

Introduction to Topology

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I. Basic Concepts of Topology

I.1. Topological Spaces

Definition I.1.1. A *topological space* is a pair $(X, \mathcal{O})$ consisting of a set X and a collection $\mathcal{O}$ of subsets of X (called the *open subsets*) such that:

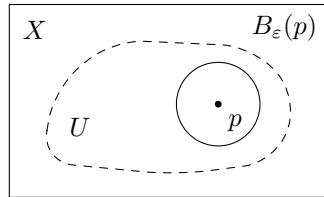
1. If $\{U_i\}_{i \in I}$ is any family of open subsets, then $\bigcup_{i \in I} U_i$ is open.
 2. If U_1 and U_2 are open, then so is $U_1 \cap U_2$.
 3. The sets $\emptyset$ and X are open.
- We call $\mathcal{O}$ a *topology* on X .
 - A subset $A \subseteq X$ is called *closed* if $X \setminus A$ is open.

Examples.

- On any set X , both $\mathcal{O} = \{\emptyset, X\}$ (the *indiscrete topology* or *trivial topology*) and $\mathcal{O} = \mathcal{P}(X) = \{U : U \subseteq X\}$ (the *discrete topology*) are topologies on X .
- Let (X, d) be a metric space. Then

$$\mathcal{O}(d) = \{U \subseteq X : \text{for every } p \in U \text{ there is an } \varepsilon > 0 \text{ such that } B_\varepsilon(p) \subseteq U\}$$

is the *topology induced* by d on X . Here $B_\varepsilon(p) = \{x \in X : d(x, p) < \varepsilon\}$.



Verification of the axioms: 1.) and 3.) are clear. 2.) Let $p \in U_1 \cap U_2$. For $i = 1, 2$, there is an ε_i with $B_{\varepsilon_i}(p) \subseteq U_i$. Then $B_{\min\{\varepsilon_1, \varepsilon_2\}}(p) \subseteq U_1 \cap U_2$.

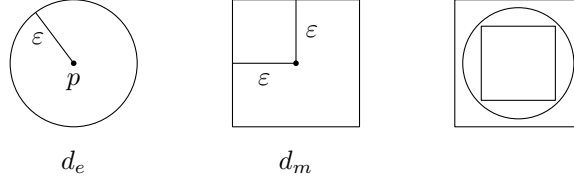
For $X = \mathbb{R}^n$ with $d_e(x, y) = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}$, we call $\mathcal{O}(d_e)$ the *standard topology* on $\mathbb{R}^n$. We have $\mathcal{O}_{\text{trivial}} \subsetneq \mathcal{O}(d_e) \subsetneq \mathcal{O}_{\text{discrete}}$ ($n \geq 1$). Note that $\bigcap_{n \geq 1} B_{1/n}(0) = \{0\}$ is not open.

Question 1. Is every topology induced by a metric? No: let $X = \{p, q\}$. For every metric d on X , the set $\{p\}$ is open, since $B_r(p) = \{p\}$ for $r = d(p, q) > 0$. Thus $(X, \mathcal{O}_{\text{trivial}})$ is *not metrizable*.

Question 2. If a topology is induced by a metric, is that metric unique? No: take $X = \mathbb{R}^n$ with $d_m(x, y) = \max\{|y_1 - x_1|, \dots, |y_n - x_n|\}$.

Claim. $\mathcal{O}(d_m) = \mathcal{O}(d_e)$.

Proof. $B_\varepsilon(p)$ for d_e ; $B_\varepsilon(p)$ for d_m :



□

Remark. Unlike in a metric space, in a topological space we can no longer express how close two points $x, y \in X$ are to one another. We can, however, still express that a sequence $(x_i)_{i=1}^\infty$ of points $x_i \in X$ approaches a point $y \in X$ arbitrarily closely:

$$\lim_{i \rightarrow \infty} x_i = y \quad :\Longleftrightarrow \quad \begin{array}{l} \text{For every open set } U \subseteq X \text{ with } y \in U \text{ there is} \\ \text{an } i_0 \text{ such that } x_i \in U \text{ for } i \geq i_0. \end{array}$$

Definition I.1.2. Let $(X, \mathcal{O})$ be a topological space, $p \in X$, and $B \subseteq X$.

- A subset $V \subseteq X$ is called a *neighborhood* of p if there is an open set $U \subseteq V$ with $p \in U$.
- The *closure* of B is the smallest closed set containing B , that is, $\overline{B} := \bigcap_{A \supseteq B \text{ closed}} A$.
- The *interior* of B is the largest open set contained in B , that is, $\overset{\circ}{B} := \bigcup_{U \subseteq B \text{ open}} U$.

Definition I.1.3. Let $(X, \mathcal{O})$ be a topological space and $B \subseteq X$ a subset. Then $\mathcal{O}|_B := \{U \cap B : U \subseteq X \text{ open}\}$ is the *subspace topology*.

Example. Let $X = \mathbb{R}$ with its standard topology and $B = [0, 2)$. Then $[0, 1) = (-1, 1) \cap [0, 2)$ is open in the subspace topology on B , but not in X .

$$[0, 1) = \begin{cases} [0, 1) & \text{with respect to } \mathcal{O}|_B, \\ (0, 1) & \text{with respect to the standard topology.} \end{cases}$$

$$\overline{[0, 1)} = [0, 1] \quad \text{with respect to both } \mathcal{O}|_B \text{ and the standard topology.}$$

Definition I.1.4. Let $(X, \mathcal{O})$ be a topological space. A collection of open sets $\mathcal{B} \subseteq \mathcal{O}$ is called a *basis* (respectively, *subbasis*) for the topology if every open set U is a union of members of $\mathcal{B}$ (respectively, of finite intersections of members of $\mathcal{B}$).

Examples.

- $\mathcal{B} = \{B_\varepsilon(p) : \varepsilon > 0, p \in \mathbb{R}^n\}$ is a basis for the standard topology on $\mathbb{R}^n$, and $\mathcal{B}' = \{B_\varepsilon(p) : \varepsilon \in \mathbb{Q}_{>0}, p \in \mathbb{Q}^n\}$ is a countable basis.
- $\mathcal{B} = \{\{x\} : x \in X\}$ is a basis for the discrete topology on X .

Let X be a set and $\mathcal{S}$ a collection of subsets of X . Then

$$\mathcal{O}(\mathcal{S}) = \{U \subseteq X : U \text{ is a union of finite intersections of members of } \mathcal{S}\}$$

is the topology generated by $\mathcal{S}$, with subbasis $\mathcal{S}$.

Definition I.1.5. Let $(X, \mathcal{O})$ be a topological space and $x_0 \in X$. A collection $\mathcal{U}$ of neighborhoods of x_0 is called a *neighborhood basis* at x_0 if every neighborhood V of x_0 contains a neighborhood $U \in \mathcal{U}$.

Example. If (X, d) is a metric space and $x_0 \in X$, then $\mathcal{U} = \{B_r(x_0) : r \in \mathbb{Q}_{>0}\}$ is a neighborhood basis at x_0 . Thus metric spaces satisfy the

First countability axiom: Every point has a countable neighborhood basis. As we saw above, $(\mathbb{R}^n, d_e)$ even satisfies the

Second countability axiom: The topology has a countable basis.

Definition I.1.6. A topological space $(X, \mathcal{O})$ is called *separable* if it has a countable subset $A \subseteq X$ that is *dense*, that is, $\overline{A} = X$.

Lemma I.1.7. Second countable $\Rightarrow$ separable.

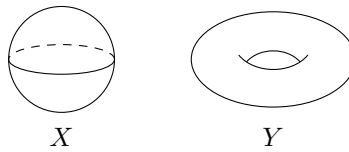
Proof. Let $\mathcal{B}$ be a countable basis. Choose one point in each $\emptyset \neq U \in \mathcal{B}$, obtaining a set $A \subseteq X$. Since $X \setminus \overline{A}$ is open and $\mathcal{B}$ is a basis, there are $U_i \in \mathcal{B}$, $i \in I$, with $X \setminus \overline{A} = \bigcup_{i \in I} U_i$. Thus $U_i \cap A = \emptyset$, so $U_i = \emptyset$, and hence $X \setminus \overline{A} = \emptyset$. $\square$

I.2. Sums and Products

Let $X \sqcup Y$ denote the disjoint union of the sets X and Y .

Definition I.2.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Then $(X \sqcup Y, \mathcal{O}(\mathcal{O}_X \sqcup \mathcal{O}_Y))$ is the *topological sum* of $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$.

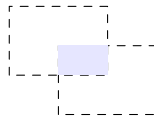
Note. $X, Y \subseteq X \sqcup Y$ are both open and closed.



Definition I.2.2. The *product* of $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is

$$(X \times Y, \mathcal{O}(\{U \times V : U \in \mathcal{O}_X, V \in \mathcal{O}_Y\})).$$

Note. We have $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$, so $\{U \times V : U \in \mathcal{O}_X, V \in \mathcal{O}_Y\}$ is a basis, but in general not a topology.



Now let $(X_i, \mathcal{O}_i)$, $i \in I$, be an arbitrary family of topological spaces.

Definition I.2.3. The *topological sum* of $(X_i, \mathcal{O}_i)_{i \in I}$ is $(\bigsqcup_{i \in I} X_i, \mathcal{O}(\bigsqcup_{i \in I} \mathcal{O}_i))$.

Definition I.2.4. The *product* of $(X_i, \mathcal{O}_i)_{i \in I}$ is

$$\left(\prod_{i \in I} X_i, \mathcal{O} \left(\{ \text{pr}_i^{-1}(U) : U \in \mathcal{O}_i, i \in I \} \right) \right),$$

where $\text{pr}_i : \prod_{j \in I} X_j \rightarrow X_i$ denotes the *projection* onto the i th factor.

Note. For $I = \{1, 2\}$, we have $\text{pr}_1^{-1}(U_1) = U_1 \times X_2$, $\text{pr}_2^{-1}(U_2) = X_1 \times U_2$, and $U_1 \times X_2 \cap X_1 \times U_2 = U_1 \times U_2$, so we recover the previous definition.

Why not use $\mathcal{O}' = \mathcal{O}(\{\prod_{i \in I} U_i : U_i \in \mathcal{O}_i\})$? Later: the universal property of the product. For now:

Example. Let $X_i = (\{0, 1\}, \mathcal{O}_{\text{discrete}})$, $i \in \mathbb{N}$. Then $\mathcal{O}'$ contains the singleton subsets of $\prod_{i \in \mathbb{N}} \{0, 1\}$, so it is discrete. Thus the sequence $x_k = (\underbrace{1, \dots, 1}_k, 0, 0, \dots)$ does not converge with

respect to $\mathcal{O}'$.

In the product topology, the *cylinder sets*

$$\{U_1 \times U_2 \times \dots : U_i \in \mathcal{O}_i, U_i = X_i \text{ for all but finitely many } i\}$$

form a basis, and hence $\lim_{k \rightarrow \infty} x_k = (1, 1, \dots)$.

I.3. Continuous Maps

Definition I.3.1. A map $f : X \rightarrow Y$ between topological spaces is called *continuous* if the preimage $f^{-1}(U) \subseteq X$ of every open set $U \subseteq Y$ is open.

Theorem I.3.2. A map $f : M \rightarrow N$ between metric spaces is continuous with respect to the induced topologies if and only if it is continuous in the ε - δ sense.

Proof. Let f be continuous as above, let $x \in M$, and let $\varepsilon > 0$. Then $U := f^{-1}(B_\varepsilon(f(x))) \subseteq M$ is open and $x \in U$, so there is a $\delta > 0$ with $B_\delta(x) \subseteq U$. Hence $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$.

Conversely, let f be continuous in the ε - δ sense and let $V \subseteq N$ be open. We must show that $f^{-1}(V) \subseteq M$ is open. Take $x \in f^{-1}(V)$. Then $f(x) \in V$, and there is an $\varepsilon > 0$ with $B_\varepsilon(f(x)) \subseteq V$. Thus there is a $\delta > 0$ with $f(B_\delta(x)) \subseteq B_\varepsilon(f(x)) \subseteq V$, that is, $B_\delta(x) \subseteq f^{-1}(V)$. $\square$

Lemma I.3.3. Let $\mathcal{B}$ be a subbasis for Y . Then $f : X \rightarrow Y$ is continuous if and only if $f^{-1}(U)$ is open for every $U \in \mathcal{B}$.

Proof. Let $U \subseteq Y$ be open. Then $U = \bigcup_{i \in I} U_1^i \cap \dots \cap U_{n_i}^i$ for suitable $U_j^i \in \mathcal{B}$, and $f^{-1}(U) = \bigcup_{i \in I} f^{-1}(U_1^i) \cap \dots \cap f^{-1}(U_{n_i}^i)$ is open. $\square$

Homework. Further characterizations.

Let $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ be continuous. Define

$$f \sqcup g : X \sqcup Y \longrightarrow Z, \quad \begin{cases} x \in X \mapsto f(x), \\ y \in Y \mapsto g(y). \end{cases}$$

Let $U \subseteq Z$ be open. Then $(f \sqcup g)^{-1}(U) = f^{-1}(U) \cup g^{-1}(U) \subseteq X \sqcup Y$ is open, so $f \sqcup g$ is continuous.

The *inclusions* $i_X : X \rightarrow X \sqcup Y$ and $i_Y : Y \rightarrow X \sqcup Y$ are also continuous: $i_X^{-1}(U \cup V) = U$.

This proves the *universal property* of the topological sum (all maps are continuous):

$$\begin{array}{ccccc} X & \xrightarrow{i_X} & X \sqcup Y & \xleftarrow{i_Y} & Y \\ & \searrow f & \downarrow \exists! & \swarrow g & \\ & & Z & & \end{array}$$

Let $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ be continuous. Define

$$f \times g : Z \longrightarrow X \times Y, \quad z \mapsto (f(z), g(z)).$$

Let $U \subseteq X$ and $V \subseteq Y$ be open. Then $(f \times g)^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V) \subseteq Z$ is open, so $f \times g$ is continuous.

The *projections* $\text{pr}_X : X \times Y \rightarrow X$ and $\text{pr}_Y : X \times Y \rightarrow Y$ are continuous. *Universal property* of the product:

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow f & \downarrow \exists! & \searrow g & \\ X & \xleftarrow{\text{pr}_X} & X \times Y & \xrightarrow{\text{pr}_Y} & Y \end{array}$$

Infinite version: for $\{f_i : Z \rightarrow X_i\}$ there is a unique $f : Z \rightarrow \prod_i X_i$ such that

$$\begin{array}{ccc} & Z & \\ f_i \swarrow & \downarrow f & \\ X_i & \xleftarrow{\text{pr}_{X_i}} & \prod_i X_i \end{array}$$

Further properties.

- If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous, then so is $g \circ f : X \rightarrow Z$.
- If X is a topological space, then id_X is continuous.
- Constant maps are continuous.
- Let X be a topological space, $B \subseteq_i X$, and $f : X \rightarrow Y$ continuous. Then $f|_B : B \rightarrow Y$ is continuous. ($f|_B = f \circ i$)
- A map $f : X \rightarrow Y$ is always continuous if X is discrete or Y is indiscrete.

Definition I.3.4. A continuous bijection $f : X \rightarrow Y$ whose inverse is continuous is called a *homeomorphism*.

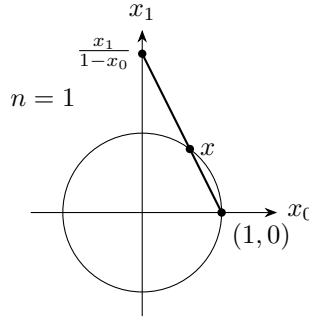
If there is a homeomorphism $f : X \rightarrow Y$, then X and Y are called *homeomorphic* (notation: $X \cong Y$). Examples:

- $(0, 1) \cong (1, \infty)$, $x \mapsto \frac{1}{x}$, $y \mapsto \frac{1}{y}$.
- $(0, 1) \cong (-\infty, \infty)$, $x \mapsto \tan(\pi(x - \frac{1}{2}))$, $y \mapsto \frac{1}{\pi} \arctan(y) + \frac{1}{2}$.
- $S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$. $S^n \setminus \{(1, 0, \dots, 0)\} \cong \mathbb{R}^n$,

$$(x_0, x_1, \dots, x_n) \mapsto \left(\frac{x_1}{1-x_0}, \dots, \frac{x_n}{1-x_0} \right),$$

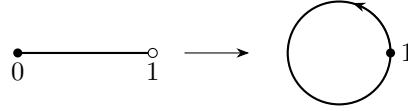
$$(y_1, \dots, y_n) \mapsto \left(\frac{s^2-1}{s^2+1}, \frac{2y_1}{s^2+1}, \dots, \frac{2y_n}{s^2+1} \right) \quad \text{where } s^2 = y_1^2 + \dots + y_n^2.$$

“Stereographic projection.”



Warning. A continuous bijection need not be a homeomorphism. Examples:

- $\text{id}_{\mathbb{R}} : (\mathbb{R}, \mathcal{O}_{\text{discrete}}) \longrightarrow (\mathbb{R}, \mathcal{O}_{\text{trivial}})$.
- $[0, 1) \longrightarrow S^1$, $t \mapsto \exp(2\pi it)$.



I.4. Connectedness

Definition I.4.1. A topological space $\emptyset \neq X$ is called *connected* if it cannot be written as a disjoint union of two nonempty open sets.

($\Leftrightarrow$ Exactly two subspaces, namely $\emptyset$ and X , are both open and closed.)

Examples.

- $\mathbb{R}$ is connected (see below).
- $\mathbb{R} \sqcup \mathbb{R}$ is disconnected.

Theorem I.4.2. Let $I \subseteq \mathbb{R}$. Then I is connected $\Leftrightarrow I$ is an interval.

Proof. “ $\Rightarrow$ ”: Suppose $I \subseteq \mathbb{R}$ is not an interval. Then there are $a < s < b$ with $a, b \in I$ and $s \notin I$. Thus $((-\infty, s) \cap I) \cup ((s, \infty) \cap I)$ expresses I as a disjoint union of nonempty open sets.

“ $\Leftarrow$ ”: Suppose $I = A \cup B$, where $A, B \subseteq I$ are open in I , $A, B \neq \emptyset$, and $A \cap B = \emptyset$. Choose $a \in A$ and $b \in B$, WLOG with $a < b$, and set $s := \inf\{x \in B : x > a\}$. Let U be a neighborhood of s . Then $U \cap B \neq \emptyset$. We also have $U \cap A \neq \emptyset$: indeed, $s \geq a$, and if

$s > a$, then $(a, s) \cap B = \emptyset$; since $(a, s) \subseteq I = A \cup B$, it follows that $(a, s) \subseteq A$. Thus s is an *accumulation point* of both A and B . We have $s \in I = A \cup B$, WLOG $s \in A$. Since $A \subseteq I$ is open, there is an open set $U \subseteq \mathbb{R}$ with $A = U \cap I$. Since $s \in U$, we have $U \cap B \neq \emptyset$, so there is a $b' \in U \cap B \subseteq U \cap I = A$, that is, $b' \in A \cap B$, a contradiction. $\square$

Definition I.4.3. A topological space $\emptyset \neq X$ is called *path-connected* if, for every $x, y \in X$, there is a continuous map $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

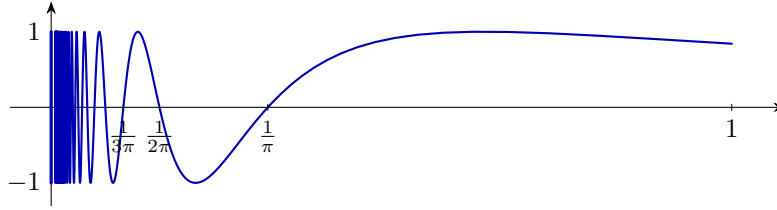
Theorem I.4.5. X path-connected $\Rightarrow X$ connected.

Proof. Let X be path-connected. If X is not connected, then $X = A \cup B$ with $A, B \subseteq X$ open, $A, B \neq \emptyset$, and $A \cap B = \emptyset$. Choose $a \in A$ and $b \in B$. There is a continuous map $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) = a$ and $\gamma(1) = b$. Then $\gamma^{-1}(A) \cup \gamma^{-1}(B)$ expresses $[0, 1]$ as a disjoint union of nonempty open sets, contradicting the [previous theorem](#). $\square$

Remark. The converse is false in general: Let

$$\bar{S} = \{(x, \sin(1/x)) : x \in (0, 1]\} \cup \{(0, y) : y \in [-1, 1]\} \subseteq \mathbb{R}^2$$

be the closed *topologist's sine curve*:



Lemma I.4.6. Let X be a topological space and $A \subseteq X$ connected (in the subspace topology). Then $\bar{A}$ is connected.

Proof. Suppose $\bar{A} = U \cup V$ with $U, V \subseteq \bar{A}$ open and $U \cap V = \emptyset$. We must show that $U = \emptyset$ or $V = \emptyset$. There are open sets $U_0, V_0 \subseteq X$ with $U_0 \cap \bar{A} = U$ and $V_0 \cap \bar{A} = V$. Thus $(A \cap U_0) \cup (A \cap V_0)$ expresses A as a disjoint union of open sets. Since A is connected, WLOG $A \cap U_0 = \emptyset$. Since $V \subseteq \bar{A}$ is closed, there is a closed set $V_1 \subseteq X$ with $V = V_1 \cap \bar{A}$. But $A = A \cap V_0 \subseteq \bar{A} \cap V_0 = V = V_1 \cap \bar{A} \subseteq V_1$. Since V_1 is closed, it follows that $\bar{A} \subseteq V_1$. Thus $V = V_1 \cap \bar{A} = \bar{A}$, so $U = \emptyset$. $\square$

Lemma I.4.7. Let $f : X \rightarrow Y$ be continuous and X connected (respectively, path-connected). Then $f(X) \subseteq Y$ is connected (respectively, path-connected).

Proof. Suppose $f(X) = (U \cap f(X)) \cup (V \cap f(X))$ is a disjoint union with $U, V \subseteq Y$ open. Then $X = f^{-1}(U) \cup f^{-1}(V)$ is a disjoint union of open sets, so WLOG $f^{-1}(U) = \emptyset \Rightarrow U \cap f(X) = \emptyset$. Let $x, y \in f(X)$. Choose $x', y' \in X$ with $f(x') = x$ and $f(y') = y$. Choose $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) = x'$ and $\gamma(1) = y'$. Then $f \circ \gamma$ joins x to y . $\square$

- For $f : (0, 1] \rightarrow \mathbb{R}^2$, $x \mapsto (x, \sin(1/x))$, we have $\bar{S} = \overline{f((0, 1])}$, so $\bar{S}$ is connected.
- Suppose there were a continuous map $\gamma : [0, 1] \rightarrow \bar{S}$ with $\gamma(0) = (0, 1)$ and $\gamma(1) = (\frac{1}{\pi}, 0)$. Set $s := \sup\{t \in [0, 1] : \gamma(t) \in \{0\} \times [-1, 1]\}$. Since γ is continuous, there is a $\delta > 0$ with

$$\gamma([s, s + \delta]) \subseteq B_{1/2}(\gamma(s)). \quad (*)$$

Let $\text{pr}_{x/y} : \mathbb{R}^2 \rightarrow \mathbb{R}$ denote the projections onto the x/y -axis. Then $\text{pr}_x \circ \gamma([s, s + \delta])$ is connected, hence an interval I , and $[0, x] \subseteq I$ with $x := \text{pr}_x(\gamma(s + \delta)) > 0$. But this implies $\text{pr}_y \circ \gamma([s, s + \delta]) = [-1, 1]$, contradicting $(*)$.

Thus $\overline{S}$ is not path-connected. In particular, the closure of a path-connected set need not be path-connected.

