, F_0) \arrow[ur] \arrow[rr, bend right] & & {}
\end{tikzcd}$$

$\square$

Proof (of theorem 2.20). We have seen in lemma 2.14 that the composition $\partial h_* P_*$ is just $h_{2\pi *} - \mathrm{id}_*$. Following the most outer way in the diagram of the preceding lemma, we see that $h_{2\pi *} - \mathrm{id}_*$ factorises as j_* followed by the three isomorphisms P_*, $\overline{h}_*$ and ∂. We define their composition to be Ψ. □

A final example will illustrate that also S^3–bundles over S^4 with Euler number different from one may occur as singularity boundaries.

2.26 Example. Let f be the complex polynomial given by

$$f(z_1, z_2, z_3, z_4, z_5) = z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 \ .$$

Again by the formula in [Mil68], p. 71, we see that f defines a fibre bundle with Milnor number $\mu = 1$ and characteristic polynomial

$$\Delta(t) = t + 1 \ .$$

Evaluating in one gives $\det H = \det S = \pm 2$ which says $H_3(K) \cong H^4(K) \cong \mathbb{Z}_2$. Thus $\frac{p_1}{2}(K) = 0$ by lemma 1.26. Similarly to the proof of lemma 1.43 one sees that the linking pairing $H^4(K) \otimes H^4(K) \to \mathbb{Z}_2$ is the nontrivial one. This is sufficient for K being (PL)–homeomorphic to an S^3–bundle over S^4 according to corollary 1.4 of [CrEs03], p. 366. Now by part 5 of theorem 1.42, computing $\mu(K)$ is enough to decide which $M_{m,n}$ is a representative of the diffeomorphism class K belongs to. Since $\sigma(\overline{F}_\theta) = \pm 1$, we have

$$\mu(K) = \pm \tfrac{1}{224}$$

with the sign indicating orientation. As of necessity $n = 2$, we only have to determine m. It turns out that the quadratic equation $\mu(K) = \mu(M_{m,2})$ with $\mu(K)$ set negative, compare lemma 1.46, has $m = -1$ as unique solution. We thus have proven:

The manifold K is diffeomorphic to the unit tangent bundle of the 4–sphere $M_{\mp 1, \pm 2}$.

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