Proposition I.4.8. Let $\{A_i\}_{i \in I}$ be a family of connected (respectively, path-connected) subspaces of a topological space X , with $\bigcap_{i \in I} A_i \neq \emptyset$. Then $\bigcup_{i \in I} A_i$ is also connected (respectively, path-connected).

Proof. Choose $x \in \bigcap_{i \in I} A_i$. Consider $\bigcup_{i \in I} A_i = U \cup V$. WLOG $x \in U$. Let $y \in V$ and $y \in A_{i_0}$. We obtain $A_{i_0} = (A_{i_0} \cap U) \cup (A_{i_0} \cap V)$. The path-connected case is clear. $\square$

Definition I.4.9. Let X be a topological space and $x \in X$. The union of all connected (respectively, path-connected) subspaces containing x is called the *connected component* (respectively, *path component*) of x .

Note. Every space is the disjoint union of its connected components, and also of its path components.

The space $\overline{S}$ has one connected component and two path components.

I.5. The Hausdorff Axiom

Definition I.5.1. A topological space X is called *Hausdorff* if any two distinct points in X have disjoint neighborhoods.

Examples.

- Every metric space (X, d) is Hausdorff: if $x, y \in X$ with $x \neq y$, then $d(x, y) := r > 0$ and $B_{r/2}(x) \cap B_{r/2}(y) = \emptyset$.
- Every discrete space is Hausdorff.

Counterexamples.

- If $|X| \geq 2$, then $(X, \mathcal{O}_{\text{trivial}})$ is not Hausdorff.
- Let k be a field. The closed sets of the *Zariski topology* on k^n are

$$\{\{x \in k^n : p(x) = 0 \text{ for all } p \in S\} : S \subseteq k[x_1, \dots, x_n]\}.$$

$\rightsquigarrow$ Algebraic geometry.

Note.

- In a Hausdorff space, a convergent sequence has a unique limit: neighborhoods of two limits contain all but finitely many terms of the sequence, so they are not disjoint.
- If X is Hausdorff, then so is every subspace $A \subseteq X$.
- Let $X, Y \neq \emptyset$. Then the following are equivalent:

1. X, Y are Hausdorff.
2. $X \sqcup Y$ is Hausdorff.
3. $X \times Y$ is Hausdorff.

I.6. Compactness

Definition I.6.1. A topological space is called *compact* if every open cover has a finite subcover. (If $X = \bigcup_{i \in I} U_i$ with the U_i open, then there are $i_1, \dots, i_n \in I$ with $X = U_{i_1} \cup \dots \cup U_{i_n}$.)

Examples.

- $(0, 1]$ and $[0, \infty)$ are not compact:

$$\bigcup_{n \geq 2} (\frac{1}{n}, 1] = (0, 1], \quad \bigcup_{n \geq 1} [0, n] = [0, \infty).$$

- Let X be discrete. Then X is compact $\Leftrightarrow |X| < \infty$.
- $(X, \mathcal{O}_{\text{trivial}})$ is compact.
- The unit interval $[0, 1]$ is compact.

Proof. Let $[0, 1] = \bigcup_{i \in I} U_i$. We claim that there is a $\delta > 0$ such that every subinterval $I \subseteq [0, 1]$ of length δ is contained in some U_i . If not, let x_n be the midpoint of a subinterval of length $\frac{1}{n}$ that is not contained in any U_i . By Bolzano–Weierstrass, $\lim_{k \rightarrow \infty} x_{n_k} = x$. But $x \in U_{i_0} \subseteq [0, 1]$ with U_{i_0} open, a contradiction. Now choose a finite cover of $[0, 1]$ by $\delta/2$ -balls. We obtain $[0, 1] = U_{i_1} \cup \dots \cup U_{i_n}$. $\square$

Compactness often allows us to deduce a global property from its local counterpart:

Lemma I.6.2. Let $f : (X, \mathcal{O}) \rightarrow (Y, d)$ be continuous and X compact. Then $f(X)$ is bounded.

Proof. For each $x \in X$, there is an open neighborhood U_x with $f(U_x) \subseteq B_1(f(x))$. Since $X = U_{x_1} \cup \dots \cup U_{x_n}$ for some $x_1, \dots, x_n$, we have $f(X) \subseteq B_1(f(x_1)) \cup \dots \cup B_1(f(x_n))$. $\square$

Theorem I.6.3. Let X be a topological space.

- (i) If X is compact and $A \subseteq X$ is closed, then A is compact.
- (ii) If X is Hausdorff and $A \subseteq X$ is compact, then A is closed.

Proof. (i) Let $A = \bigcup_{i \in I} (U_i \cap A)$ with $U_i \subseteq X$ open. Then $X = X \setminus A \cup \bigcup_{i \in I} U_i$ is an open cover of X .

(ii) Let $A \subsetneq X$ be compact. Choose $x \in X \setminus A$. For each $y \in A$, let U_y and V_y be neighborhoods of x and y with $U_y \cap V_y = \emptyset$. Consider $A = \bigcup_{y \in A} (V_y \cap A)$. Since A is compact, there are $y_1, \dots, y_n \in A$ with $A \subseteq V_{y_1} \cup \dots \cup V_{y_n}$. Set $U = U_{y_1} \cap \dots \cap U_{y_n}$ and $V = V_{y_1} \cup \dots \cup V_{y_n}$. Then $U \cap V = \emptyset$; in particular, $U \cap A = \emptyset$. Thus x is an *interior point* of $X \setminus A$. Since $x \in X \setminus A$ was arbitrary, $X \setminus A$ is open, so A is closed. $\square$

Theorem I.6.4. Let $f : X \rightarrow Y$ be continuous.

- (i) If X is compact, then so is $f(X)$.
- (ii) If f is bijective, X is compact, and Y is Hausdorff, then f is a homeomorphism.

Proof. (i) Let $f(X) = \bigcup_{i \in I} (U_i \cap f(X))$. Then

$$X = \bigcup_{i \in I} f^{-1}(U_i) = f^{-1}(U_{i_1}) \cup \cdots \cup f^{-1}(U_{i_n}) = f^{-1}(U_{i_1} \cup \cdots \cup U_{i_n}),$$

so $f(X) = (U_{i_1} \cap f(X)) \cup \cdots \cup (U_{i_n} \cap f(X))$.

(ii) Let f be bijective. Then:

$$f^{-1} \text{ continuous} \iff (A \subseteq X \text{ closed} \Rightarrow f(A) \subseteq Y \text{ closed}).$$

Let $A \subseteq X$ be closed. Since X is compact, A is compact, so $f(A)$ is compact. Since Y is Hausdorff, $f(A)$ is closed. $\square$

Note. Let $X, Y \neq \emptyset$. Then the following are equivalent:

- 1. X, Y are compact.
- 2. $X \sqcup Y$ is compact.
- 3. $X \times Y$ is compact.

Proof. 1.) $\Leftrightarrow$ 2.): clear. 3.) $\Rightarrow$ 1.): apply the [last theorem](#) to $\text{pr}_X : X \times Y \rightarrow X$ and $\text{pr}_Y : X \times Y \rightarrow Y$.

1.) $\Rightarrow$ 3.): Let $X \times Y = \bigcup_{i \in I} U_i$. For each $(x, y) \in X \times Y$, there are $i(x, y) \in I$ and open sets $V_{(x, y)} \subseteq X$, $W_{(x, y)} \subseteq Y$ with $(x, y) \in V_{(x, y)} \times W_{(x, y)} \subseteq U_{i(x, y)}$. Fix x . Then $Y = \bigcup_{y \in Y} W_{(x, y)} = W_{(x, y_1(x))} \cup \cdots \cup W_{(x, y_{n_x}(x))}$. Set $V_x := V_{(x, y_1(x))} \cap \cdots \cap V_{(x, y_{n_x}(x))}$. Then $X = \bigcup_{x \in X} V_x = V_{x_1} \cup \cdots \cup V_{x_m}$ and

$$X \times Y = \bigcup_{i=1}^m \bigcup_{j=1}^{n_{x_i}} V_{x_i} \times W_{(x_i, y_j(x_i))}.$$

$\square$

Remark. For an infinite family $\{X_i\}$ of compact spaces, the product $\prod_{i \in I} X_i$ is also compact (Tychonoff's theorem).

Theorem (Heine–Borel) I.6.5. Let $K \subseteq \mathbb{R}^n$. Then the following are equivalent:

- (i) K is compact.
- (ii) K is bounded and closed.

Proof. (i) $\Rightarrow$ (ii): Theorem I.6.3(ii) and the [lemma](#) with $f = \|\cdot\|$.

(ii) $\Rightarrow$ (i): For $r \gg 0$, we have $K \subseteq [-r, r]^n$. By the example, the note, and Theorem I.6.3(i), the set K is compact. $\square$

I.7. The Quotient Topology

Let “ $\sim$ ” be an equivalence relation on the topological space X . We want to equip the set $X/\sim$ of equivalence classes with a topology satisfying the following universal property: If $f : X \rightarrow Y$ is continuous and $f(x_1) = f(x_2)$ whenever $x_1 \sim x_2$, then

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ p \downarrow & \nearrow \exists! \bar{f} & \\ X/\sim & & \end{array} \quad (\text{All maps continuous})$$

Thus $X/\sim$ must carry the finest topology for which $p : X \rightarrow X/\sim$ is still continuous, that is,

$$U \subseteq X/\sim \text{ open} \iff p^{-1}(U) \subseteq X \text{ open},$$

because

- p must be continuous (so the topology cannot be finer).
- (The topology cannot be coarser either.)

$$\begin{array}{ccc} X & \xrightarrow{p} & X/\sim \\ p \downarrow & \nearrow \text{id} & \\ X/\sim & & \end{array}$$

Observation. Let $f : X \rightarrow Y$ be continuous and surjective. Then $x_1 \sim x_2 \Leftrightarrow f(x_1) = f(x_2)$ defines an equivalence relation on X for which $\bar{f}$ is a continuous bijection. Moreover, $\bar{f}$ is a homeomorphism if and only if Y carries the quotient topology:

Theorem I.7.1. For a surjective map $f : X \rightarrow Y$, the following are equivalent:

- (i) $\bar{f} : X/\sim \xrightarrow{\cong} Y$.
- (ii) For every map $g : Y \rightarrow Z$ of sets, $g \circ f$ is continuous if and only if g is continuous.
- (iii) $U \subseteq Y$ is open $\Leftrightarrow f^{-1}(U) \subseteq X$ is open.
- (iv) $A \subseteq Y$ is closed $\Leftrightarrow f^{-1}(A) \subseteq X$ is closed.

Definition I.7.2. A surjective map $f : X \rightarrow Y$ satisfying one (and hence all) of these properties is called a *quotient map*, or an *identification map*.

Proof. (iii) $\Leftrightarrow$ (iv): clear. (i) $\Rightarrow$ (ii): Let $g : Y \rightarrow Z$ be a map of sets. If g is continuous, then so is $g \circ f = g \circ \bar{f} \circ p$. If $g \circ f$ is continuous, then so is $g \circ \bar{f}$ by the universal property. Thus $g = (g \circ \bar{f}) \circ \bar{f}^{-1}$ is continuous. (iii) $\Rightarrow$ (i): By (iii), f is continuous, so $\bar{f}$ is a continuous bijection by the universal property. Let $U \subseteq X/\sim$ be open. Then $V = p^{-1}(U) \subseteq X$ is open and $f^{-1}(f(V)) = V$, so $\bar{f}(U) = f(V) \subseteq Y$ is open by (iii).

(ii) $\Rightarrow$ (iii): Consider $(Y, \mathcal{O}_Y)$ and $(Y, \mathcal{O}'_Y)$, where $\mathcal{O}'_Y$ is defined by (iii). Since we already know (iii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (ii), condition (ii) also holds for $(Y, \mathcal{O}'_Y)$. Applying (ii) to $g = \text{id}_{(Y, \mathcal{O}_Y)}$ and $g = \text{id}_{(Y, \mathcal{O}'_Y)}$ shows that f is continuous for both topologies. Thus $\text{id}_Y \circ f = f$ is continuous for both $\text{id}_Y : (Y, \mathcal{O}_Y) \rightarrow (Y, \mathcal{O}'_Y)$ and $\text{id}_Y : (Y, \mathcal{O}'_Y) \rightarrow (Y, \mathcal{O}_Y)$. By (ii), both identity maps are therefore continuous, that is, $\mathcal{O}_Y = \mathcal{O}'_Y$. $\square$

Note. Injective quotient maps are homeomorphisms.

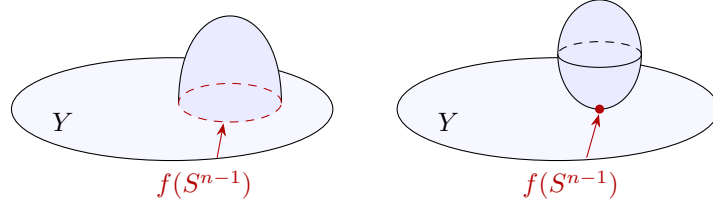
Examples. 1.) Gluing spaces.

For $A \subseteq X$ and $f : A \rightarrow Y$ continuous, let “ $\sim$ ” be the finest equivalence relation on $X \sqcup Y$ for which $a \sim f(a)$ for every $a \in A$. Then

$$X \sqcup_f Y := X \sqcup Y / \sim$$

is obtained by *attaching* X to Y along the *attaching map* f .

Example. $S^{n-1} \subseteq D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, $f : S^{n-1} \rightarrow Y$.



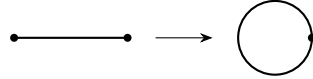
“Attaching an n -cell.”

Lemma I.7.3. There is a canonical inclusion $Y \subseteq X \sqcup_f Y$ (but in general not $X \subseteq X \sqcup_f Y$).

Proof. Injective ✓ Continuous ✓. Let $U \subseteq Y$ be open. We must show that $p(U) \subseteq p(Y)$ is open. Since $f^{-1}(U) \subseteq A$ is open, there is an open set $V \subseteq X$ with $f^{-1}(U) = V \cap A$. Thus $p^{-1}(p(V \sqcup U)) = V \sqcup U \subseteq X \sqcup Y$ is open, so $p(V \sqcup U) \subseteq X \sqcup_f Y$ is open, and $p(U) = p(V \sqcup U) \cap p(Y)$. $\square$

2.) Collapsing a subspace to a point. This is the special case of attaching with $Y = \bullet$. Notation: $X/A := X \sqcup_f \bullet$ with $f : A \rightarrow \bullet$.

Example. We claim that $[0, 1]/\{0, 1\} \cong S^1$.



Proof.

$$\begin{array}{ccc}
 [0, 1] \sqcup \bullet & \xrightarrow{f} & S^1 \\
 \downarrow p & \nearrow \bar{f} & \\
 [0, 1]/\{0, 1\} & &
 \end{array}$$

$\bullet \xrightarrow{1} \bullet$
 $t \mapsto e^{2\pi i t}$

$\bar{f}$ is a continuous bijection.

- $[0, 1]/\{0, 1\} = p([0, 1] \sqcup \bullet)$ is compact.
- $S^1 \subseteq \mathbb{R}^2$ is Hausdorff.

$\square$

Similarly, $D^n/S^{n-1} \cong S^n$.

Note:

- For $x_0 \in X$, we have $X/\{x_0\} \cong X$.
- $X/\emptyset \cong X \sqcup \bullet$; in particular, $\emptyset/\emptyset = \bullet$.

3.) Mapping tori and self-identifications. Let $f : X \xrightarrow{\cong} X$ be a homeomorphism. Then

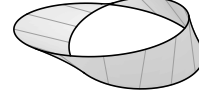
$$T_f := X \times [0, 1] / (x, 1) \sim (f(x), 0)$$

is called the *mapping torus* of f . Examples:

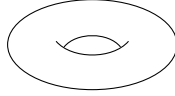
- $f = \text{id}_{[-1,1]}$, $T_f =$ “cylinder.”



- $f = -\text{id}_{[-1,1]}$, $T_f =$ “Möbius strip.”



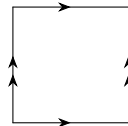
- $f = \text{id}_{S^1}$, $T_f =$ torus.



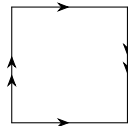
- $f : S^1 \rightarrow S^1$, $(x, y) \mapsto (-x, y)$, $T_f =$ “Klein bottle.”



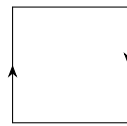
Self-identifications are usually represented schematically:



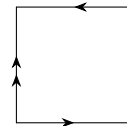
Torus



Klein
bottle



Möbius strip



$\mathbb{R}P^2$

$$D^2 / S^1 \ni x \sim -x \in S^1 \cong S^2 / x \sim -x \cong \mathbb{R}P^2 \quad (\text{Boy's surface}).$$

4.) Orbit spaces of group actions.

Definition I.7.4. A *topological group* is a group G that is also a topological space, such that both $G \times G \rightarrow G$, $(g_1, g_2) \mapsto g_1 g_2$, and $G \rightarrow G$, $g \mapsto g^{-1}$ are continuous.

Remark. One usually also requires the space to be Hausdorff.

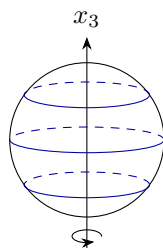
Definition I.7.5. An action $G \curvearrowright X$ of a topological group G on a topological space X is called continuous if $G \times X \rightarrow X$, $(g, x) \mapsto gx$ is continuous.

For $x \in X$, the set $Gx = \{gx : g \in G\}$ is called the *orbit* of x . “Lying in the same orbit” defines an equivalence relation “ $\sim$ ” on X , and

$$X/G := X/\sim$$

is called the *orbit space* of $G \curvearrowright X$.

Example. $\text{SO}(2) \curvearrowright S^2$ by rotation about the x_3 -axis:



$$\begin{array}{ccc} S^2 & \xrightarrow{\text{pr}_3} & [-1, 1] \\ \downarrow & \nearrow \cong & \\ S^2/\text{SO}(2) & & \end{array}$$

For $x \in X$, the subgroup $G_x = \{g \in G : gx = x\}$ is called the *stabilizer* of x .

Lemma I.7.6. Let G be compact and X Hausdorff. Then $G/G_x \xrightarrow{\cong} Gx$, $gG_x \mapsto gx$.

Proof. Well-defined: ✓ Surjective: ✓ Injective:

$$gx = hx \Rightarrow h^{-1}gx = x \Rightarrow h^{-1}g \in G_x \Rightarrow gG_x = hG_x.$$

Continuous: consider $G \rightarrow G/G_x \rightarrow Gx$, $g \mapsto gx$, and apply part (ii) of the [theorem above](#). Now G/G_x is compact and $Gx \subseteq X$ is Hausdorff. It follows that $G/G_x \cong Gx$. $\square$

Warning 1. Compactness, connectedness, and path-connectedness pass to quotients; the Hausdorff property generally does not.

Warning 2. A quotient map need not be *open* or *closed*. For example, $f : [0, 3) \rightarrow S^1$, $t \mapsto \exp(\pi it)$ is a quotient map. But $f([0, 1)) = f([2, 3)) \subseteq S^1$ is neither open nor closed.

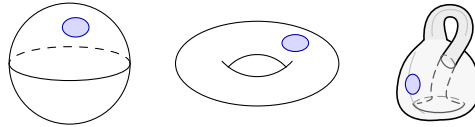
II. Manifolds and Surfaces

II.1. Manifolds

Definition II.1.1. An n -dimensional *manifold* is a second-countable Hausdorff space M that is *locally Euclidean*: every $p \in M$ has a neighborhood $U_p \subseteq M$ with $U_p \cong \mathbb{R}^n$.

Remark. 1.) We have

$$\{x \in \mathbb{R}^n : \|x\| < 1\} \xrightarrow{\cong} \mathbb{R}^n, \quad x \mapsto \frac{x}{1 - \|x\|}.$$



2.) A homeomorphism $\varphi : U_p \xrightarrow{\cong} \mathbb{R}^n$ is called a *chart* around p . Another chart $\psi : V_q \xrightarrow{\cong} \mathbb{R}^n$ is *smoothly compatible* with φ if

$$\psi \circ \varphi^{-1} \Big|_{\varphi(U_p \cap V_q)} : \varphi(U_p \cap V_q) \longrightarrow \psi(U_p \cap V_q)$$

is a *diffeomorphism*, that is, smooth (C^∞) with a smooth inverse. An *atlas* is a collection of charts covering M . A *smooth structure* on M is a maximal atlas of smoothly compatible charts.

$\rightsquigarrow$ differential topology

Theorem (Kervaire–Milnor). There are 28 essentially different smooth structures on S^7 .

A smooth structure allows us to define the *tangent bundle*, on which we study geometric structures (Riemannian metrics, symplectic forms, ...) $\rightsquigarrow$ differential geometry.

Examples.

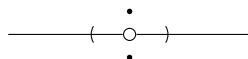
1. Every discrete space with at most countably many points is a 0-dimensional manifold.
2. $\mathbb{R}$ and S^1 are 1-dimensional manifolds.
3. Let M be an m -dimensional manifold and N an n -dimensional manifold. Then $M \times N$ is an $(m + n)$ -dimensional manifold.

$$S^1 \times S^1 = \bigcirc$$

4. If M and N are n -dimensional manifolds, then $M \sqcup N$ is an n -dimensional manifold.

Counterexamples.

1. $\overline{S}$ is not locally Euclidean.
2. $(\mathbb{R}, \mathcal{O}_{\text{discrete}})$ is locally Euclidean and Hausdorff, but not second-countable.
3. $\mathbb{R} \sqcup \mathbb{R} /_{\iota_1(x) \sim \iota_2(x) \text{ for } x \neq 0}$ is locally Euclidean and second-countable, but not Hausdorff.



Theorem II.1.2. $\mathbb{R}P^n$ is an n -dimensional manifold.

Proof. Second-countable: $p : S^n \rightarrow \mathbb{R}P^n$ is open, since for open $U \subseteq S^n$, the set $p^{-1}(p(U)) = U \cup -U$ is open. Continuous open surjections $f : X \rightarrow Y$ map bases to bases: let $\mathcal{B}$ be a basis of X and let $U \subseteq Y$ be open. Write $f^{-1}(U) = \bigcup_{V \in \mathcal{B}' \subseteq \mathcal{B}} V$. Then

$$U = f(f^{-1}(U)) = \bigcup_V f(V).$$

Hausdorff: let $[x], [y] \in \mathbb{R}P^n$, $[x] \neq [y]$. Then there are disjoint open neighborhoods $U, V \subseteq S^n$ of x, y such that $\pm U, \pm V$ are disjoint, so $p(U), p(V)$ are disjoint neighborhoods of $[x], [y]$.

Locally Euclidean: let $x = (x_0 : x_1 : \dots : x_n) \in \mathbb{R}P^n$. Then there is an i with $x_i \neq 0$, and

$$U = \{(y_0 : \dots : y_n) : y_i \neq 0\}$$

is an open neighborhood of x .

Claim:

$$\varphi : U \xrightarrow{\cong} \mathbb{R}^n, \quad (y_0 : \dots : y_n) \mapsto \left(\frac{y_0}{y_i}, \dots, \frac{\widehat{y_i}}{y_i}, \dots, \frac{y_n}{y_i} \right).$$

Proof: well-defined $\checkmark$. Since $U \subseteq \mathbb{R}P^n$ is open, U carries the quotient topology induced by $p|_{p^{-1}(U)}$. Defining

$$\varphi' : p^{-1}(U) \rightarrow \mathbb{R}^n, \quad (y_0, \dots, y_n) \mapsto \left(\frac{y_0}{y_i}, \dots, \frac{\widehat{y_i}}{y_i}, \dots, \frac{y_n}{y_i} \right),$$

gives $\varphi \circ p|_{p^{-1}(U)} = \varphi'$. Since φ' is continuous, φ is continuous by Theorem I.7.1. The inverse map

$$\varphi^{-1} : \mathbb{R}^n \rightarrow U, \quad (x_1, \dots, x_n) \mapsto (x_1 : \dots : x_i : 1 : x_{i+1} : \dots : x_n)$$

is also continuous, since it is the composite

$$\mathbb{R}^n \rightarrow \{x \in \mathbb{R}^{n+1} : x_i = 1\} \rightarrow \{x \in S^n : x_i > 0\} \xrightarrow{p} U.$$

□

Theorem II.1.3. Every connected 1-dimensional manifold is homeomorphic to either $\mathbb{R}$ or S^1 .

Proof. [David Gale, The classification of 1-manifolds]

□

II.2. Surfaces

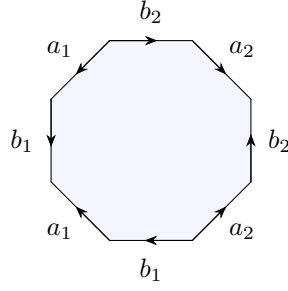
Definition II.2.1. A 2-dimensional manifold is also called a *surface*.

Constructing surfaces: consider a word, for example

$$w = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$$

- in an alphabet of n letters,
- in which each letter occurs exactly twice,
- with arbitrary choices of exponents ± 1 .

Label the boundary of a $2n$ -gon with this word.



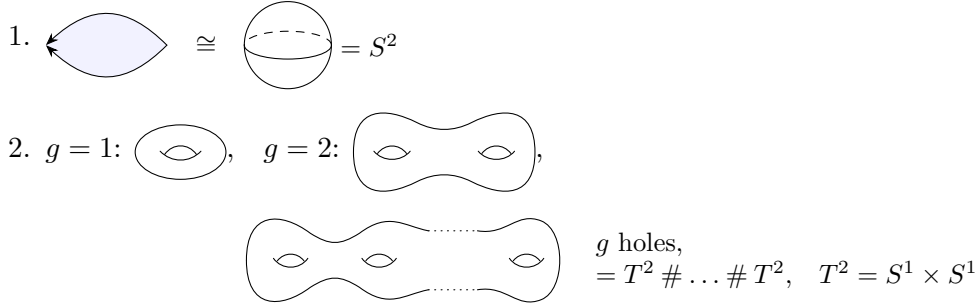
Fact I. The resulting quotient is a connected compact surface.

Fact II. For every connected compact surface F , there is a word w as above with $F \cong F_w$. (Choose a triangulation, order the triangles so that each new one adjoins the preceding ones, and glue along just one edge at each step to obtain a $2n$ -gon.)

Theorem II.2.2. (Classification of surfaces) The surfaces F_w associated with the following words w form a complete set of representatives of the homeomorphism classes of connected compact surfaces:

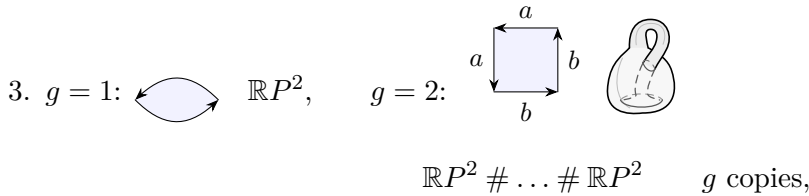
1. aa^{-1} ,
2. $a_1b_1a_1^{-1}b_1^{-1} \dots a_gb_ga_g^{-1}b_g^{-1} \quad (g \geq 1)$,
3. $a_1a_1a_2a_2 \dots a_ga_g \quad (g \geq 1)$.

Pictures:



$M \# N = \text{connected sum} =$ remove open n -balls from M and N , then glue the boundaries.

Orientable surface of genus g . Notation: Σ_g .



Nonorientable surface of genus g . Notation: N_g .

Proof. Two words w and w' are called *equivalent* ($w \sim w'$) if $F_w \cong F_{w'}$. The following operations produce equivalent words:

1. Cyclic permutation, for example $abc^{-1} \sim bc^{-1}a$,
2. Taking the inverse: $abc \sim c^{-1}b^{-1}a^{-1}$,

3. Inverting a fixed letter everywhere, for example

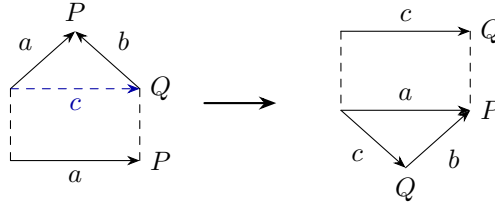
$$ab^{-1}cbab^{-1} \sim abcb^{-1}ab,$$

4. Deleting (proper) subwords of the form aa^{-1} , for example

$$baa^{-1}c \sim bc.$$

Claim: Every word w can be replaced by an equivalent word w' for which all vertices are identified.

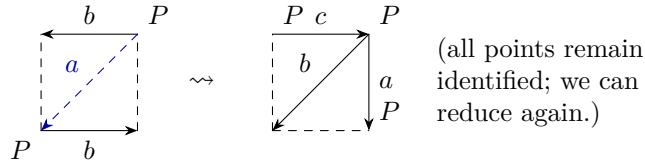
Proof: Let w have no subword of the form “ aa^{-1} .” Let P, Q be adjacent vertices in different equivalence classes,



$\rightsquigarrow$ the Q -class gains one point, and the P -class loses one point. □

We may therefore assume that w is *reduced* (no “ aa^{-1} ” subwords, all vertices identified).

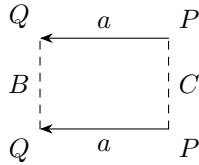
Make pairs of sides of *type I* ($\dots a \dots a \dots$) adjacent:



Claim. If w has at least four letters and a *type II* pair of sides ($\dots a \dots a^{-1} \dots$), then w has the form

$$\dots b \dots a \dots b^{-1} \dots a^{-1} \dots$$

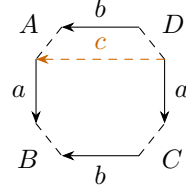
Proof: We have the following $2n$ -gon with $B, C \neq \emptyset$ (w is reduced).



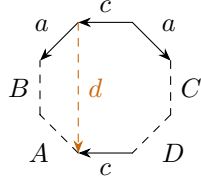
If there were only one type II pair of sides, no edge in B would be identified with an edge in C , since type I pairs are adjacent. Thus $P \neq Q$, a contradiction. Hence B and C must each contain an edge with the same letter and opposite exponents. □

Claim: $\dots b \dots a \dots b^{-1} \dots a^{-1} \dots \sim \dots bab^{-1}a^{-1} \dots$

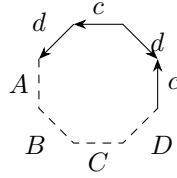
Proof:



If A, B, C, D are empty, we are done.
WLOG, let $A \neq \emptyset$. Cut along c ,
glue along b , and reduce.



If BA is empty, invert c and we are done.
Otherwise: cut along d , glue
along a , and reduce.



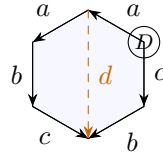
Invert d and we are done.

□

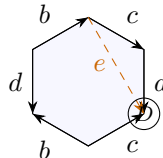
Thus we can transform w into a product of squares and commutators.

Claim: $aabcb^{-1}c^{-1}D \sim aabbccD$.

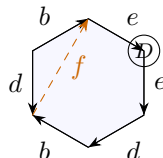
Proof:



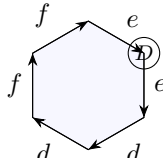
Cut along d , glue
along a .



Cut along e , glue
along c .



Cut along f , glue along b .



Make the e -edges adjacent as above.
Permute cyclically and
rename d, e, f as a, b, c .

□

By cyclic permutation, we can place commutators after squares and replace them by three squares as in the last step. We thus obtain one of the words in 1.), 2.), 3.). The corresponding surfaces are all distinct $\rightarrow$ fundamental group, see Section VI.3. □

III. Homotopy and the Fundamental Group

III.1. Homotopy, Homotopy Equivalence, and Deformation Retracts

Set $I := [0, 1]$.

Definition III.1.1. Let X, Y be topological spaces. Continuous maps $f, g : X \rightarrow Y$ are called *homotopic* if there is a continuous map

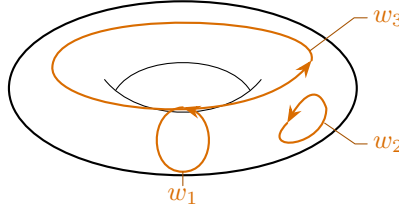
$$H : X \times I \rightarrow Y$$

with $H(x, 0) = f(x)$, $H(x, 1) = g(x)$ for all $x \in X$.

- The map H is called a *homotopy* from f to g . (Notation: $f \simeq_H g$, or simply $f \simeq g$.)
- For $t \in I$, let $H_t : X \rightarrow Y$, $x \mapsto H(x, t)$.
- We can think of H as a “movie” that continuously transforms f into g .

Example. (Homotopies of paths)

- Let $X = S^1$, $Y = T^2$. Then $w_1, w_2, w_3 : X \rightarrow Y$ are *not* homotopic. (Proof later)



- Any two maps $f, g : X \rightarrow \mathbb{R}^n$ are homotopic, since $f \simeq_H (X \rightarrow \mathbb{R}^n, x \mapsto 0)$ with $H(x, t) = (1 - t) \cdot f(x)$, and similarly for g . Moreover, we have

Lemma III.1.2. “ $\simeq$ ” is an equivalence relation on

$$C(X, Y) = \{f : X \rightarrow Y : f \text{ continuous}\}.$$

Proof. Reflexivity: $f \simeq_H f$ with $H(x, t) = f(x)$. Symmetry: if $f \simeq_H g$, then $g \simeq_{\overline{H}} f$ with $\overline{H}(x, t) = H(x, 1 - t)$.

Transitivity: let $f \simeq_H g$ and $g \simeq_{H'} h$. Then $f \simeq_{H''} h$ with

$$H''(x, t) = \begin{cases} H(x, 2t) & t \in [0, \frac{1}{2}], \\ H'(x, 2t - 1) & t \in [\frac{1}{2}, 1]. \end{cases}$$

□

Note.

1. Let $f_1, f_2 : Y \rightarrow Z$, $g_1, g_2 : X \rightarrow Y$, $f_1 \simeq_H f_2$, $g_1 \simeq_{H'} g_2$. Then $f_1 \circ g_1 \simeq_{H''} f_2 \circ g_2$, where $H''_t = H_t \circ H'_t$.

2. Let $f_1, f_2 : X \rightarrow Y$, $g_1, g_2 : Z \rightarrow W$, $f_1 \simeq_H f_2$, $g_1 \simeq_{H'} g_2$. Then for $f_1 \times g_1, f_2 \times g_2 : X \times Z \rightarrow Y \times W$, we have $f_1 \times g_1 \simeq_{H''} f_2 \times g_2$ with $H''_t = H_t \times H'_t$.

Definition III.1.3. A continuous map $f : X \rightarrow Y$ is a *homotopy equivalence* if there is a continuous map $g : Y \rightarrow X$ (a *homotopy inverse*) such that

$$g \circ f \simeq \text{id}_X \quad \text{and} \quad f \circ g \simeq \text{id}_Y.$$

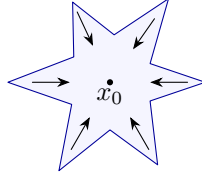
- If such a homotopy equivalence $f : X \rightarrow Y$ exists, X and Y are called *homotopy equivalent*. (Notation: $X \simeq Y$.)
- “ $X \simeq Y$ ” is an equivalence relation, and $X \simeq Y$ and $X' \simeq Y'$ imply $X \times X' \simeq Y \times Y'$.
- Special case: if $f : X \rightarrow Y$ is the inclusion of a subspace and $g : Y \rightarrow X$ is a *retraction*, that is, $g \circ f = \text{id}_X$ (not just $g \circ f \simeq \text{id}_X$), and $f \circ g \simeq_H \text{id}_Y$, then X is called a *deformation retract* of Y , and H is a *deformation retraction*. If, in addition, $H(x, t) = x$ for $x \in X$ and $t \in I$, then X is a *strong deformation retract*, and H is a *strong deformation retraction*.
- A topological space X is called *contractible* if a point in X is a deformation retract of X (in particular, $X \neq \emptyset$). Equivalently: $X \simeq \bullet$.

Examples:

- $\{0\} \hookrightarrow \mathbb{R}^n$ is a strong deformation retract:

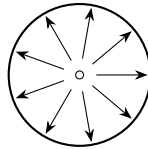
$$H : \mathbb{R}^n \times I \longrightarrow \mathbb{R}^n, \quad (x, t) \longmapsto t \cdot x.$$

- More generally: if $U \subseteq \mathbb{R}^n$ is *star-shaped*, that is, there is an $x_0 \in U$ such that $\{x_0 + t(x - x_0) : t \in I\} \subseteq U$ for all $x \in U$, then $\{x_0\} \hookrightarrow U$ is a strong deformation retract.

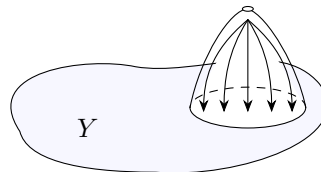


- $S^{n-1} \xrightarrow{j} D^n$ is not even a *retract*: there is no continuous map $D^n \xrightarrow{r} S^{n-1}$ with $r \circ j = \text{id}_{S^{n-1}}$. (The proof for $n = 2$ comes later, and for $n \geq 3$ in Topology I.)
- $S^{n-1} \hookrightarrow D^n \setminus \{0\}$ is a *strong deformation retract*

$$H : (D^n \setminus \{0\}) \times I \longrightarrow D^n \setminus \{0\}, \quad (x, t) \longmapsto (1 - t) \cdot x + t \cdot \frac{x}{\|x\|}.$$



- For $S^{n-1} \subseteq D^n \setminus \{0\}$ and $\varphi : S^{n-1} \rightarrow Y$, let $(D^n \setminus \{0\}) \cup_{\varphi} Y$ be the result of attaching an n -cell “with a hole.” Then $Y \subseteq (D^n \setminus \{0\}) \cup_{\varphi} Y$ is a strong deformation retract.

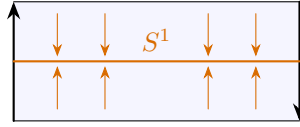


- Let M be the Möbius strip. Its core circle $S^1 \subseteq M$ is a strong deformation retract.

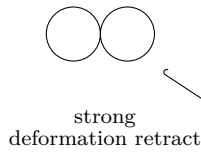
$$M = [-1, 1]^2 / (-1, x) \sim (1, -x),$$

$$j : S^1 = [-1, 1] / \{-1, 1\} \longrightarrow M, \quad [x] \longmapsto [(x, 0)],$$

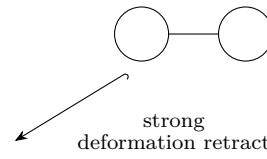
$$r : M \longrightarrow S^1 = [-1, 1] / \{-1, 1\}, \quad [(x, y)] \longmapsto [x].$$



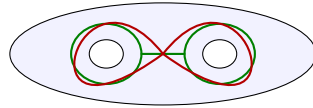
$$r \circ j = \text{id}_{S^1}, \quad j \circ r \simeq_H \text{id}_M, \quad H_t([(x, y)]) = [(x, t \cdot y)].$$



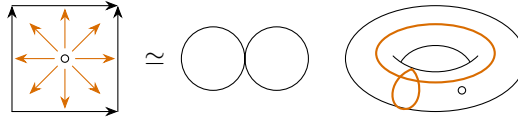
$\simeq$



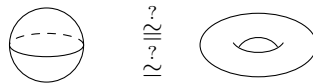
•



- $T^2 \setminus \{\text{point}\}$



III.2. The Fundamental Group



Definition III.2.1. Two continuous maps $f, g : X \rightarrow Y$ are called *homotopic relative to* $A \subseteq X$ if there is a homotopy $f \simeq_H g$ with $H(a, t) = f(a) = g(a)$ for all $a \in A, t \in I$. Notation: $f \simeq_A g$. If $A = \{x_0\}$, then H is called a *pointed homotopy*.

If $X = I, A = \{0, 1\}$, then H is called a *homotopy relative to the endpoints*.



- A continuous map $\gamma : I \rightarrow X$ is called a *path*. If two paths $\gamma_1, \gamma_2 : I \rightarrow X$ satisfy $\gamma_1(1) = \gamma_2(0)$, then

$$\gamma_1 \gamma_2 : I \longrightarrow X, \quad t \longmapsto \begin{cases} \gamma_1(2t) & \text{for } t \in [0, \frac{1}{2}], \\ \gamma_2(2t - 1) & \text{for } t \in [\frac{1}{2}, 1], \end{cases}$$

is called the *concatenation* of γ_1 and γ_2 .



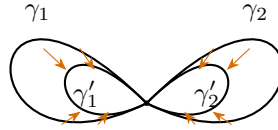
A path $\gamma : I \rightarrow X$ with $\gamma(0) = \gamma(1)$ is called a *loop*.

Theorem III.2.2. Let X be a topological space and let $x_0 \in X$ be a basepoint. Concatenation defines a well-defined group structure on the set

$$\pi_1(X, x_0) := \{\gamma : I \rightarrow X : \gamma(0) = \gamma(1) = x_0\} / \simeq_{\{0,1\}}$$

of homotopy classes relative to the endpoints of loops in X starting and ending at x_0 . We call $\pi_1(X, x_0)$ the *fundamental group* of X at the basepoint x_0 .

Proof. • *Well-defined.* Let $\gamma_1 \simeq_{\{0,1\}} \gamma'_1$, $\gamma_2 \simeq_{\{0,1\}} \gamma'_2$,



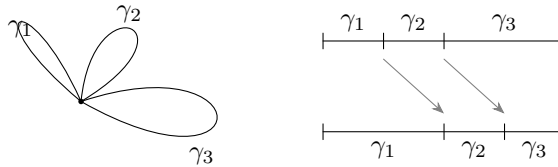
so $\gamma_1 \gamma_2 \simeq_{\{0,1\}} \gamma'_1 \gamma'_2$.

Observation: for every continuous map $\varphi : [0, 1] \rightarrow [0, 1]$ with $\varphi(0) = 0$ and $\varphi(1) = 1$, and every $\gamma : I \rightarrow X$, we have $\gamma \simeq_{\{0,1\}} \gamma \circ \varphi$ via

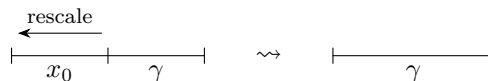
$$H : I \times I \longrightarrow X, \quad (s, t) \longmapsto \gamma((1 - t) \cdot s + t \cdot \varphi(s)).$$

(Rescaling the time parameter)

- *Associativity.* Rescaling the time parameter gives $(\gamma_1 \gamma_2) \gamma_3 \simeq_{\{0,1\}} \gamma_1 (\gamma_2 \gamma_3)$.



- *Identity.* Let $c_{x_0} : I \rightarrow X$, $t \mapsto x_0$, be the constant path. Then $c_{x_0} \gamma \simeq_H \gamma \simeq \gamma c_{x_0}$.



- *Inverses.* For $\gamma : I \rightarrow X$, let $\bar{\gamma} : I \rightarrow X$, $t \mapsto \gamma(1 - t)$.

$$\bar{\gamma}\gamma \simeq c_{x_0} \simeq_H \gamma\bar{\gamma}, \quad H_t(s) = \begin{cases} \gamma(2ts) & \text{for } s \in [0, \frac{1}{2}], \\ \gamma(2t(1-s)) & \text{for } s \in [\frac{1}{2}, 1]. \end{cases}$$



□

Let $A \subseteq X$ and $B \subseteq Y$ be subspaces. A map $f : X \rightarrow Y$ with $f(A) \subseteq B$ is called a map of *pairs of spaces* (notation: $f : (X, A) \rightarrow (Y, B)$). In the special case $A = \{x_0\}$, $B = \{y_0\}$, we write $f : (X, x_0) \rightarrow (Y, y_0)$ for a *pointed map of pointed spaces*.

Proposition III.2.3. A continuous map $f : (X, x_0) \rightarrow (Y, y_0)$ induces a group homomorphism

$$\pi_1(f) : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0), \quad [\gamma] \longmapsto [f \circ \gamma].$$

Proof.

$$\begin{aligned} \pi_1(f)([\alpha][\beta]) &= \pi_1(f)([\alpha\beta]) = [f \circ (\alpha\beta)] \\ &= [(f \circ \alpha)(f \circ \beta)] = [f \circ \alpha][f \circ \beta] = \pi_1(f)([\alpha])\pi_1(f)([\beta]). \end{aligned}$$

□

Proposition III.2.4. Let $(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$ be continuous maps. Then $\pi_1(g \circ f) = \pi_1(g) \circ \pi_1(f)$.

Proof.

$$\begin{aligned} \pi_1(g \circ f)([\gamma]) &= [(g \circ f) \circ \gamma] = [g \circ (f \circ \gamma)] = \\ &= \pi_1(g)([f \circ \gamma]) = \pi_1(g)(\pi_1(f)([\gamma])). \end{aligned}$$

□

Proposition III.2.5. $\pi_1(\text{id}_{(X, x_0)}) = \text{id}_{\pi_1(X, x_0)}$.

Proof. $\pi_1(\text{id}_{(X, x_0)})([\gamma]) = [\text{id}_{(X, x_0)} \circ \gamma] = [\gamma]$.

□

Proposition III.2.6. Let $f, g : (X, x_0) \rightarrow (Y, y_0)$, $f \simeq_{\{x_0\}}^H g$. Then $\pi_1(f) = \pi_1(g)$.

Proof. Let $[\gamma] \in \pi_1(X, x_0)$. Define

$$H^\gamma : I \times I \longrightarrow Y, \quad (s, t) \longmapsto H(\gamma(s), t).$$

Then $H_0^\gamma = f \circ \gamma$, $H_1^\gamma = g \circ \gamma$, $H^\gamma(0, t) = H^\gamma(1, t) = y_0$ for all $t \in I$, that is,

$$f \circ \gamma \simeq_{\{0,1\}} g \circ \gamma \iff [f \circ \gamma] = [g \circ \gamma] \iff \pi_1(f)([\gamma]) = \pi_1(g)([\gamma]).$$

□

Proposition III.2.7. Let A be a strong deformation retract of X , with inclusion $j : A \hookrightarrow X$, and let $a \in A$. Then

$$\pi_1(j) : \pi_1(A, a) \xrightarrow{\cong} \pi_1(X, a).$$

Proof. Let $r : X \rightarrow A$ be the retraction with $r \circ j = \text{id}_A$ and $j \circ r \simeq_A \text{id}_X$. Then $\pi_1(r) \circ \pi_1(j) = \text{id}_{\pi_1(A, a)}$ and $\pi_1(j) \circ \pi_1(r) = \pi_1(\text{id}_X) = \text{id}_{\pi_1(X, a)}$. $\square$

Example. $\pi_1(\mathbb{R}^n, 0) \cong \pi_1(\{0\}, 0) \cong \{1\}$.

Proposition III.2.8. If $f : (X, x_0) \rightarrow (Y, y_0)$ with $f : X \xrightarrow{\cong} Y$, then $\pi_1(f)$ is an isomorphism.

Proof. Exercise. $\square$

We define $(X, x_0) \times (Y, y_0) := (X \times Y, (x_0, y_0))$.

Proposition III.2.9. $\pi_1((X, x_0) \times (Y, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Proof.

$$\begin{aligned} (\gamma : I \rightarrow X \times Y) &\mapsto (\text{pr}_X \circ \gamma, \text{pr}_Y \circ \gamma), \\ (\gamma : I \rightarrow X, \gamma' : I \rightarrow Y) &\mapsto \left(\gamma \times \gamma' : \begin{array}{l} I \longrightarrow X \times Y \\ t \longmapsto (\gamma(t), \gamma'(t)) \end{array} \right). \end{aligned}$$

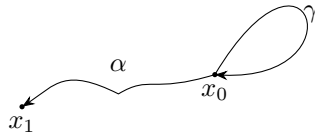
$\square$

Example. We will show later that $\pi_1(S^1, \bullet) \cong \mathbb{Z}$, so

$$\pi_1(T^n, \bullet) \cong \pi_1((S^1, \bullet) \times \cdots \times (S^1, \bullet)) \cong \mathbb{Z}^n.$$

Proposition III.2.10. Let $x_0, x_1 \in X$ and let $\alpha : I \rightarrow X$ satisfy $\alpha(0) = x_0$ and $\alpha(1) = x_1$. Then

$$\Phi_\alpha : \pi_1(X, x_0) \xrightarrow{\cong} \pi_1(X, x_1), \quad [\gamma] \mapsto [\bar{\alpha}\gamma\alpha].$$



Proof. Well-defined $\checkmark$.

Homomorphism:

$$\begin{aligned} \Phi_\alpha([\gamma][\gamma']) &= \Phi_\alpha([\gamma\gamma']) = [\bar{\alpha}\gamma\gamma'\alpha] = [\bar{\alpha}\gamma\alpha\bar{\alpha}\gamma'\alpha] \\ &= [\bar{\alpha}\gamma\alpha][\bar{\alpha}\gamma'\alpha] = \Phi_\alpha([\gamma])\Phi_\alpha([\gamma']). \end{aligned}$$

Inverse: $[\gamma] \mapsto [\alpha\gamma\bar{\alpha}]$. $\square$

Definition III.2.11. A topological space X is called *simply connected* if X is path-connected and $\pi_1(X, x_0) = \{1\}$ for some (and hence every) $x_0 \in X$.

Theorem III.2.12. For $n \geq 2$, S^n is simply connected.

Lemma III.2.13. (Lebesgue) Let (X, d) be a compact metric space with an open cover $\mathcal{U}$. Then there is a $\delta > 0$ such that every subspace $A \subseteq X$ of diameter $< \delta$ is contained in some $U \in \mathcal{U}$.

Proof. Consider $X = \bigcup_{x \in X} B_{\varepsilon(x)}(x)$, with $B_{2\varepsilon(x)}(x) \subseteq U_i$ for some $U_i \in \mathcal{U}$. Obtain $X = \bigcup_{i=1}^n B_{\varepsilon(x_i)}(x_i)$ and set $\delta := \min\{\varepsilon(x_1), \dots, \varepsilon(x_n)\}$. If a ball $B \subseteq X$ of radius $< \delta$ were not contained in any of $B_{2\varepsilon(x_1)}(x_1), \dots, B_{2\varepsilon(x_n)}(x_n)$, its center would lie outside $\bigcup_{i=1}^n B_{\varepsilon(x_i)}(x_i) = X$, a contradiction. $\square$

Such a δ is called a *Lebesgue number* for $(X, \mathcal{U})$.

Lemma III.2.14. Let $n \geq 2$ and $\gamma : (I, \{0, 1\}) \rightarrow (S^n, x_0)$. Then there is a nonsurjective map $\gamma' : (I, \{0, 1\}) \rightarrow (S^n, x_0)$ with $\gamma \simeq_{\{0,1\}} \gamma'$.

Proof. Let $\mathcal{U}$ be the cover of S^n by open hemispheres. Choose a Lebesgue number δ for $(\gamma(I), \mathcal{U}|_{\gamma(I)})$. Since γ is continuous on a compact space, it is uniformly continuous. Hence there is an $m \in \mathbb{N}$ such that

$$\underbrace{\gamma\left(\left[\frac{k}{m}, \frac{k+1}{m}\right]\right)}_{\text{diameter} < \delta} \subseteq U_k \in \mathcal{U} \quad \text{for } k = 0, \dots, m-1.$$

Let $\gamma_k : [\frac{k}{m}, \frac{k+1}{m}] \rightarrow \mathbb{R}^{n+1}$ be the line segment from $\gamma(\frac{k}{m})$ to $\gamma(\frac{k+1}{m})$. Define $H : I \times I \rightarrow S^n$ by

$$H(s, t) = \frac{(1-t)\gamma(s) + t\gamma_k(s)}{\|(1-t)\gamma(s) + t\gamma_k(s)\|} \quad \text{for } s \in \left[\frac{k}{m}, \frac{k+1}{m}\right],$$

The denominator is $\neq 0$, since γ and γ_k lie in the same open half-space. Then $\gamma' := H_1$ is not surjective for $n \geq 2$, and $\gamma \simeq_{\{0,1\}} \gamma'$. $\square$

Proof of Theorem III.2.12. For $[\gamma] \in \pi_1(S^n, x_0)$, let $\gamma' \simeq_{\{0,1\}} \gamma$ and $y_0 \notin \text{im}(\gamma')$. Consider

$$\begin{array}{ccc} S^n \setminus \{y_0\} & \xrightarrow{j} & S^n \\ & \nwarrow \gamma'' & \uparrow \gamma' \\ & & I \end{array}$$

We have

$$[\gamma] = [\gamma'] = [j \circ \gamma''] \stackrel{\text{III.2.3}}{=} \pi_1(j)([\gamma'']) = 1,$$

since

$$\pi_1(S^n \setminus \{y_0\}, x_0) \stackrel{\text{stereographic projection}}{\cong} \pi_1(\mathbb{R}^n, x_1) \stackrel{\text{III.2.10}}{\cong} \pi_1(\mathbb{R}^n, 0) \stackrel{\text{III.2.7}}{\cong} 1,$$

so $[\gamma''] = 1$. $\square$

Since $\pi_1(T^2, \bullet) = \mathbb{Z}^2$ but $\pi_1(S^2, \bullet) \cong \{1\}$, we indeed have

$$\text{---} \not\cong \text{---}$$

and, a fortiori,

$$\text{---} \not\cong \text{---}.$$

(We still owe a proof that $\pi_1(S^1, \bullet) \cong \mathbb{Z} \rightarrow$ next chapter).

IV. Covering Spaces

IV.1. Fiber Bundles and Covering Spaces

Observation. A map $p: X \rightarrow Y$ induces a disjoint decomposition $X = \bigcup_{y \in Y} p^{-1}(y)$ of X into the *fibers* $p^{-1}(y)$ over $y \in Y$.

- p surjective: Every fiber contains at least one element.
- p injective: Every fiber contains at most one element.

Definition IV.1.1. A *fiber bundle* consists of a surjective continuous map $p: E \rightarrow B$ (the *bundle projection*) and a topological space F (the *fiber*) such that for every $b \in B$, there are an open neighborhood $U \subseteq B$ of b and a homeomorphism $\varphi: p^{-1}(U) \xrightarrow{\cong} U \times F$ such that the diagram

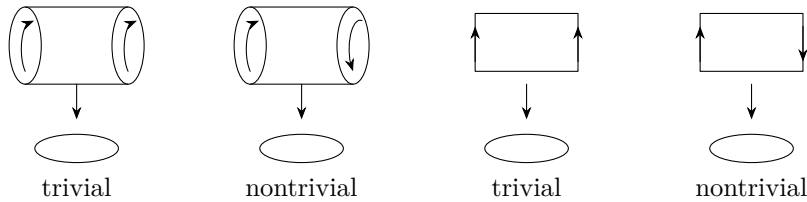
$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow[\cong]{\varphi} & U \times F \\ & \searrow p|_{p^{-1}(U)} & \swarrow \text{pr}_U \\ & U & \end{array}$$

commutes (the *local triviality condition*).

- A fiber bundle is called (*globally*) *trivial* if there is a commuting diagram

$$\begin{array}{ccc} E & \xrightarrow[\cong]{\varphi} & B \times F \\ & \searrow p & \swarrow \text{pr}_B \\ & B & \end{array}$$

- Examples:

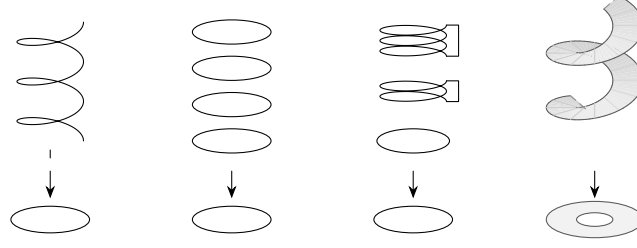


The space E is called the *total space*, and the space B is called the *base space* of the fiber bundle.

Proposition IV.1.2. The bundle projection of a fiber bundle is an open quotient map.

Proof. Homework. □

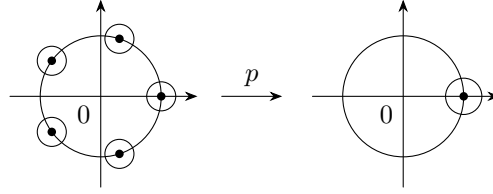
Definition IV.1.3. A *covering* is a fiber bundle $p: E \rightarrow B$ with discrete fiber F .



Note. The number $|F| \in \{1, 2, \dots\} \cup \{\infty\}$ is called the *number of sheets* of the covering.

Example. For $n \in \mathbb{N}$, the map $p: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \setminus \{0\}$, $z \mapsto z^n$, is an n -sheeted covering. For $z = r \exp(2\pi i \varphi) \in \mathbb{C} \setminus \{0\}$, we have

$$p^{-1}(z) = \left\{ \sqrt[n]{r} \exp\left(2\pi i \frac{\varphi}{n}\right), \dots, \sqrt[n]{r} \exp\left(2\pi i \frac{n-1+\varphi}{n}\right) \right\}.$$



The map $p: S^1 \rightarrow S^1$, $z \mapsto z^n$, is an n -sheeted covering.

- The map $p: \mathbb{R} \rightarrow S^1$, $t \mapsto \exp(2\pi i t)$, is an ∞ -sheeted covering.
- Finite products of coverings are coverings, for example

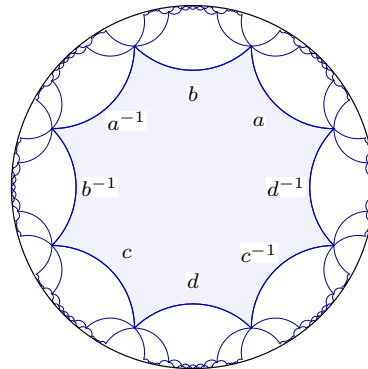
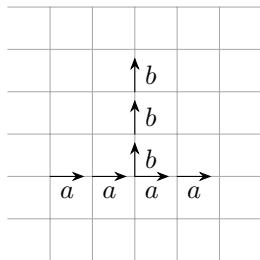
$$p: \mathbb{R}^n \rightarrow T^n, \quad (t_1, \dots, t_n) \mapsto (\exp(2\pi i t_1), \dots, \exp(2\pi i t_n)).$$

- $p: S^n \rightarrow \mathbb{R}P^n$, $(x_0, \dots, x_n) \mapsto (x_0 : \dots : x_n)$. For $x = (x_0 : \dots : x_n) \in \mathbb{R}P^n$, assume WLOG that $x_0 \neq 0$. Then $U = \{(x'_0 : \dots : x'_n) : x'_0 \neq 0\}$ is an open neighborhood of x , and

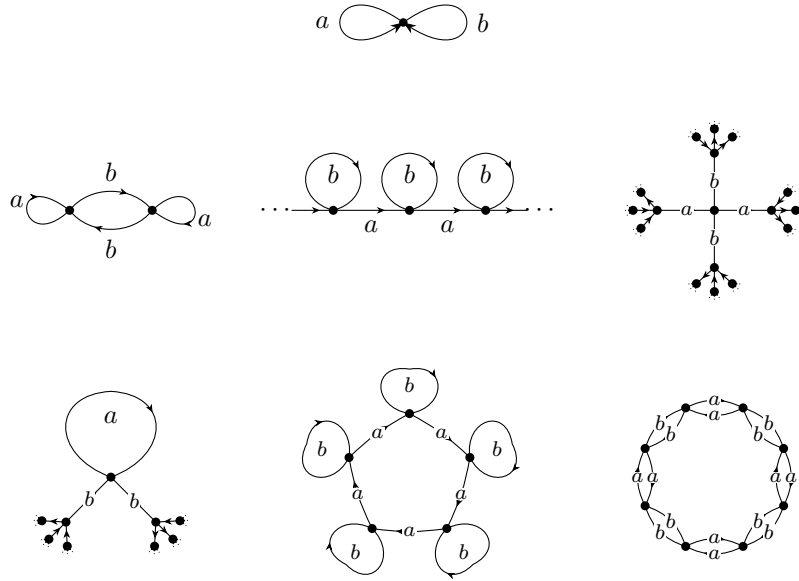
$$p^{-1}(U) = \{(x_0, \dots, x_n) \in S^n : x_0 < 0\} \cup \{(x_0, \dots, x_n) \in S^n : x_0 > 0\},$$

so p is a two-sheeted covering.

- For $g \geq 1$, there is a covering $p: \mathbb{R}^2 \rightarrow \Sigma_g$.



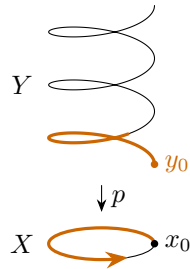
- The covering space theory of $S^1 \vee S^1$ is particularly rich:



IV.2. Lifts

Let $p: Y \rightarrow X$ be a covering. The lifting problem for paths:

$$\begin{array}{ccc} & (Y, y_0) & \\ \exists! \tilde{\gamma} \nearrow & \downarrow p & \\ (I, 0) & \xrightarrow{\gamma} & (X, x_0) \end{array}$$



Proposition IV.2.1. For every path $\gamma: I \rightarrow X$ with $\gamma(0) = x_0$ and every point $y_0 \in Y$ with $p(y_0) = x_0$, there is exactly one path $\tilde{\gamma}: I \rightarrow Y$ with $\tilde{\gamma}(0) = y_0$ and $p \circ \tilde{\gamma} = \gamma$.

Proof. Uniqueness. Let $\tilde{\gamma}': I \rightarrow Y$ with $\tilde{\gamma}'(0) = y_0$ be another path with $\tilde{\gamma}'(0) = y_0$ and $p \circ \tilde{\gamma}' = \gamma$. Then the sets

$$\{t \in I : \tilde{\gamma}(t) = \tilde{\gamma}'(t)\} \quad \text{and} \quad \{t \in I : \tilde{\gamma}(t) \neq \tilde{\gamma}'(t)\}$$

are both open, since $\gamma(t) \in X$ has an open neighborhood $U \subseteq X$ with

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\cong} & U \times F \\ \downarrow p & & \downarrow \text{pr}_U \\ U & & U \end{array}$$

Thus the U -coordinate is locally determined by the lifting condition $p \circ \tilde{\gamma} = \gamma$, while the F -coordinate must remain constant because F is discrete. Since I is connected and $0 \in \{t \in I : \tilde{\gamma}(t) = \tilde{\gamma}'(t)\}$, it follows that $\tilde{\gamma}(t) = \tilde{\gamma}'(t)$ for all $t \in I$. *Existence.* Let

$$t_0 = \sup\{t \in I : \exists \tilde{\gamma} : [0, t] \rightarrow Y, \tilde{\gamma}(0) = y_0, p \circ \tilde{\gamma} = \gamma\}.$$

Local triviality gives $t_0 > 0$. We must show that $t_0 = 1$. Suppose $t_0 < 1$. Let

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow[\varphi]{\cong} & U \times F \\ & \searrow p & \swarrow \text{pr}_U \\ & U & \end{array}$$

be a local trivialization around $\gamma(t_0) \in X$. Then there is $0 < \delta$ with $\gamma([t_0 - \delta, t_0 + \delta]) \subseteq U$. Let $\tilde{\gamma} : [0, t_0 - \delta] \rightarrow Y$ with $\tilde{\gamma}(0) = y_0$. Set $f := \text{pr}_F(\varphi(\tilde{\gamma}(t_0 - \delta)))$. Then

$$[t_0 - \delta, t_0 + \delta] \rightarrow Y, \quad t \mapsto \varphi^{-1}(\gamma(t), f)$$

extends the lift $\tilde{\gamma}$ to $[0, t_0 + \delta]$, a contradiction. $\square$

The lifting problem for homotopies:

$$\begin{array}{ccc} Z \times \{0\} & \xrightarrow{\tilde{H}_0} & Y \\ \downarrow & \exists! \tilde{H} \nearrow & \downarrow p \\ Z \times I & \xrightarrow{H} & X \end{array}$$

For $Z = \bullet$, this is the lifting problem for paths. For $z \in Z$, set

$$H^z : I \rightarrow X, \quad t \mapsto H(z, t)$$

and let $\tilde{H}^z$ be the unique lift of H^z with $\tilde{H}^z(0) = \tilde{H}_0(z)$.

Proposition IV.2.2. The map $\tilde{H} : Z \times I \rightarrow Y$, $(z, t) \mapsto \tilde{H}^z(t)$, is continuous; that is, the lifting problem for homotopies has a unique solution.

Proof. For $t \in I$ and $\varepsilon > 0$, let $I_\varepsilon(t) = (t - \varepsilon, t + \varepsilon) \cap I$. Observation: Suppose that, for a neighborhood $U \subseteq Z$ of $z \in Z$, the image $H(U \times I_\varepsilon(t))$ lies in a neighborhood $V \subseteq X$ of $H(z, t)$ over which p is trivial. Then, for $\tilde{H}|_{U \times I_\varepsilon(t)}$ to be continuous, it suffices that $\tilde{H}|_{U \times \{t'\}}$ be continuous for some $t' \in I_\varepsilon(t)$. Indeed:

$$\begin{array}{ccc} & & V \times F \\ & \nearrow \tilde{H} & \downarrow p = \text{pr}_V \\ U \times I_\varepsilon(t) & \xrightarrow{H} & V. \end{array}$$

We see that $\text{pr}_V \circ \tilde{H} = H$ is continuous and $\text{pr}_F \circ \tilde{H}$ is independent of the time coordinate: for $z' \in U$, the map $\text{pr}_F(\tilde{H}(z', \cdot)) = \text{pr}_F \circ \tilde{H}^{z'}|_{I_\varepsilon(t)} : I_\varepsilon(t) \rightarrow F$ is continuous and F is discrete. Hence $(z', t) \mapsto \text{pr}_F(\tilde{H}(z', t)) = \text{pr}_F(\tilde{H}^{z'}(t))$ is continuous as well. Now consider

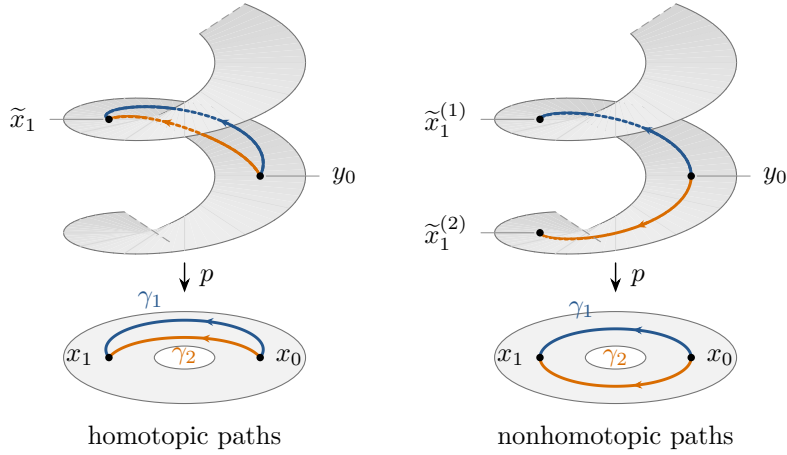
$$T = \left\{ t \in I \mid \begin{array}{l} \exists U \times I_\varepsilon(t), V \subseteq X, t' \in I_\varepsilon(t) : \\ H(U \times I_\varepsilon(t)) \subseteq V, p|_{p^{-1}(V)} \text{ trivial,} \\ \tilde{H}|_{U \times \{t'\}} \text{ continuous} \end{array} \right\}.$$

By the observation, T is open, and $0 \in T$ because the given map $\tilde{H}_0$ is continuous. It remains to show that T is closed. Let $t \in \bar{T}$. Since H is continuous, there is $U \times I_\varepsilon(t)$ with $H(U \times I_\varepsilon(t)) \subseteq V$ and $p|_{p^{-1}(V)}$ trivial. Choose $t' \in I_\varepsilon(t) \cap T$. Then there is $U' \subseteq U$ with $z \in U'$ such that $\tilde{H}|_{U' \times I_\varepsilon(t')}$ is continuous. It follows that $H((U \cap U') \times I_\varepsilon(t)) \subseteq V$ and $\tilde{H}|_{(U \cap U') \times \{t'\}}$ is continuous, so $t \in T$. $\square$

Lemma IV.2.3. (Monodromy lemma) Let $\gamma_1, \gamma_2: I \rightarrow X$ be paths with $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(1) = \gamma_2(1)$, and $\gamma_1 \simeq_{\{0,1\}}^H \gamma_2$. Let $y_0 \in Y$ with $p(y_0) = \gamma_1(0) = \gamma_2(0)$, let $\tilde{\gamma}_1, \tilde{\gamma}_2: I \rightarrow Y$ be the unique lifts with $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0) = y_0$, and let $\tilde{H}$ be the unique lift of H with initial lift $\tilde{\gamma}_1$. Then $\tilde{\gamma}_1 \simeq_{\{0,1\}}^{\tilde{H}} \tilde{\gamma}_2$ and, in particular, $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$.

Proof. The map $\tilde{H}$ solves the lifting problem

$$\begin{array}{ccc} I \times \{0\} & \xrightarrow{\tilde{\gamma}_1} & Y \\ \downarrow & \searrow \tilde{H} & \downarrow p \\ I \times I & \xrightarrow{H} & X. \end{array}$$



We have $p \circ \tilde{H}_t(0) = H_t(0) = \gamma_1(0) = \gamma_2(0) =: x_0$ for $t \in I$. Thus we obtain $I \rightarrow p^{-1}(x_0)$, $t \mapsto \tilde{H}_t(0)$. Since the fiber $p^{-1}(x_0)$ is discrete, this map is constant, so $\tilde{H}_0(0) = \tilde{H}_1(0)$. Similarly, $\tilde{H}_0(1) = \tilde{H}_1(1)$. Hence $\tilde{H}_1(0) = \tilde{H}_0(0) = \tilde{\gamma}_1(0) = y_0$ and $p \circ \tilde{H}_1 = H_1 = \gamma_2$. Uniqueness of path lifting gives $\tilde{H}_1 = \tilde{\gamma}_2$. Therefore $\tilde{\gamma}_2(1) = \tilde{H}_1(1) = \tilde{H}_0(1) = \tilde{\gamma}_1(1)$. $\square$

Now let $p: (Y, y_0) \rightarrow (X, x_0)$ be a pointed covering.

Note. The homomorphism $\pi_1(p): \pi_1(Y, y_0) \rightarrow \pi_1(X, x_0)$ is injective.

Proof. If a loop $\gamma: (I, \{0,1\}) \rightarrow (Y, y_0)$ satisfies $c_{x_0} \simeq_{\{0,1\}}^H p \circ \gamma$, then the lift $\tilde{H}$ starting at c_{y_0} shows, by the [monodromy lemma](#), that $c_{y_0} \simeq_{\{0,1\}}^{\tilde{H}} \gamma$. $\square$

Definition IV.2.4. The subgroup

$$G(Y, y_0) := \text{im}(\pi_1(p)) \leq \pi_1(X, x_0)$$

is called the *associated subgroup* of the pointed covering p . We now wish to solve the lifting problem

$$\begin{array}{ccc} & & (Y, y_0) \\ & \nearrow \exists! \tilde{f} & \downarrow p \\ (Z, z_0) & \xrightarrow{f} & (X, x_0) \end{array}$$

A necessary condition is

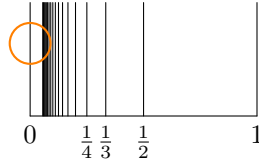
$$\begin{aligned} \text{im}(\pi_1(f)) &= \text{im}(\pi_1(p \circ \tilde{f})) \\ &= \text{im}(\pi_1(p) \circ \pi_1(\tilde{f})) \\ &\leq \text{im}(\pi_1(p)). \end{aligned}$$

$$\begin{array}{ccc} & & (\mathbb{R}, 0) \\ & \nearrow \nexists & \downarrow p \\ (S^1, 1) & \xrightarrow{\text{id}} & (S^1, 1) \end{array}$$

Goal: Find a condition on Z under which this criterion is also sufficient.

Definition IV.2.5. A topological space is called *locally path-connected* if every neighborhood of every point contains a path-connected neighborhood of that point.

Example. (The topologist's comb)



- path-connected
- not locally path-connected
- Every manifold is locally path-connected.

Theorem IV.2.6. (Lifting criterion) Let Z be path-connected and locally path-connected. Then a map $f: (Z, z_0) \rightarrow (X, x_0)$ has a unique lift $\tilde{f}: (Z, z_0) \rightarrow (Y, y_0)$ with $p \circ \tilde{f} = f$ if and only if $\text{im}(\pi_1(f)) \leq G(Y, y_0)$.

$$\begin{array}{ccc} & & (Y, y_0) \\ & \nearrow \exists! & \downarrow p \\ (Z, z_0) & \xrightarrow{f} & (X, x_0) \end{array} \iff \begin{array}{ccc} & & \pi_1(Y, y_0) \\ & \nearrow \exists! & \downarrow \pi_1(p) \\ \pi_1(Z, z_0) & \xrightarrow{\pi_1(f)} & \pi_1(X, x_0) \end{array}$$

Proof. Uniqueness: Let $\tilde{f}'$ be a second lift. For $z \in Z$, choose $\gamma: I \rightarrow Z$ with $\gamma(0) = z_0$ and $\gamma(1) = z$. Then $\tilde{f} \circ \gamma$ and $\tilde{f}' \circ \gamma$ are two lifts of the path $f \circ \gamma$ starting at y_0 . Hence $\tilde{f} \circ \gamma = \tilde{f}' \circ \gamma$ and, in particular, $\tilde{f}(z) = \tilde{f}(\gamma(1)) = \tilde{f}'(\gamma(1)) = \tilde{f}'(z)$.

Existence: For $z \in Z$, choose $\gamma: I \rightarrow Z$ with $\gamma(0) = z_0$ and $\gamma(1) = z$ as above. Let $\widetilde{f \circ \gamma}: I \rightarrow Y$ be the unique lift of $f \circ \gamma$ with $\widetilde{f \circ \gamma}(0) = y_0$. Set $\widetilde{f}(z) := \widetilde{f \circ \gamma}(1)$. This is well-defined: if $\gamma': I \rightarrow Z$ is another path with $\gamma'(0) = z_0$ and $\gamma'(1) = z$, then $[f \circ (\gamma\overline{\gamma'})] \in G(Y, y_0)$, and therefore

$$\widetilde{f \circ (\gamma\overline{\gamma'})} = \widetilde{(f \circ \gamma)(f \circ \overline{\gamma'})}$$

is a loop. Thus $\widetilde{f \circ \gamma}(1) = \widetilde{f \circ \gamma'}(1)$. Moreover,

$$p \circ \widetilde{f}(z) = p(\widetilde{f \circ \gamma}(1)) = f \circ \gamma(1) = f(z).$$

It remains to prove continuity. Let $z_1 \in Z$, and let $U \subseteq Y$ be a neighborhood of $\widetilde{f}(z_1)$, WLOG small enough that $p|_U$ is a homeomorphism. Then there is a path-connected neighborhood V of z_1 with $f(V) \subseteq p(U)$. Fix a path $\gamma: I \rightarrow Z$, $\gamma(0) = z_0$, $\gamma(1) = z_1$. For $z \in V$, choose $\gamma': I \rightarrow V$, $\gamma'(0) = z_1$, $\gamma'(1) = z$. Then

$$\widetilde{f}(z) = \widetilde{f \circ \gamma\gamma'}(1) = (\widetilde{f \circ \gamma}(p|_U^{-1}(f \circ \gamma')))(1) \in U.$$

□

Example. Let $g \geq 1$, $n \geq 2$. Then every map $S^n \xrightarrow{f} \Sigma_g$ is *null-homotopic* (that is, homotopic to a constant map). Since $\pi_1(S^n, \bullet) \stackrel{\text{III.2.12}}{=} \{1\}$, we have the diagram

$$\begin{array}{ccc} & (\mathbb{R}^2, 0) & \xrightarrow[r \simeq]{(0,0)} \\ \exists! \widetilde{f} \nearrow & \downarrow p & \nwarrow i \\ (S^n, \bullet) & \xrightarrow{f} & (\Sigma_g, \bullet) \end{array}$$

so $f = p \circ \widetilde{f} = p \circ \text{id}_{\mathbb{R}^2} \circ \widetilde{f} \simeq p \circ i \circ r \circ \widetilde{f} = \text{const.}$

IV.3. Classification of Covering Spaces

Let (X, x_0) be path-connected and locally path-connected.

Goal. Find all path-connected coverings $p: (Y, y_0) \rightarrow (X, x_0)$ up to *basepoint-preserving isomorphism*:

$$\begin{array}{ccc} (Y, y_0) & \xrightarrow{\cong} & (Y', y'_0) \\ & \searrow p & \swarrow p' \\ & (X, x_0) & \end{array}$$

Desired correspondence.

$$\begin{aligned} \Phi: \{p: (Y, y_0) \rightarrow (X, x_0) \text{ path-connected}\} / \cong &\xrightarrow{1:1} \{G \leq \pi_1(X, x_0)\}, \\ [p: (Y, y_0) \rightarrow (X, x_0)] &\longmapsto G(Y, y_0). \end{aligned}$$

Theorem IV.3.1. (Uniqueness theorem) Let $p: (Y, y_0) \rightarrow (X, x_0)$ and $p': (Y', y'_0) \rightarrow (X, x_0)$ be path-connected coverings. Then:

$$\begin{array}{ccc} (Y, y_0) & \xrightarrow[\varphi]{\cong} & (Y', y'_0) \\ & \searrow p & \swarrow p' \\ & (X, x_0) & \end{array} \iff G(Y, y_0) = G(Y', y'_0).$$

(“ $\Rightarrow$ ”: Φ is well-defined; “ $\Leftarrow$ ”: Φ is injective.)

Proof. “ $\Rightarrow$ ”: $G(Y, y_0) = \text{im}(\pi_1(p)) = \text{im}(\pi_1(p' \circ \varphi)) = \text{im}(\pi_1(p')) = G(Y', y'_0)$.

“ $\Leftarrow$ ”:

$$\begin{array}{ccc} & (Y', y'_0) & \\ \exists! \varphi \nearrow & \downarrow p' & \\ (Y, y_0) & \xrightarrow{p} & (X, x_0) \end{array} \quad \begin{array}{ccc} & (Y, y_0) & \\ \exists! \psi \nearrow & \downarrow p & \\ (Y', y'_0) & \xrightarrow{p'} & (X, x_0) \end{array}$$

$$\begin{array}{ccc} & (Y, y_0) & \\ \psi \circ \varphi \nearrow & \downarrow p & \\ (Y, y_0) & \xrightarrow{p} & (X, x_0) \end{array} \quad \begin{array}{ccc} & (Y', y'_0) & \\ \varphi \circ \psi \nearrow & \downarrow p' & \\ (Y', y'_0) & \xrightarrow{p'} & (X, x_0) \end{array}$$

Note: If X is locally path-connected, then so is Y . Uniqueness of lifting implies $\psi \circ \varphi = \text{id}_{(Y, y_0)}$ and $\varphi \circ \psi = \text{id}_{(Y', y'_0)}$. $\square$

Now let $G \leq \pi_1(X, x_0)$ be given. For $x \in X$, set

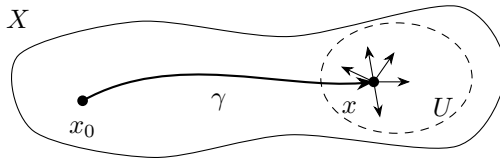
$$Y_x := \{\gamma: I \rightarrow X : \gamma(0) = x_0, \gamma(1) = x\} / \sim,$$

where $\gamma \sim \gamma' :\iff [\gamma\gamma'] \in G$. Let $Y := \coprod_{x \in X} Y_x$ and define $p: Y \rightarrow X, y \mapsto x$ for $y \in Y_x$, and $y_0 = [c_{x_0}]$. Then $p(y_0) = x_0$, and p is surjective. To do:

1. Define a path-connected topology on Y .
2. Show that p is a covering (continuous, locally trivial, with discrete fibers).
3. Show that $G(Y, y_0) = G$.

1. For $y := [\gamma] \in Y$ and an open path-connected neighborhood $U \subseteq X$ of $x := \gamma(1)$, let

$$V(U, y) = \{[\gamma\alpha] \sim : \alpha: I \rightarrow U, \alpha(0) = x\}.$$



This is well-defined, since $[\gamma\gamma'] \in G$ implies

$$[\gamma\alpha\overline{\gamma'\alpha}] = [\gamma\alpha\overline{\gamma'}] = [\gamma\gamma'] \in G, \quad \text{so } [\gamma\alpha] = [\gamma'\alpha].$$

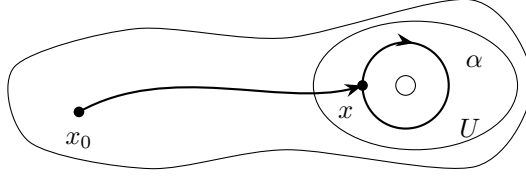
We declare $V \subseteq Y$ to be *open* if every $y \in V$ admits a set $V(U, y)$ as above with $V(U, y) \subseteq V$. This defines a topology: $\emptyset, Y$ are open, unions of open sets are open, and the intersection of two open sets V_1, V_2 is open. Indeed, if $y \in V_1 \cap V_2$ and $V(U_1, y) \subseteq V_1, V(U_2, y) \subseteq V_2$,

then $U_1 \cap U_2$ contains a path-connected neighborhood of $p(y)$ because X is locally path-connected. By construction, the sets $V(U, y)$ form a neighborhood basis at y . The space Y is path-connected, since for $y = [\gamma]_\sim \in Y$ the map

$$I \rightarrow Y, \quad t \mapsto [I \rightarrow X, s \mapsto \gamma(s \cdot t)]_\sim$$

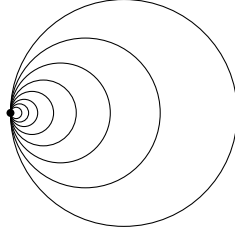
is a continuous path from y_0 to y .

2. The map $p: Y \rightarrow X$ is continuous: for $y \in Y$ and a path-connected neighborhood U of $p(y)$, we have $p(V(U, y)) = U$. We now wish to show that $p: Y \rightarrow X$ is locally trivial with discrete fibers. The latter condition means that for every $x \in X$ and $y \in Y_x$, there is an open path-connected neighborhood $U \subseteq X$ of x with $Y_x \cap V(U, y) = \{y\}$. Thus, for $y = [\gamma]_\sim$, we require $[\gamma\alpha]_\sim = [\gamma]_\sim$ for every $\alpha: I \rightarrow U$ with $\alpha(0) = \alpha(1) = x$; that is, $[\gamma\alpha\bar{\gamma}] \in G$.

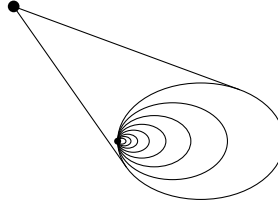


For this to hold even when $G = \{1\}$, for every $x \in X$, there must be a neighborhood $U \subseteq X$ of x such that every loop based at x in U is null-homotopic in X . We call such a space *semilocally simply connected*.

- The *Hawaiian earring* is not semilocally simply connected.



- The *cone* on the Hawaiian earring is semilocally simply connected, but not locally simply connected.

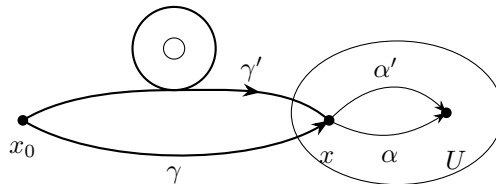


- Manifolds are even locally simply connected.

From now on, assume that X is semilocally simply connected. For sufficiently small U , we then have $\gamma\alpha\bar{\gamma} \simeq_{\{0,1\}} c_{x_0}$, so $[\gamma\alpha\bar{\gamma}] = 1 \in G$. Such an open set U also has the following property: if

$$z \in V(U, y) \cap V(U, y') \quad \text{for } y, y' \in Y_x, \quad y = [\gamma]_\sim, \quad y' = [\gamma']_\sim,$$

that is, $z = [\gamma\alpha]_\sim = [\gamma'\alpha']_\sim$,



then $[\gamma]_{\sim} = [\gamma']_{\sim}$, so $y = y'$. Consequently, $V(U, y) \cap V(U, y') = \emptyset$ for $y \neq y'$. Moreover, all sets $V(U, y)$ are open, and $p|_{V(U, y)}$ is a continuous open bijection, hence a homeomorphism. We therefore obtain a homeomorphism

$$p^{-1}(U) = \bigcup_{y \in Y_x} V(U, y) \cong \coprod_{y \in Y_x} V(U, y) \cong U \times Y_x,$$

taking $p|_{p^{-1}(U)}$ to pr_U .

3. For $[\gamma] \in \pi_1(X, x_0)$, we have $[\gamma] \in G(Y, y_0)$ if and only if the lift $\tilde{\gamma}$ with $\tilde{\gamma}(0) = y_0$ also satisfies $\tilde{\gamma}(1) = y_0$. Since $\tilde{\gamma}(1) = [\gamma]_{\sim}$, this is equivalent to

$$[\gamma]_{\sim} = [c_{x_0}]_{\sim} \iff [\gamma \overline{c_{x_0}}] \in G \iff [\gamma] \in G.$$

We have thus proved:

Theorem IV.3.2. (Existence theorem) Let (X, x_0) be path-connected, locally path-connected, and semilocally simply connected. Then every subgroup $G \leq \pi_1(X, x_0)$ admits a path-connected covering $p: (Y, y_0) \rightarrow (X, x_0)$ with $G(Y, y_0) = G$.

(Φ is surjective if X is also semilocally simply connected.)

Remark. As we already know, this covering is unique up to pointed isomorphism. It is also locally path-connected and semilocally simply connected (by lifting a null-homotopy).

IV.4. Deck Transformations and the Galois Correspondence

Definition IV.4.1. Let $p: Y \rightarrow X$ be a covering. A *deck transformation* of p is a homeomorphism $\varphi: Y \xrightarrow{\cong} Y$ with $p \circ \varphi = p$.

$$\begin{array}{ccc} Y & \xrightarrow[\cong]{\varphi} & Y \\ & \searrow p & \swarrow p \\ & X & \end{array}$$

The set $D(p)$ of deck transformations clearly forms a group under composition, called the *deck transformation group* (or the automorphism group of p).

Examples.

$$D(\mathbb{R} \rightarrow S^1) \cong \mathbb{Z}, \quad D(\mathbb{R}^n \rightarrow T^n) \cong \mathbb{Z}^n, \quad D(S^n \rightarrow \mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}.$$

$$D\left(\begin{array}{c} \text{three sheets} \\ \rightarrow \end{array} \right) \cong \mathbb{Z}/3\mathbb{Z}$$

$$D\left(\begin{array}{c} \text{figure-eight} \\ \rightarrow \end{array} \right) \cong \mathbb{Z}/2\mathbb{Z}$$

$$D\left(\begin{array}{c} \text{Y-junction} \\ \rightarrow \end{array} \right) \cong \{1\}$$

(All of these are stated without proof for now.) We have an action $D(p) \curvearrowright Y$ given by

$$D(p) \times Y \rightarrow Y, \quad (\varphi, y) \mapsto \varphi(y).$$

If Y is path-connected and locally path-connected, then this action is *free* ($G \curvearrowright X$ is free $:\iff gx \neq x$ for every $x \in X$ and every $g \in G \setminus \{1\}$): If $\varphi(y) = y$ for some $y \in Y$, then $\varphi = \text{id}_Y$ by the uniqueness assertion of the [lifting criterion](#):

$$\begin{array}{ccc} & & (Y, y) \\ & \nearrow \varphi & \downarrow p \\ (Y, y) & \xrightarrow{p} & (X, p(y)) \end{array}$$

For each $x_0 \in X$, restriction gives $D(p) \curvearrowright Y_{x_0} := p^{-1}(x_0)$. Let $y_0, y_1 \in Y_{x_0}$. By the [uniqueness theorem](#),

$$\begin{array}{ccc} (Y, y_0) & \xrightarrow[\cong]{\varphi} & (Y, y_1) \\ & \searrow p \quad \swarrow p & \\ & (X, x_0) & \end{array}$$

there is $\varphi \in D(p)$ with $\varphi(y_0) = y_1$ if and only if $G(Y, y_0) = G(Y, y_1)$. Let $\tilde{\gamma}: I \rightarrow Y$ with $\tilde{\gamma}(0) = y_0$ and $\tilde{\gamma}(1) = y_1$. Set $\gamma := p \circ \tilde{\gamma}$ to obtain

$$\begin{array}{ccc} \pi_1(Y, y_0) & \xrightarrow[\cong]{[\tilde{\gamma}(\cdot)]} & \pi_1(Y, y_1) \\ \pi_1(p) \downarrow & & \downarrow \pi_1(p) \\ \pi_1(X, x_0) & \xrightarrow[\cong]{[\gamma]^{-1}(\cdot)[\gamma]} & \pi_1(X, x_0). \end{array}$$

Thus $G(Y, y_1) = [\gamma]^{-1}G(Y, y_0)[\gamma]$; that is, y_1 lies in the $D(p)$ -orbit of y_0 if and only if $[\gamma]$ lies in the *normalizer* of $G(Y, y_0)$.

Definition IV.4.2. Let G be a group and $H \leq G$ a subgroup. Then

$$N_H := N_H^G := \{g \in G : g^{-1}Hg = H\}$$

is the *normalizer* of H in G .

Definition IV.4.3. A subgroup $H \leq G$ is called *normal*, or a *normal subgroup*, if $N_H^G = G$ (notation: $H \trianglelefteq G$).

Examples.

- $\{1\}, G \trianglelefteq G$.
- $H \trianglelefteq N_H^G \leq G$ (N_H^G is maximal with this property).
- If G is abelian, then every $H \leq G$ is normal.
- For every field K , we have $\text{SL}_n(K) \trianglelefteq \text{GL}_n(K)$.
- The subgroup $\{\text{id}, (1, 2)\} \leq S_3$ is not normal; $\{\text{id}, (1, 2, 3), (1, 3, 2)\} \trianglelefteq S_3$.

Lemma IV.4.4. For a homomorphism $\varphi: G \rightarrow H$, we have $\ker \varphi \trianglelefteq G$.

Proof. Let $g_0 \in \ker \varphi$ and $g \in G$. Then $\varphi(g^{-1}g_0g) = \varphi(g)^{-1} \cdot 1 \cdot \varphi(g) = 1$, so $g^{-1}g_0g \in \ker \varphi$. $\square$

Note. In general, $\text{im}(\varphi) \leq H$ is merely a subgroup. For $H \leq G$, let $G/H = \{gH : g \in G\}$ denote the set of *left cosets* of H . The number $[G : H] := |G/H|$ is the *index* of H in G . If $H \trianglelefteq G$, then $gH = Hg$, so

$$(g_1H)(g_2H) = g_1(Hg_2)H = g_1(g_2H)H = g_1g_2H,$$

that is, multiplication in G induces a well-defined group structure on G/H , and the group G/H is called the *factor group*, or *quotient group*, of G by H .

Theorem IV.4.5. (First isomorphism theorem) Let $\varphi: G \rightarrow H$ be a homomorphism. Then

$$\bar{\varphi}: G/\ker \varphi \longrightarrow \text{im } \varphi, \quad g(\ker \varphi) \longmapsto \varphi(g)$$

is a well-defined isomorphism.

Proof. Well-definedness: For $g_1 \in \ker \varphi$, we have $\varphi(gg_1) = \varphi(g)\varphi(g_1) = \varphi(g)$. Homomorphism: clear, since φ is a homomorphism. Injectivity:

$$\bar{\varphi}(g \ker \varphi) = 1 \iff \varphi(g) = 1 \iff g \in \ker \varphi \iff g \ker \varphi = \ker \varphi.$$

$\square$

Theorem IV.4.6. Let $p: (Y, y_0) \rightarrow (X, x_0)$ be a path-connected and locally path-connected covering, and let $G := G(Y, y_0)$. For $[\gamma] \in N_G^{\pi_1(X, x_0)}$, let $\varphi_{[\gamma]} \in D(p)$ be the unique deck transformation with $\varphi_{[\gamma]}(y_0) = \tilde{\gamma}(1)$. Then

$$N_G \rightarrow D(p), \quad [\gamma] \mapsto \varphi_{[\gamma]}$$

defines a homomorphism, which induces an isomorphism $N_G/G \xrightarrow{\cong} D(p)$.

Proof. We have

$$\begin{aligned} \varphi_{[\alpha]}(\varphi_{[\beta]}(y_0)) &= (\varphi_{[\alpha]} \circ \tilde{\beta}^{y_0})(1) \stackrel{\text{uniqueness of path lifting}}{=} \tilde{\beta} \varphi_{[\alpha]}(y_0)(1) \\ &= \tilde{\beta} \tilde{\alpha}^{y_0}(1) = \tilde{\alpha} \tilde{\beta}^{y_0}(1) = \varphi_{[\alpha][\beta]}(y_0). \end{aligned}$$

The uniqueness assertion of the [lifting criterion](#) gives $\varphi_{[\alpha][\beta]} = \varphi_{[\alpha]} \circ \varphi_{[\beta]}$. The homomorphism is surjective: for $\varphi \in D(p)$, take $\tilde{\gamma}: I \rightarrow Y$ with $\tilde{\gamma}(0) = y_0$, $\tilde{\gamma}(1) = \varphi(y_0)$, and let $\gamma := p \circ \tilde{\gamma}$. Then $[\gamma] \in N_G$ and $\varphi = \varphi_{[\gamma]}$. Moreover,

$$\varphi_{[\gamma]} = \text{id}_Y \iff \tilde{\gamma}^{y_0}(1) = y_0 \iff [\gamma] \in G,$$

that is, G is the kernel of the homomorphism. $\square$

Theorem IV.4.7. For a path-connected and locally path-connected covering $p: Y \rightarrow X$, the following are equivalent:

- (i) For every $x \in X$, the action $D(p) \curvearrowright Y_x$ is *transitive* (that is, for any two points $y_1, y_2 \in Y_x$, there is $\varphi \in D(p)$ with $\varphi(y_1) = y_2$; equivalently, the fibers Y_x are the orbits of $D(p) \curvearrowright Y$).
- (ii) For every $y \in Y$, we have $G(Y, y) \trianglelefteq \pi_1(X, p(y))$.

Proof. As we have already seen, there is $\varphi \in D(p)$ with $\varphi(y_0) = y_1$ if and only if, for $\tilde{\gamma}: I \rightarrow Y$, $\tilde{\gamma}(0) = y_0$, $\tilde{\gamma}(1) = y_1$, we have $[p \circ \tilde{\gamma}] \in N_G^{\pi_1(X, x_0)}$. $\square$

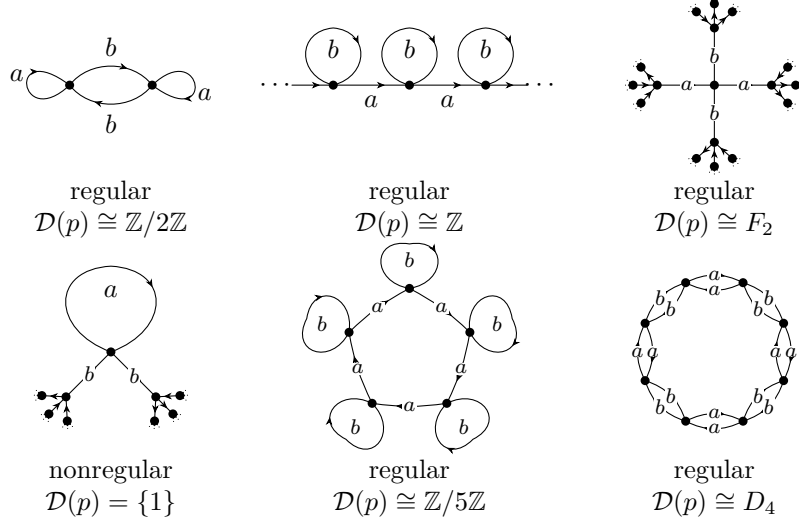
Definition IV.4.8. Such a covering is called *regular*, *normal*, or *Galois*.

Corollary IV.4.9. For a regular covering $p: (Y, y_0) \rightarrow (X, x_0)$, we have

- (i) $D(p) \cong \pi_1(X, x_0)/G(Y, y_0)$,
- (ii) $|Y_{x_0}| = [\pi_1(X, x_0) : G(Y, y_0)]$,
- (iii) $Y/D(p) \cong X$.

□

Examples.



For the third example: $D(p) \cong F_2$ (see [Chapter V](#)). For the last example:

$$D(p) \cong D_4 \cong \langle r, s \mid r^4, s^2, (rs)^2 \rangle \cong \mathbb{Z}/4 \rtimes \mathbb{Z}/2 \quad (\text{see V.5}).$$

Theorem IV.4.10. We have $\pi_1(S^1, 1) \cong \mathbb{Z}$.

Proof. For the covering $p: (\mathbb{R}, 0) \rightarrow (S^1, 1)$, $t \mapsto \exp(2\pi it)$, we have $G(\mathbb{R}, 0) = \{1\} \leq \pi_1(S^1, 1)$, so $\pi_1(S^1, 1) \cong D(p)$. But

$$T: \mathbb{Z} \rightarrow D(p), \quad n \mapsto (x \mapsto x + n)$$

is an isomorphism, since T is

- a homomorphism: clear.
- injective: $x = x + n$ implies $n = 0$.
- surjective: Let $\varphi \in D(p)$, so $p \circ \varphi = p$, that is, $\exp(2\pi i\varphi(x)) = \exp(2\pi ix)$, and hence $x - \varphi(x) = n_x \in \mathbb{Z}$. But $\mathbb{R} \rightarrow \mathbb{Z}$, $x \mapsto n_x = x - \varphi(x)$, is continuous and therefore constant. Thus there is $n \in \mathbb{Z}$ with $\varphi(x) = x + n$, so $T(n) = \varphi$.

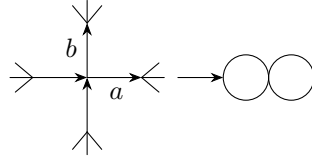
□

We have now proved that $S^n \not\cong T^n$ for $n \neq 1$.

Definition IV.4.11. A path-connected and locally path-connected covering $p: Y \rightarrow X$ is called *universal* if Y is simply connected.

Examples.

- $\mathbb{R} \rightarrow S^1$,
- $\mathbb{R}^n \rightarrow T^n$,
- $\mathbb{R}^2 \rightarrow \Sigma_g$,
- $\mathbb{R}^2 \rightarrow K$ (Klein bottle),
- $\mathbb{R} \times I \rightarrow M$ (Möbius strip),
- $S^n \rightarrow \mathbb{R}P^n$ for $n \geq 2$.



Terminology. We call a topological space X *well-connected* if it is path-connected, locally path-connected, and semilocally simply connected.

Corollary IV.4.12. Let X be well-connected and $x_0 \in X$. Then there is exactly one universal covering $(\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ up to pointed isomorphism.

Why universal? Let $(Y_1, y_1) \xrightarrow{p_1} (X, x_0)$ and $(Y_2, y_2) \xrightarrow{p_2} (X, x_0)$ be path-connected coverings of a well-connected space (X, x_0) . Suppose $G(Y_1, y_1) \leq G(Y_2, y_2)$.

$$\begin{array}{ccc} & (Y_2, y_2) & \\ \exists! f \nearrow & \downarrow p_2 & \\ (Y_1, y_1) & \xrightarrow{p_1} & (X, x_0) \end{array}$$

Proposition IV.4.13. The map $f: (Y_1, y_1) \rightarrow (Y_2, y_2)$ is a covering, and it is regular if and only if $G(Y_1, y_1) \trianglelefteq G(Y_2, y_2)$.

Proof. Since $\pi_1(Y_2, y_2) \cong G(Y_2, y_2)$, the second part of the statement has already been proved. The subgroup $\pi_1(p_2)^{-1}(G(Y_1, y_1)) \leq \pi_1(Y_2, y_2)$ corresponds to a covering $p_3: (Y_3, y_3) \rightarrow (Y_2, y_2)$.

$$\begin{array}{ccc} (Y_3, y_3) & \xrightarrow{p_3} & (Y_2, y_2) \\ \downarrow h \quad \uparrow g & \nearrow f & \downarrow p_2 \\ (Y_1, y_1) & \xrightarrow{p_1} & (X, x_0) \end{array}$$

Let g be the lift of f along p_3 , and let h be the lift of $p_2 \circ p_3$ along p_1 . Then

$$p_2 \circ p_3 \circ g \circ h = p_2 \circ f \circ h = p_2 \circ p_3 \quad \text{and} \quad p_1 \circ h \circ g = p_2 \circ p_3 \circ g = p_2 \circ f = p_1.$$

Thus $g \circ h$ is a lift of $p_2 \circ p_3$ along $p_2 \circ p_3$, and $h \circ g$ is a lift of p_1 along p_1 . Uniqueness of lifting gives $g \circ h = \text{id}$ and $h \circ g = \text{id}$. Hence the homeomorphism h identifies the map f with p_3 ; that is, f is a covering. $\square$

Thus, if $G(Y_1, y_1) \leq G(Y_2, y_2)$, there is a commutative triangle of coverings

$$\begin{array}{ccc}
 (Y_1, y_1) & & \\
 \downarrow p_1 & \searrow p_3 & \\
 & (Y_2, y_2) & \\
 & \swarrow p_2 & \\
 (X, x_0) & &
 \end{array}$$

In the special case $G(\tilde{X}, \tilde{x}_0) = \{1\}$, we obtain:

Theorem IV.4.14. (Universal covering) Let (X, x_0) be well-connected. Then $(\tilde{X}, \tilde{x}_0)$ is, in a unique way, a regular covering of every other path-connected covering (Y, y_0) , with

$$\begin{array}{ccc}
 (\tilde{X}, \tilde{x}_0) & & \\
 \downarrow & \searrow & \\
 & (Y, y_0) & \\
 & \swarrow p & \\
 (X, x_0) & &
 \end{array}$$

□

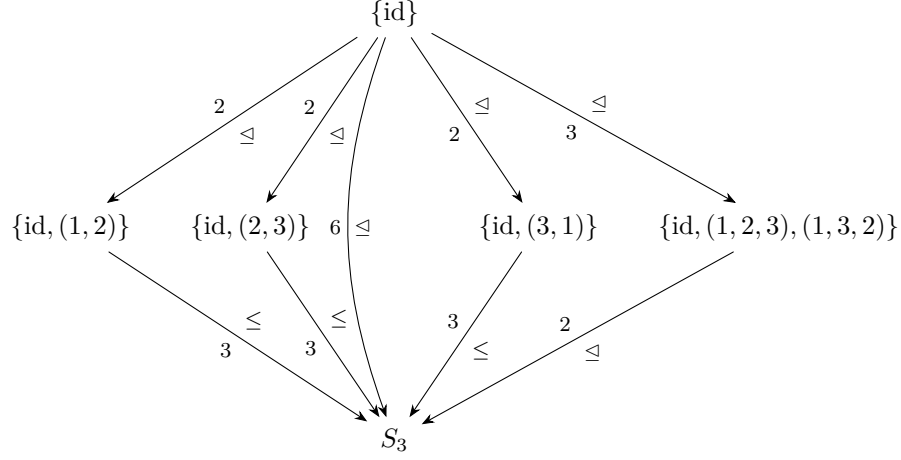
In particular, every path-connected covering (Y, y_0) is obtained as

$$(\tilde{X}, \tilde{x}_0)/G(Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)/\pi_1(X, x_0)$$

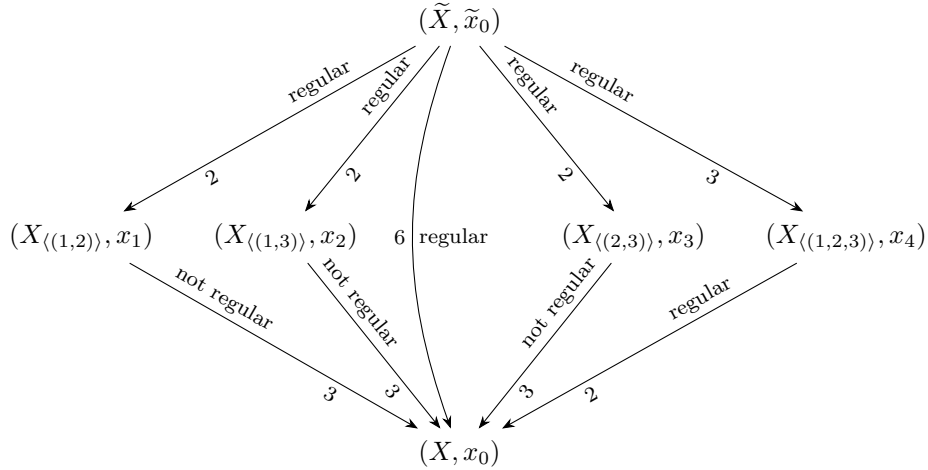
with $G(Y, y_0) \leq \pi_1(X, x_0) \cong D((\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0))$.

Conclusion. For a well-connected space (X, x_0) , there is a *Galois correspondence* between the directed system of subgroups of $\pi_1(X, x_0)$ and the directed system of pointed path-connected coverings of X . It takes subgroup indices to numbers of sheets and takes precisely the normal subgroups to normal coverings. The whole group $\pi_1(X, x_0)$ corresponds to the trivial covering, and the trivial subgroup corresponds to the universal covering.

Example. $\pi_1(X, x_0) \cong S_3$.



Index



Number of sheets

Since $S^n \xrightarrow{p} \mathbb{R}P^n$ is a two-sheeted universal covering for $n \geq 2$, it also follows that $\pi_1(\mathbb{R}P^n, x_0) \cong D(p) \cong \mathbb{Z}/2\mathbb{Z}$.

Lemma IV.4.15. We have $(\mathbb{R}P^3, x_0) \cong (\text{SO}(3), \text{id})$.

Proof. Let $\varphi: D^3 \rightarrow \text{SO}(3)$ be the map with $\varphi(0) = \text{id}$ that sends $v \in D^3 \setminus \{0\}$ to rotation about the axis $\mathbb{R}v$ through the angle $|v| \cdot \pi$ (right-hand rule). This map is surjective, and since $\varphi(v) = \varphi(-v)$ for $v \in S^2$, it induces a continuous bijection $\bar{\varphi}: \mathbb{R}P^3 \rightarrow \text{SO}(3)$. Because $\mathbb{R}P^3$ is compact and $\text{SO}(3)$ is Hausdorff, $\bar{\varphi}$ is a homeomorphism. $\square$

Thus $\pi_1(\text{SO}(3), \text{id}) \cong \mathbb{Z}/2\mathbb{Z}$, which can be illustrated by the *belt trick*.

V. Introduction to Category Theory

V.1. Categories

Many mathematical concepts, such as products “ $X \times Y$,” are introduced in exactly the same formal way, whether we are dealing with groups, vector spaces, topological spaces, and so on.

- Find an abstract definition that yields these special cases.
- To define “ $X \times Y$,” however, we must specify that X and Y are two “instances” (*objects*) of the same “type” (*category*).

Definition V.1.1. A category $\mathcal{C}$ consists of a class of *objects* $\text{Ob}(\mathcal{C})$ and, for each pair of objects $X, Y \in \text{Ob}(\mathcal{C})$, a class of *morphisms* (or *arrows*) $\text{Hom}_{\mathcal{C}}(X, Y)$. We write $(f: X \rightarrow Y) \in \text{Hom}_{\mathcal{C}}(X, Y)$ and call X the *source* and Y the *target* of the arrow f . The following conditions must hold:

- (i) For $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, there is a specified arrow $g \circ f: X \rightarrow Z$, the *composition* of f with g , and we have $(h \circ g) \circ f = h \circ (g \circ f)$.
- (ii) For $X \in \text{Ob}(\mathcal{C})$, there is an arrow $\text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$, the *identity* on X , such that $f \circ \text{id}_X = f$ for all $f: X \rightarrow Y$ and $\text{id}_X \circ g = g$ for all $g: Y \rightarrow X$.

Notes.

- We use “class” rather than set because of logical difficulties (“the set of all sets”!?).
- For $X \in \text{Ob}(\mathcal{C})$, the arrow $\text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$ is unique. Proof: $\text{id}'_X = \text{id}_X \circ \text{id}'_X = \text{id}_X$.
- Arrows of the form $f: X \rightarrow X$ are called *endomorphisms*.
- An arrow $f: X \rightarrow Y$ is called an *isomorphism* if it has an *inverse* $g: Y \rightarrow X$ with $g \circ f = \text{id}_X$, $f \circ g = \text{id}_Y$.
- If f has an inverse, it is unique. Proof:

$$g' = g' \circ \text{id}_Y = g' \circ (f \circ g) = (g' \circ f) \circ g = \text{id}_X \circ g = g.$$

- If f is both an endomorphism and an isomorphism, f is called an *automorphism*.

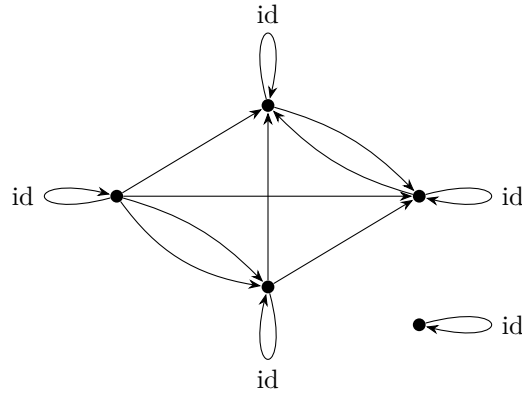
Examples.

Category	Objects	Morphisms	Isomorphisms
Set	Sets	Maps	Bijections
$R\text{-mod}$	R -modules	R -linear maps	R -isomorphisms
$K\text{-Vect}$	K -vector spaces	K -linear maps	K -isomorphisms
Group	Groups	Group homomorphisms	Group isomorphisms
Ab	Abelian groups	Group homomorphisms	Group isomorphisms
Top	Topological spaces	Continuous maps	Homeomorphisms
$\text{Top}_\bullet$	Pointed topological spaces	Basepoint-preserving continuous maps	Basepoint-preserving homeomorphisms
HoTop	Topological spaces	Homotopy classes of continuous maps	Homotopy classes of homotopy equivalences
$\text{HoTop}_\bullet$	Pointed topological spaces	Pointed homotopy classes of basepoint-preserving continuous maps	Pointed homotopy classes of basepoint-preserving homotopy equivalences
$\text{Top}^{(2)}$	Pairs (X, A) of topological spaces with inclusion $A \subseteq X$	Continuous maps preserving the distinguished subspaces	Homeomorphisms restricting to homeomorphisms

Thus, a category is a directed graph with loops and multiple edges

+ associative composition

+ identities



Definition V.1.2. Let $\mathcal{C}$ be a category. Its *opposite category* $\mathcal{C}^{\text{op}}$ is given by $\text{Ob}(\mathcal{C}^{\text{op}}) = \text{Ob}(\mathcal{C})$ and $\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$, with the composition $f \circ g$ in $\mathcal{C}^{\text{op}}$ given by $g \circ f$ in $\mathcal{C}$.

V.2. Functors

Definition V.2.1. A (*covariant*) *functor* $F: \mathcal{C} \rightarrow \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ assigns an object $F(X) \in \text{Ob}(\mathcal{D})$ to each $X \in \text{Ob}(\mathcal{C})$ and a morphism $F(f) \in \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ to each $f \in \text{Hom}_{\mathcal{C}}(X, Y)$, such that

$$(i) \quad F(f \circ g) = F(f) \circ F(g),$$

$$(ii) \quad F(\text{id}_X) = \text{id}_{F(X)}.$$

A *contravariant functor* $F: \mathcal{C} \rightarrow \mathcal{D}$ is a covariant functor from $\mathcal{C}^{\text{op}}$ to $\mathcal{D}$; that is, for $f: X \rightarrow Y$ we have $F(f) \in \text{Hom}_{\mathcal{D}}(F(Y), F(X))$ and $F(f \circ g) = F(g) \circ F(f)$.

Examples.

- (i) Every category $\mathcal{C}$ has an *identity functor* $\text{id}_{\mathcal{C}}$.
- (ii) *Forgetful functors* $K\text{-Vect} \rightarrow \text{Ab}$, $\text{Ab} \rightarrow \text{Group}$, ...
- (iii) The fundamental group defines $\pi_1: \text{Top}_{\bullet} \rightarrow \text{Group}$. For $f: (X, x_0) \rightarrow (Y, y_0)$, we obtain $\pi_1(f): \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$.
- (iv) We have functors $\text{Top} \rightarrow \text{HoTop}$ and $\text{Top}_{\bullet} \rightarrow \text{HoTop}_{\bullet}$ that send f to $[f]$.
- (v) We have already seen that the functor π_1 factors through $\text{HoTop}_{\bullet}$:

$$\begin{array}{ccc} & \text{HoTop}_{\bullet} & \\ \nearrow & & \searrow \pi_1 \\ \text{Top}_{\bullet} & \xrightarrow{\pi_1} & \text{Group} \end{array}$$

(Here we use the fact that functors $F: \mathcal{C} \rightarrow \mathcal{D}$, $G: \mathcal{D} \rightarrow \mathcal{E}$ can be *composed* to give $G \circ F: \mathcal{C} \rightarrow \mathcal{E}$.)

- (vi) *Abelianization* is a functor $(\cdot)_{\text{ab}}: \text{Group} \rightarrow \text{Ab}$ that sends G to the abelian group $G/[G, G]$, where $[G, G] \leq G$ is generated by all *commutators* $g_1 g_2 g_1^{-1} g_2^{-1}$. Clearly, $[G, G] \trianglelefteq G$ and $f([G, G]) \subseteq [H, H]$ for $f: G \rightarrow H$, so we obtain $f_{\text{ab}}: G_{\text{ab}} \rightarrow H_{\text{ab}}$.
- (vii) The *free vector space* is a functor $F: \text{Set} \rightarrow K\text{-Vect}$ that assigns to a set X the vector space of formal K -linear combinations

$$F(X) = \left\{ \sum_{x \in X} \lambda_x x : \lambda_x \in K, \lambda_x = 0 \text{ for almost all } x \in X \right\}.$$

Thus, $X \subseteq F(X)$ is a basis, and $f: X \rightarrow Y$ induces $F(f): F(X) \rightarrow F(Y)$ by unique K -linear extension.

- (viii) *Dualization* is a contravariant functor $D: K\text{-Vect} \rightarrow K\text{-Vect}$ with $D(V) = V^* = \text{Hom}_K(V, K)$, and for $f: V \rightarrow W$ we have $D(f): W^* \rightarrow V^*$, $\delta \mapsto \delta \circ f$.
- (ix) Fix a K -vector space W . Then $(\cdot) \otimes_K W: K\text{-Vect} \rightarrow K\text{-Vect}$ is a covariant functor. For $f: V \rightarrow U$, we obtain $f \otimes \text{id}_W: V \otimes_K W \rightarrow U \otimes_K W$ with $f \otimes \text{id}_W(v \otimes w) = f(v) \otimes w$ for elementary tensors $v \otimes w \in V \otimes_K W$.

$$\text{Construction: } V \otimes_K W = \frac{F(V \times W)}{\left\langle \begin{array}{l} (\lambda u + \mu v, w) - \lambda(u, w) - \mu(v, w), \\ (v, \lambda u + \mu w) - \lambda(v, u) - \mu(v, w) \end{array} \right\rangle}.$$

Slogan. A functorial construction must be independent of choices.

V.3. Natural Transformations

Let V be a finite-dimensional K -vector space. Choosing a basis $B \subseteq V$ gives the dual basis $B^* \subseteq V^*$ defined by $b_i^*(b_j) = \delta_{ij}$, and $b_i \mapsto b_i^*$ defines an isomorphism $V \xrightarrow{\cong} V^*$. An isomorphism $V \xrightarrow{\cong} V^{**}$ exists “naturally,” without making any choices:

$$\text{ev}_V: v \mapsto (V^* \rightarrow K, \delta \mapsto \delta(v)).$$

The advantage is that, for every K -linear map $f: V \rightarrow W$, the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \text{ev}_V \downarrow & & \downarrow \text{ev}_W \\ V^{**} & \xrightarrow{f^{**}} & W^{**}, \end{array}$$

commutes, since for $\delta \in W^*$ and $v \in V$ we have

$$\text{ev}_W(f(v))(\delta) = \delta(f(v)) = \text{ev}_V(v)(f^*(\delta)) = f^{**}(\text{ev}_V(v))(\delta),$$

that is, $\text{ev}_W \circ f = f^{**} \circ \text{ev}_V$.

Definition V.3.1. Let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be functors. A *natural transformation* $\eta: F \rightarrow G$ assigns to each $A \in \text{Ob}(\mathcal{C})$ a morphism $\eta_A \in \text{Hom}_{\mathcal{D}}(F(A), G(A))$ such that, for every arrow $f \in \text{Hom}_{\mathcal{C}}(A, B)$, the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \eta_A \downarrow & & \downarrow \eta_B \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

commutes in $\mathcal{D}$.

We call η_A the *component* of η at A .

Examples.

- (i) We have just seen $\eta = \text{ev}: \text{id}_{K\text{-Vect}} \rightarrow DD = (\cdot)^{**}$.
- (ii) A group G defines a category $\underline{G}$ with exactly one object “ $\bullet$ ” and $\text{Hom}_{\underline{G}}(\bullet, \bullet) = G$, with group multiplication as composition and $\text{id}_{\bullet} = 1 \in G$. A G -action $G \curvearrowright X$ is now the same thing as a functor $F: \underline{G} \rightarrow \text{Set}$. The set is $X := F(\bullet)$, with action $g \cdot x := F(g)(x)$. We need to show:

$$1.) \quad g \cdot (h \cdot x) = (g \cdot h) \cdot x \quad 2.) \quad 1 \cdot x = x.$$

$$\begin{aligned} 1.) \quad g \cdot (h \cdot x) &= g \cdot (F(h)(x)) = F(g)(F(h)(x)) \\ &= F(gh)(x) = (g \cdot h) \cdot x. \end{aligned}$$

$$2.) \quad 1 \cdot x = F(1)(x) = F(\text{id}_{\bullet})(x) = \text{id}_{F(\bullet)}(x) = \text{id}_X(x) = x.$$

For $F, G: \underline{G} \rightarrow \text{Set}$, a natural transformation $\eta: F \rightarrow G$ is an *equivariant map*: for $x \in X$, the identity $(\eta_{\bullet} \circ F(g))(x) = (G(g) \circ \eta_{\bullet})(x)$ means precisely that $\eta_{\bullet}(g \cdot x) = g \cdot \eta_{\bullet}(x)$.

Definition V.3.2. A *natural isomorphism* $\eta: F \xrightarrow{\cong} G$ is a natural transformation whose components are isomorphisms.

Trick question. Is $\text{ev}: \text{id}_{K\text{-Vect}} \rightarrow (\cdot)^{**}$ a natural isomorphism? No, since $V^{**} \not\cong V$ when V is infinite-dimensional. However, ev is a natural isomorphism if we restrict id and $(\cdot)^{**}$ to the category of finite-dimensional K -vector spaces.

Definition V.3.3. An *equivalence* of categories consists of functors $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ such that $G \circ F$ and $F \circ G$ are naturally isomorphic to $\text{id}_{\mathcal{C}}$ and $\text{id}_{\mathcal{D}}$.

Example.

- $\text{Cov}_{(X, x_0)} \rightarrow \text{Sub}_{\pi_1(X, x_0)}$ (see homework)
- Let L/K be a finite Galois extension.

$$\begin{aligned} \text{Intermediate fields}_{L/K} &\longrightarrow \text{Sub}_{\text{Gal}(L/K)}, \\ Z &\longmapsto \{\sigma \in \text{Gal}(L/K) : \sigma|_Z = \text{id}_Z\}, \\ \text{Sub}_{\text{Gal}(L/K)} &\longrightarrow \text{Intermediate fields}_{L/K}, \quad H \longmapsto L^H. \end{aligned}$$

$$\left(\text{Hom}_{\text{Intermediate fields}_{L/K}}(L, L') = \begin{cases} \{L/L'\} & \text{if } L' \subseteq L, \\ \emptyset & \text{otherwise.} \end{cases} \right)$$

V.4. Adjunction

Given a forgetful functor $G: \mathcal{D} \rightarrow \mathcal{C}$, we ask for the most general and efficient construction in the reverse direction.

Example.

- $K\text{-Vect} \rightarrow \text{Set} \rightsquigarrow \text{Free vector space}$ ([Example \(vii\)](#))
- $\text{Ab} \rightarrow \text{Group} \rightsquigarrow \text{Abelianization}$ ([Example \(vi\)](#))

These functors $F: \mathcal{C} \rightarrow \mathcal{D}$ are “left adjoint” to G .

Definition V.4.1. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ be functors. Then F is called *left adjoint* to G (and G is called *right adjoint* to F) if, for every $A \in \text{Ob}(\mathcal{C})$ and $B \in \text{Ob}(\mathcal{D})$, there is a bijection

$$\text{Hom}_{\mathcal{D}}(F(A), B) \cong \text{Hom}_{\mathcal{C}}(A, G(B)), \quad f \longmapsto \bar{f}$$

that is natural in A and B .

“Natural in A ”: for fixed $B \in \text{Ob}(\mathcal{D})$, the functors $\text{Hom}_{\mathcal{D}}(F(\cdot), B)$ and $\text{Hom}_{\mathcal{C}}(\cdot, G(B)): \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ are naturally isomorphic. Here (and elsewhere) we assume that $\mathcal{C}$ and $\mathcal{D}$ are *locally small*; that is, $\text{Hom}(X, Y)$ is always an actual set. Thus, for $f: A' \rightarrow A$ in $\mathcal{C}$ and $\varphi \in \text{Hom}_{\mathcal{D}}(F(A), B)$, we have

$$\bar{\varphi} \circ f = \overline{\varphi \circ F(f)}.$$

“Natural in B ” means that, for fixed $A \in \text{Ob}(\mathcal{C})$, $g: B \rightarrow B'$ in $\mathcal{D}$, and $\varphi \in \text{Hom}_{\mathcal{D}}(F(A), B)$, we have

$$G(g) \circ \bar{\varphi} = \overline{g \circ \varphi}.$$

The term “adjunction” comes from the formal similarity to Hilbert space operators:

$$\langle f(v), w \rangle = \langle v, f^*(w) \rangle.$$

Example.

- The free vector space functor $F: \text{Set} \rightarrow K\text{-Vect}$ is left adjoint to the forgetful functor $G: K\text{-Vect} \rightarrow \text{Set}$:

$$\text{Hom}_{K\text{-Vect}}(F(X), V) \cong \text{Hom}_{\text{Set}}(X, G(V)), \quad \varphi \longmapsto \varphi|_X.$$

This is a bijection by unique K -linear extension. Naturality follows from the formulas above.

- The *free group* functor $F: \mathbf{Set} \rightarrow \mathbf{Group}$ is left adjoint to the forgetful functor $G: \mathbf{Group} \rightarrow \mathbf{Set}$. For a set S , let $F(S)$ consist of all words, such as $a^3b^{-2}aab^{-5}b^5c$, in the alphabet S , subject to the obvious identifications, for example $aa^{-1}b = b$, $c^2c^{-3} = c^{-1}$, ... Composition is given by concatenation. The identity element is the empty word “1” or “ e .” A map $f: S_1 \rightarrow S_2$ has a unique extension $F(f): F(S_1) \rightarrow F(S_2)$,

$$F(f)(abc^{-3}b) = f(a)f(b)f(c)^{-3}f(b).$$

The functor properties are clear, and the natural bijection of Hom-sets follows in the same way.

For every group H , we obtain a group homomorphism

$$\varepsilon_H: F(G(H)) \longrightarrow H,$$

that restricts to id_H on the alphabet H . This defines a natural transformation $\varepsilon: F \circ G \rightarrow \text{id}_{\mathbf{Group}}$ (the “*counit*” of the adjunction). We obtain an isomorphism

$$\bar{\varepsilon}_H: F(G(H)) / \ker \varepsilon_H \xrightarrow{\cong} H,$$

that is, every group is a quotient of a free group!

Definition V.4.2. Let S be a set and $R \subseteq F(S)$. The pair (S, R) is called a *presentation* of the group G if

$$G \cong F(S) / \langle\langle R \rangle\rangle,$$

where $\langle\langle R \rangle\rangle$ denotes the smallest normal subgroup of $F(S)$ containing R . Notation: $G = \langle S \mid R \rangle$.

By means of $\bar{\varepsilon}_H$, every group H has a presentation. We seek efficient presentations, however; for example,

$$\mathbb{Z}^2 \cong \langle a, b \mid aba^{-1}b^{-1} \rangle.$$

Proof. By the adjunction, $a \mapsto (1, 0)$, $b \mapsto (0, 1)$ extends to a surjective homomorphism $p: F(\{a, b\}) \rightarrow \mathbb{Z}^2$, and $\langle\langle aba^{-1}b^{-1} \rangle\rangle \leq \ker p$. We therefore obtain

$$\bar{p}: F(\{a, b\}) / \langle\langle aba^{-1}b^{-1} \rangle\rangle \longrightarrow \mathbb{Z}^2.$$

We need to show that $\bar{p}$ is injective. Let $x = [a]^{n_1}[b]^{m_1} \cdots [a]^{n_k}[b]^{m_k} \in \ker \bar{p}$. Then $\bar{p}(x) = 0$ implies

$$\begin{pmatrix} n_1 + \cdots + n_k \\ m_1 + \cdots + m_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

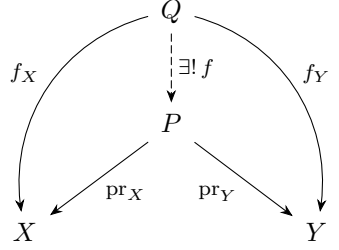
so $x = [a]^{n_1+\cdots+n_k}[b]^{m_1+\cdots+m_k} = 1$. □

V.5. Limits and Colimits

Recall that we seek a categorical definition of the product “ $X \times Y$.” Observe that we have *projections*

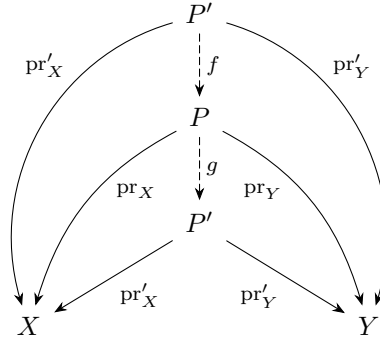
$$\begin{array}{ccc} & X \times Y & \\ \text{pr}_X \swarrow & & \searrow \text{pr}_Y \\ X & & Y. \end{array}$$

Definition V.5.1. Let $\mathcal{C}$ be a category and $X, Y \in \text{Ob}(\mathcal{C})$. A *product* of X and Y is an object $P \in \text{Ob}(\mathcal{C})$ with arrows $\text{pr}_X: P \rightarrow X$ and $\text{pr}_Y: P \rightarrow Y$ satisfying the universal property



That is, given $f_X: Q \rightarrow X$ and $f_Y: Q \rightarrow Y$, there is exactly one arrow $f: Q \rightarrow P$ with $f_X = \text{pr}_X \circ f$ and $f_Y = \text{pr}_Y \circ f$. Notation: $f = f_X \times f_Y$ or $f = (f_X, f_Y)$.

Products need not exist, but when they do, they are unique:



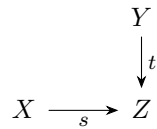
The uniqueness clause in the universal property implies $g \circ f = \text{id}_{P'}$. Similarly, $f \circ g = \text{id}_P$. (“*Abstract nonsense*”)

Example. In **Set**, **Group**, **K-Vect**, **R-mod**, **Top**, and **HoTop**, the categorical product is the usual product.

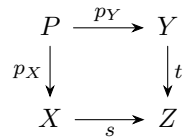
Caution. In **Top**⁽²⁾, we have $(X, A) \times (Y, B) = (X \times Y, A \times B)$, but some authors write $(X, A) \times (Y, B)$ for $(X \times Y, (X \times B) \cup (A \times Y))$.

Products are special cases of *fiber products*.

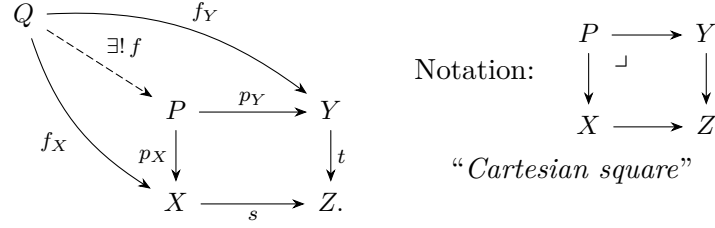
Definition V.5.2. The *fiber product* or *pullback* of the diagram



in $\mathcal{C}$ consists of $P \in \text{Ob}(\mathcal{C})$ and arrows $p_X: P \rightarrow X$ and $p_Y: P \rightarrow Y$ such that



commutes and is universal, that is,



As above, fiber products are unique.

Example.

- Fiber products always exist in **Set**. For

$$\begin{array}{ccc} & Y & \\ & \downarrow t & \\ X & \xrightarrow{s} & Z \end{array} \quad \text{we have} \quad P = \{(x, y) \in X \times Y : s(x) = t(y)\}$$

and $p_X = \text{pr}_X|_P$, $p_Y = \text{pr}_Y|_P$. For $X \xleftarrow{f_X} Q \xrightarrow{f_Y} Y$, the map is $f: Q \rightarrow P$, $q \mapsto (f_X(q), f_Y(q))$.

- In **Top**, use the same construction with the subspace topology inherited from the product topology.

Homework: if $p: E \rightarrow B$ is a fiber bundle and $f: B' \rightarrow B$ is continuous, then $p': f^*E \rightarrow B'$ in

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ p' \downarrow & \lrcorner & \downarrow p \\ B' & \xrightarrow{f} & B \end{array}$$

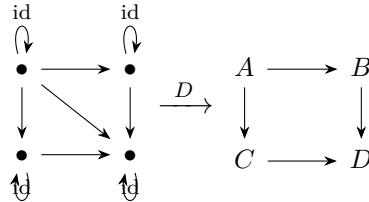
is a fiber bundle ($\leadsto$ a functor $f^*: \text{Cov}_B \rightarrow \text{Cov}_{B'}$).

We seek a further generalization of fiber products.

A category is called *small* if its class of objects and its classes of morphisms are all sets.

Definition V.5.3. A *diagram* in $\mathcal{C}$ is a functor $D: I \rightarrow \mathcal{C}$ from a small category I (the *index category*).

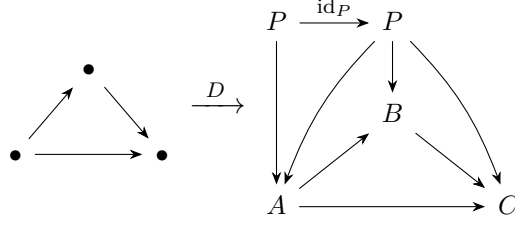
Example.



- Let $P \in \text{Ob}(\mathcal{C})$.

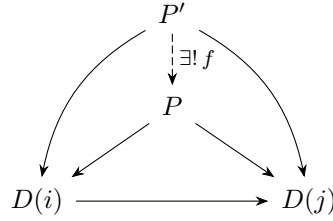
$$\begin{aligned} \mathbf{1}_P^I: I &\longrightarrow \mathcal{C}, \\ X &\longmapsto P, \\ (f: X \rightarrow Y) &\longmapsto \text{id}_P. \end{aligned}$$

Definition V.5.4. A *cone* over the diagram $D: I \rightarrow \mathcal{C}$ consists of an object $P \in \text{Ob}(\mathcal{C})$ and a natural transformation $\eta: \mathbf{1}_P^I \rightarrow D$.



Thus, a cone consists of its apex P and arrows from the apex to every object of the diagram, such that all triangles (the sides of the cone) commute.

Definition V.5.5. A *limit* of $D: I \rightarrow \mathcal{C}$ is a universal cone $\eta: \mathbf{1}_P^I \rightarrow D$: for every other cone $\eta': \mathbf{1}_{P'}^I \rightarrow D$, there is a unique arrow $f: P' \rightarrow P$ with $\eta'_i = \eta_i \circ f$ for all $i \in I$.



By abstract nonsense, limits are unique up to unique isomorphism. Notation: $P = \lim D = \varprojlim D$.

Example.

- The product of $X, Y \in \text{Ob}(\mathcal{C})$ is the limit of the diagram

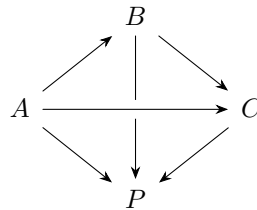
$$(\bullet \quad \bullet) \xrightarrow{D} (X \quad Y).$$

- The fiber product of $X \xrightarrow{s} Z \xleftarrow{t} Y$ is the limit of

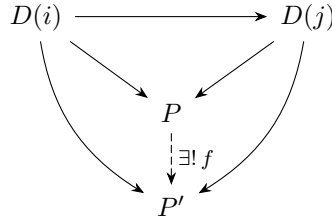
$$\begin{array}{ccc} & \bullet & Y \\ & \downarrow & \downarrow t \\ \bullet & \longrightarrow \bullet & X \xrightarrow{s} Z \end{array} \xrightarrow{D}$$

Categorical concepts have *dual* concepts, obtained by “reversing all arrows.”

Example. A *cocone* over $D: I \rightarrow \mathcal{C}$ is a natural transformation $\eta: D \rightarrow \mathbf{1}_P^I$.



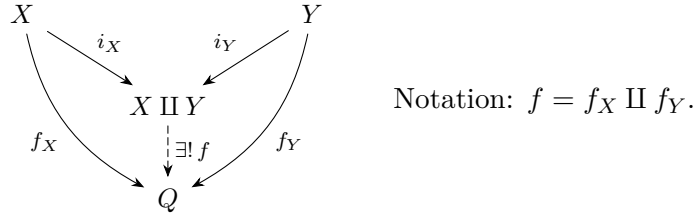
Definition V.5.6. A *colimit* of $D: I \rightarrow \mathcal{C}$ is a universal cocone $\eta: D \rightarrow \mathbf{1}_P^I$:



Again, a colimit is unique up to unique isomorphism. Notation: $P = \operatorname{colim} D = \varinjlim D$.

Example.

- The colimit of $(\bullet \rightarrow \bullet) \xrightarrow{D} (X \rightarrow Y)$ is called the *coproduct* of X and Y . Notation: $X \amalg Y$. Thus, $X \amalg Y$ comes with morphisms $X \xrightarrow{i_X} X \amalg Y \xleftarrow{i_Y} Y$ satisfying the universal property



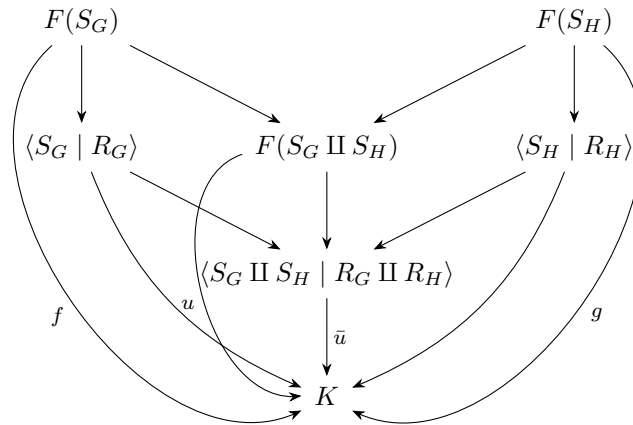
Example.

- In **Set**, the coproduct is the disjoint union.
- In **Top** and **HoTop**, the coproduct of spaces X and Y is the topological sum $X \amalg Y = X \sqcup Y$.
- In **Top_•** and **HoTop_•**, the coproduct $(X, x_0) \amalg (Y, y_0)$ is the *one-point union* (or *wedge sum*) $(X \vee Y, \bullet)$.
- In $R\text{-Mod}$, we have $M \amalg N = M \oplus N$. Note that $M \oplus N \cong M \times N$, but for infinite I we generally have $\bigoplus_{i \in I} M_i \not\cong \prod_{i \in I} M_i$.

Lemma V.5.7. Coproducts exist in **Group**. The coproduct of G and H is called their *free product* (notation $G * H$), and for $G = \langle S_G \mid R_G \rangle$ and $H = \langle S_H \mid R_H \rangle$ we have

$$G * H = \langle S_G \amalg S_H \mid R_G \amalg R_H \rangle.$$

Proof. We verify the universal property:



The homomorphisms f and g are determined by their restrictions to S_G and S_H (the $\mathbf{Group} \leftrightarrow \mathbf{Set}$ adjunction). By the universal property of the coproduct in $\mathbf{Set}$, we obtain $S_G \amalg S_H \rightarrow K$ and hence, by the $\mathbf{Group} \leftrightarrow \mathbf{Set}$ adjunction, $u: F(S_G \amalg S_H) \rightarrow K$. Since f and g factor through $\langle S_G \mid R_G \rangle$ and $\langle S_H \mid R_H \rangle$, respectively, we obtain $\bar{u}: \langle S_G \amalg S_H \mid R_G \amalg R_H \rangle \rightarrow K$. Moreover, $\bar{u}$ is uniquely determined by $\bar{u}|_{S_G \amalg S_H}$, and thus by f and g . $\square$

The colimit of a diagram

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & & X \end{array} \xrightarrow{D} \begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & & \end{array}$$

in $\mathcal{C}$ is called an *amalgamated sum* or *pushout*. Thus, it consists of a $Z \in \text{Ob}(\mathcal{C})$ and arrows $X \xrightarrow{i_X} Z \xleftarrow{i_Y} Y$ satisfying the universal property

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow & & \downarrow i_Y \\ X & \xrightarrow{i_X} & Z \end{array} \quad \begin{array}{c} \text{Notation:} \\ \begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow \lrcorner & & \downarrow \\ X & \longrightarrow & Z \end{array} \\ \text{“cocartesian square”} \end{array}$$

$\begin{array}{ccc} & & f_Y \\ & \searrow & \downarrow \\ & & Q \end{array}$
 $\begin{array}{ccc} & & \exists! \\ & \searrow & \downarrow \\ & & Q \end{array}$
 $\begin{array}{ccc} & & f_X \\ & \searrow & \downarrow \\ & & Q \end{array}$

Lemma V.5.8. Pushouts exist in $\mathbf{Top}$ and are given by

$$\begin{array}{ccc} A & \xrightarrow{t} & Y \\ \downarrow s & \lrcorner & \downarrow \\ X & \longrightarrow & (X \amalg Y)/\sim \end{array}$$

where “ $\sim$ ” denotes the finest equivalence relation on $X \amalg Y$ for which $s(a) \sim t(a)$ holds for all $a \in A$.

Proof. We need to show:

$$\begin{array}{ccc} A & \xrightarrow{t} & Y \\ \downarrow s & & \downarrow \\ X & \longrightarrow & (X \amalg Y)/\sim \end{array} \quad \begin{array}{c} \text{Notation:} \\ \begin{array}{ccc} A & \xrightarrow{t} & Y \\ \downarrow s & \lrcorner & \downarrow \\ X & \longrightarrow & (X \amalg Y)/\sim \end{array} \end{array}$$

$\begin{array}{ccc} & & f_Y \\ & \searrow & \downarrow \\ & & Q \end{array}$
 $\begin{array}{ccc} & & \exists! \\ & \searrow & \downarrow \\ & & Q \end{array}$
 $\begin{array}{ccc} & & f_X \\ & \searrow & \downarrow \\ & & Q \end{array}$

By the universal property of the coproduct, there is a map $f_X \amalg f_Y: X \amalg Y \rightarrow Q$, and we have

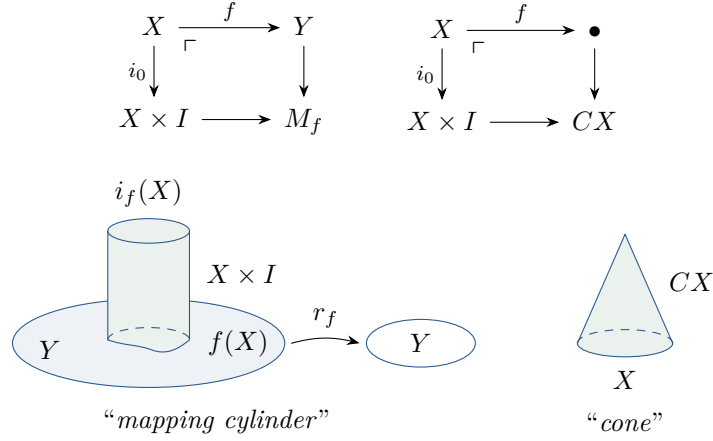
$$(f_X \amalg f_Y)(s(a)) = f_X(s(a)) = f_Y(t(a)) = (f_X \amalg f_Y)(t(a)).$$

By the universal property of the quotient topology, we obtain $\overline{f_X \amalg f_Y}: (X \amalg Y)/\sim \rightarrow Q$ as desired. If $g: (X \amalg Y)/\sim \rightarrow Q$ is another arrow as above, it factors $f_X \amalg f_Y$ through $(X \amalg Y)/\sim$, so $g = \overline{f_X \amalg f_Y}$ by uniqueness in the universal property of the quotient topology. $\square$

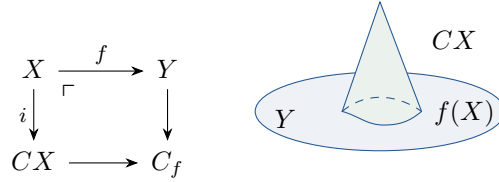
If $s: A \hookrightarrow X$ is the inclusion of a subspace, then $(X \amalg Y)/\sim$ is obtained by attaching X to Y along t .

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & X \cup_f Y \end{array} \quad \text{In particular:} \quad \begin{array}{ccc} A & \longrightarrow & \bullet \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & X/A \end{array}$$

Special cases:



The inclusion $i: X \hookrightarrow CX$ of the base shows that every space embeds in a contractible space. Moreover, $f = r_f \circ i_f$; that is, every continuous map f is a composition of an inclusion and a homotopy equivalence.



“Mapping cone” (“an improvement on $Y/\text{im } f$ ”)

Proposition V.5.9. Let K be a compact Hausdorff space. If

$$\begin{array}{ccc}
 A & \xrightarrow{f_2} & Y \\
 f_1 \downarrow & \lrcorner & \downarrow g_2 \\
 X & \xrightarrow{g_1} & Z
 \end{array}
 \text{ is cocartesian in } \mathbf{Top}, \text{ then so is }
 \begin{array}{ccc}
 A \times K & \xrightarrow{f_2 \times \text{id}} & Y \times K \\
 f_1 \times \text{id} \downarrow & \lrcorner & \downarrow g_2 \times \text{id} \\
 X \times K & \xrightarrow{g_1 \times \text{id}} & Z \times K.
 \end{array}$$

Proof. By the [previous lemma](#), $g_1 \amalg g_2: X \amalg Y \rightarrow Z$ is a quotient map. By Problem 2 on Problem Sheet 4/5, so is

$$g_1 \times \text{id}_K \amalg g_2 \times \text{id}_K: X \times K \amalg Y \times K \longrightarrow Z \times K.$$

Let $\sim$ be the equivalence relation on $X \amalg Y$ generated by $f_1(a) \sim f_2(a)$ for $a \in A$, and let $\sim_K$ be the equivalence relation on $X \times K \amalg Y \times K$ generated by

$$(f_1(a), k) \sim_K (f_2(a), k) \quad (a \in A, k \in K).$$

For $u, v \in X \amalg Y$, we have

$$(u, k_1) \sim_K (v, k_2) \iff u \sim v \text{ and } k_1 = k_2 :$$

Every finite chain of generating identifications leaves the K -coordinate unchanged; conversely, every such chain in $X \amalg Y$ lifts to a chain in the product at fixed k . Thus, the $\sim_K$ -classes are exactly the fibers of the quotient map above. We therefore obtain an induced homeomorphism

$$(X \times K \amalg Y \times K) / \sim_K \xrightarrow{\cong} Z \times K.$$

The [lemma](#) now completes the proof. □

Lemma V.5.10. Pushouts exist in **Group** and are given by

$$\begin{array}{ccc} A & \xrightarrow{t} & H \\ s \downarrow & & \downarrow \\ G & \longrightarrow & (G * H)/N, \end{array}$$

where $N = \langle\langle s(a)t(a)^{-1} : a \in A \rangle\rangle \trianglelefteq G * H$.

Proof. We need to show:

$$\begin{array}{ccccc} A & \xrightarrow{t} & H & & \\ s \downarrow & & \downarrow & \searrow g & \\ G & \longrightarrow & (G * H)/N & \xrightarrow{\exists!} & Q. \\ & \searrow f & & & \end{array}$$

By the universal property of the coproduct, we obtain

$$f * g: G * H \longrightarrow Q.$$

Since $(f * g)(s(a)t(a)^{-1}) = f(s(a))g(t(a))^{-1} = 1$, we obtain

$$\overline{f * g}: (G * H)/N \longrightarrow Q.$$

As above, uniqueness follows from the universal property of the quotient group. □

Thus, for $G = \langle S_1 \mid R_1 \rangle$ and $H = \langle S_2 \mid R_2 \rangle$, we have

$$(G * H)/N = \langle S_1 \amalg S_2 \mid R_1 \amalg R_2 \amalg \{s(a)t(a)^{-1} : a \in A\} \rangle.$$

In the special case where s and t are inclusions of subgroups,

$$(G * H)/N =: G *_A H$$

is called the *amalgamated (free) product* of G and H .

Example. $\mathbb{Z}/6 *_{\mathbb{Z}/2} \mathbb{Z}/4 \cong \mathrm{SL}(2; \mathbb{Z})$.

VI. Computing Fundamental Groups

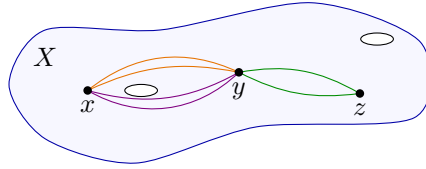
VI.1. The Fundamental Groupoid

Definition VI.1.1. A *groupoid* is a small category in which every arrow is an isomorphism.

Definition VI.1.2. Let X be a topological space. The *fundamental groupoid* of X is the small category $\Pi(X)$ with $\text{Ob}(\Pi(X)) = X$ and, for $x, y \in X$,

$$\text{Hom}_{\Pi(X)}(x, y) = \{\gamma: I \rightarrow X : \gamma(0) = x, \gamma(1) = y\} / \simeq_{\{0,1\}}$$

with concatenation as composition (for $x \xrightarrow{[\gamma_1]} y \xrightarrow{[\gamma_2]} z$, we have $[\gamma_2] \circ [\gamma_1] = [\gamma_1 \gamma_2]$) and identity morphisms $\text{id}_x = [c_x]$ for $x \in X$.



Note.

$$[\gamma]^{-1} = [\bar{\gamma}], \quad \text{Hom}_{\Pi(X)}(x_0, x_0) = \pi_1(X, x_0).$$

Definition VI.1.3. A category is called *connected* if any two objects can be connected by a zigzag of arrows:

$$A \longleftarrow B \longrightarrow C \longleftarrow D$$

($\text{Hom}(A, D) = \emptyset$ is allowed; for groupoids, however, $\text{Hom}(A, D) \neq \emptyset$ follows.)

Note. $\Pi(X)$ is connected $\iff X$ is path-connected.

Lemma VI.1.4. Let $\mathcal{G}$ be a nonempty connected groupoid and let $x \in \text{Ob}(\mathcal{G})$. Then the inclusion functor

$$I_x: \underline{\text{Aut}_{\mathcal{G}}(x)} = \underline{\text{Hom}_{\mathcal{G}}(x, x)} \longrightarrow \mathcal{G}$$

$$\bullet \longmapsto x$$

$$g: x \rightarrow x \longmapsto g: x \rightarrow x$$

is an equivalence of categories.

Proof. Choose $f_y: x \rightarrow y$ for every $y \in \text{Ob}(\mathcal{G})$ (using the axiom of choice and the fact that $\mathcal{G}$ is small), with $f_x = \text{id}_x$. Define

$$\begin{aligned} R: \mathcal{G} &\longrightarrow \underline{\text{Aut}_{\mathcal{G}}(x)} \\ y &\longmapsto \bullet \\ g: y \rightarrow z &\longmapsto f_z^{-1} \circ g \circ f_y \end{aligned} \quad \begin{array}{ccc} y & \xrightarrow{g} & z \\ & \swarrow f_y \quad \searrow f_z & \\ & x & \end{array}$$

We have $R \circ I_x = \text{id}_{\text{Aut}_{\mathcal{G}}(x)}$. Moreover, $f: I_x \circ R \rightarrow \text{id}_{\mathcal{G}}$ is a natural isomorphism, since for $g: y \rightarrow z$ we have

$$\begin{array}{ccccc} x & \xrightarrow{f_y} & y & \xrightarrow{g} & z & \xrightarrow{f_z^{-1}} & x \\ f_y \downarrow \cong & & & & & & \cong \downarrow f_z \\ & & y & \xrightarrow{g} & z & & \end{array}$$

□

Corollary VI.1.5. Let X be a path-connected topological space. Then the inclusion functor $\pi_1(X, x_0) \rightarrow \Pi(X)$ is an equivalence of categories. □

VI.2. The Seifert–van Kampen Theorem

Let **Groupoid** be the category of groupoids with functors as morphisms. We have functors

$$(_) : \mathbf{Group} \rightarrow \mathbf{Groupoid}, \quad \Pi : \mathbf{Top} \rightarrow \mathbf{Groupoid}.$$

Theorem VI.2.1. (Seifert–van Kampen–Brown) Let X be a topological space and let $\mathcal{O}$ be an open cover of X such that $U \cap V \in \mathcal{O}$ for all $U, V \in \mathcal{O}$. Regard $\mathcal{O}$ as a small category with inclusions as morphisms. Then restricting Π to $\mathcal{O}$ defines a diagram $\Pi|_{\mathcal{O}} : \mathcal{O} \rightarrow \mathbf{Groupoid}$, and $\Pi(X) = \text{colim } \Pi|_{\mathcal{O}}$.

Note. Writing “ $\text{colim}_{U \in \mathcal{O}} U$ ” for “ $\text{colim}(\mathcal{O} \hookrightarrow \mathbf{Top})$ ” and “ $\text{colim}_{U \in \mathcal{O}} \Pi(U)$ ” for “ $\text{colim } \Pi|_{\mathcal{O}}$ ”, the theorem states that

$$\Pi(\text{colim}_{U \in \mathcal{O}} U) = \text{colim}_{U \in \mathcal{O}} \Pi(U),$$

which is a “cocontinuity property” of the functor Π .

Proof of the theorem. Clearly, the inclusions $\Pi(U) \rightarrow \Pi(X)$ define a cocone $\iota : \Pi|_{\mathcal{O}} \rightarrow \mathbf{1}_{\Pi(X)}^{\mathcal{O}}$. We must show that for every other cocone $\eta : \Pi|_{\mathcal{O}} \rightarrow \mathcal{G}$ there is a unique functor $F : \Pi(X) \rightarrow \mathcal{G}$ such that $\eta_U = F \circ \iota_U$ for all $U \in \mathcal{O}$.

$$\begin{array}{ccc} \Pi(U) & \xrightarrow{\iota_U} & \Pi(X) \\ & \searrow \eta_U & \downarrow F \\ & & \mathcal{G} \end{array}$$

For $x \in X$, choose $U \in \mathcal{O}$ with $x \in U$ and set $F(x) = \eta_U(x)$. This is well defined: for $V \in \mathcal{O}$ with $x \in V$, we have $U \cap V \in \mathcal{O}$ and

$$\begin{array}{ccccc} \Pi(U) & \longleftarrow & \Pi(U \cap V) & \longrightarrow & \Pi(V) \\ & \searrow \eta_U & \downarrow \eta_{U \cap V} & \swarrow \eta_V & \\ & & \mathcal{G} & & \end{array}$$

thus $\eta_V(x) = \eta_{U \cap V}(x) = \eta_U(x)$. Now let $[\gamma] \in \text{Hom}_{\Pi(X)}(x, y)$. To define $F([\gamma]) \in \text{Hom}_{\mathcal{G}}(F(x), F(y))$, choose a Lebesgue number δ for the open cover $\{\gamma^{-1}(U)\}_{U \in \mathcal{O}}$ of the compact space I . Subdivide I into n intervals of length $< \delta$ to obtain a concatenation

$$\gamma = \gamma_1 \gamma_2 \cdots \gamma_n$$

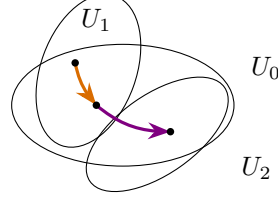
with $\text{im}(\gamma_i) \subseteq U_i$ for some $U_i \in \mathcal{O}$. Set

$$F([\gamma]) = \eta_{U_n}([\gamma_n]) \circ \cdots \circ \eta_{U_1}([\gamma_1]).$$

We must show that $F([\gamma])$ is independent of

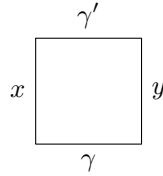
- (i) the subdivision $\gamma = \gamma_1 \cdots \gamma_n$ and the choice of $U_1, \dots, U_n$,
- (ii) the choice of representative γ of $[\gamma]$.

For (i). We show that $F([\gamma])$ remains unchanged when we refine the subdivision and change the U_i . To this end, let $\gamma = \gamma_1 \gamma_2$ with $\text{im}(\gamma) \subseteq U_0$ and $\text{im}(\gamma_i) \subseteq U_i$, $i = 1, 2$. Then

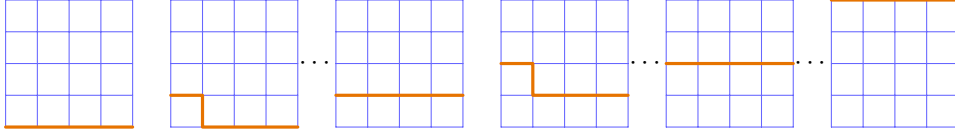


$$\begin{aligned}
 \eta_{U_2}([\gamma_2]) \circ \eta_{U_1}([\gamma_1]) &= \eta_{U_2 \cap U_0}([\gamma_2]) \circ \eta_{U_1 \cap U_0}([\gamma_1]) \\
 &= \eta_{U_0}([\gamma_2]) \circ \eta_{U_0}([\gamma_1]) = \eta_{U_0}([\gamma_2] \circ [\gamma_1]) \\
 &= \eta_{U_0}([\gamma_1 \gamma_2]) = \eta_{U_0}([\gamma]).
 \end{aligned}$$

For (ii). Consider $\gamma \simeq_{\{0,1\}}^H \gamma'$ with $H: I \times I \rightarrow X$:



Choose a Lebesgue number δ for $\{H^{-1}(U)\}_{U \in \mathcal{O}}$ and subdivide $I \times I$ into squares of diameter $< \delta$. We obtain a sequence of paths homotopic relative to their endpoints:



Two consecutive paths have the form $\gamma_A \gamma_1 \gamma_E$ and $\gamma_A \gamma_2 \gamma_E$, with $\gamma_1 \simeq_{\{0,1\}} \gamma_2$ in some $U \in \mathcal{O}$. It follows that

$$F([\gamma_A \gamma_1 \gamma_E]) = \cdots \circ \eta_U([\gamma_1]) \circ \cdots = \cdots \circ \eta_U([\gamma_2]) \circ \cdots = F([\gamma_A \gamma_2 \gamma_E]).$$

By construction, F factors the cocone η through ι and is uniquely determined by η . □

Theorem VI.2.2. (Seifert–van Kampen – group version) Let X be a nonempty topological space, let $x_0 \in X$, and let $\mathcal{O}$ be a cover by open path-connected sets such that $x_0 \in U$ for every $U \in \mathcal{O}$ and $U \cap V \in \mathcal{O}$ for all $U, V \in \mathcal{O}$. Regard $\mathcal{O}$ as a small category with pointed inclusions as morphisms. Then restricting π_1 to $\mathcal{O}$ defines a diagram $\pi_1|_{\mathcal{O}}: \mathcal{O} \rightarrow \mathbf{Group}$, and $\pi_1(X, x_0) = \text{colim}_{(U, x_0) \in \mathcal{O}} \pi_1(U, x_0)$. Equivalently:

$$\pi_1 \left(\text{colim}_{(U, x_0) \in \mathcal{O}} (U, x_0) \right) \cong \text{colim}_{(U, x_0) \in \mathcal{O}} \pi_1(U, x_0).$$

Proof. The inclusions define the cocone $i: \pi_1|_{\mathcal{O}} \rightarrow \mathbf{1}_{\pi_1(X, x_0)}^{\mathcal{O}}$. We must show that i is universal. Clearly, X is path-connected, so $I: \pi_1(X, x_0) \rightarrow \Pi(X)$ is an equivalence of categories. We

prove only the case $|\mathcal{O}| < \infty$. For every $y \in X$, choose a path $\gamma_y: x_0 \rightarrow y$ within $(\bigcap_{U \ni y} U) \in \mathcal{O}$, with $\gamma_{x_0} = c_{x_0}$. A quasi-inverse to I is given by

$$R([\gamma: y \rightarrow z]) = [x_0 \xrightarrow{\gamma_y} y \xrightarrow{\gamma_z} z \xrightarrow{\gamma_z^{-1}} x_0]$$

and similarly we obtain $R_U: \Pi(U) \rightarrow \pi_1(U, x_0)$ for every $U \in \mathcal{O}$, such that

$$\begin{array}{ccc} \Pi(U) & \xrightarrow{R_U} & \pi_1(U, x_0) \\ \downarrow & & \downarrow \\ \Pi(V) & \xrightarrow{R_V} & \pi_1(V, x_0) \end{array}$$

commutes for $U \subseteq V$. Thus $R: \Pi|_{\mathcal{O}} \rightarrow \pi_1|_{\mathcal{O}}$ is a natural transformation. Now let $\eta: \pi_1|_{\mathcal{O}} \rightarrow \mathbf{1}_G^{\mathcal{O}}$ be any cocone. It induces the cocones

$$\underline{\eta}: \pi_1|_{\mathcal{O}} \longrightarrow \mathbf{1}_G^{\mathcal{O}}, \quad \underline{\eta} \circ R: \Pi|_{\mathcal{O}} \longrightarrow \mathbf{1}_G^{\mathcal{O}}.$$

The groupoid version of the theorem gives a unique functor $F: \Pi(X) \rightarrow \underline{G}$ with $\underline{\eta}_U \circ R_U = F \circ \iota_U$:

$$\begin{array}{ccccc} \Pi(U) & \xrightarrow{R_U} & \pi_1(U, x_0) & \xrightarrow{\eta_U} & \underline{G} \\ \downarrow \iota_U & & & \nearrow F & \\ \Pi(X) & & & & \end{array}$$

We extend the diagram to

$$\begin{array}{ccccccc} \pi_1(U, x_0) & \xrightarrow{I_U} & \Pi(U) & \xrightarrow{R_U} & \pi_1(U, x_0) & \xrightarrow{\eta_U} & \underline{G} \\ \downarrow i_U & & \downarrow \iota_U & & & \nearrow F & \\ \pi_1(X, x_0) & \xrightarrow{I} & \Pi(X) & & & & \end{array}$$

Now let $f: \pi_1(X, x_0) \rightarrow \underline{G}$ be the group homomorphism with $\underline{f} = F \circ I$. Since $R_U \circ I_U = \text{id}_{\pi_1(U, x_0)}$, we have $\eta_U = f \circ i_U$ for every $U \in \mathcal{O}$, as desired. For uniqueness, the diagram

$$\begin{array}{ccccc} \Pi(U) & \xrightarrow{R_U} & \pi_1(U, x_0) & \xrightarrow{\eta_U} & \underline{G} \\ \downarrow \iota_U & & \downarrow i_U & \nearrow \underline{g} & \\ \Pi(X) & \xrightarrow{R} & \pi_1(X, x_0) & & \end{array}$$

gives $\underline{g} \circ R = F$, since $\underline{g} \circ R$ factors the cocone $\underline{\eta} \circ R$ through ι , and F is the unique such factorization. Precomposing with I yields $\underline{g} = F \circ I = \underline{f}$, so $g = f$. $\square$

VI.3. Examples of Fundamental Group Computations

Consider the simplest nontrivial case of the theorem: $\mathcal{O} = \{U_1, U_2, U_1 \cap U_2\}$, $x_0 \in U_1 \cap U_2$. Then

$$\begin{array}{ccc} \pi_1(U_1 \cap U_2, x_0) & \longrightarrow & \pi_1(U_1, x_0) \\ \downarrow & \lrcorner & \downarrow \\ \pi_1(U_2, x_0) & \longrightarrow & \pi_1(X, x_0). \end{array}$$

Examples.

- Let $X = S^n$ with $n \geq 2$. Choose $x_0 = (1, 0, \dots, 0)$ and

$$U_1 = S^n \setminus \{(0, \dots, 0, -1)\}, \quad U_2 = S^n \setminus \{(0, \dots, 0, 1)\}.$$

Then

$$\begin{array}{ccc} \pi_1(U_1 \cap U_2, x_0) & \xrightarrow{\quad} & \{1\} \\ \downarrow & \lrcorner & \downarrow \\ \{1\} & \xrightarrow{\quad} & \pi_1(S^n, x_0), \end{array}$$

that is, $\pi_1(S^n, x_0) = \{1\}$ (as a quotient of $\{1\} * \{1\} = \{1\}$). Note that for $n = 1$, the intersection $U_1 \cap U_2$ is disconnected!

- $X = S^1 \vee S^1$,

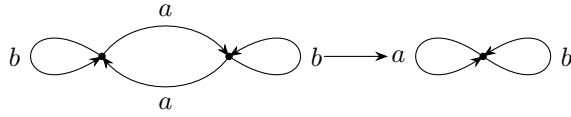


$$U_1 \cap U_2 = \times \simeq \bullet,$$

$$\begin{array}{ccc} \{1\} & \xrightarrow{\quad} & \mathbb{Z} \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{Z} & \xrightarrow{\quad} & \pi_1(S^1 \vee S^1, \bullet), \end{array}$$

hence $\pi_1(S^1 \vee S^1, \bullet) \cong \mathbb{Z} * \mathbb{Z} \cong F_2$. By induction, $\pi_1(\bigvee_{i=1}^n S^1, \bullet) \cong F_n$.

Note:

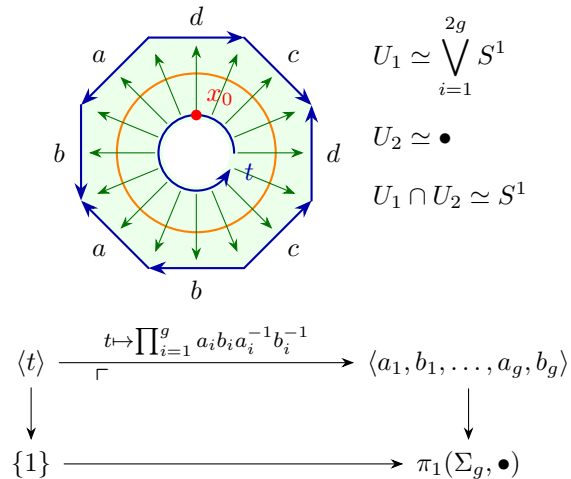


is a two-sheeted covering. Thus

$$\underbrace{\langle b, a^2, aba^{-1} \rangle}_{\cong F_3} \leq \underbrace{F(\{a, b\})}_{\cong F_2}$$

and $[F_2 : F_3] = 2$. More generally, let $[F_n : H] = e$. Then the covering space X_H of $X = \bigvee_{i=1}^n S^1$ is a graph with e vertices and $e \cdot n$ edges. Collapsing a spanning tree shows that $H \cong F_m$ with $m = e \cdot n - (e - 1) = e(n - 1) + 1$ (the Nielsen–Schreier formula).

- $X = \Sigma_g$.



It follows that

$$\pi_1(\Sigma_g, \bullet) = \left\langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1} \right\rangle$$

for the *surface group of genus g* . Similarly,

$$\pi_1(N_g, \bullet) = \langle a_1, \dots, a_g \mid a_1^2 \cdots a_g^2 \rangle.$$

Note:

$$\pi_1(\Sigma_g, \bullet)_{\text{ab}} \cong \mathbb{Z}^{2g}, \quad \pi_1(N_g, \bullet)_{\text{ab}} \cong \mathbb{Z}^g / \langle k \cdot (2, \dots, 2)^T : k \in \mathbb{Z} \rangle \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}.$$

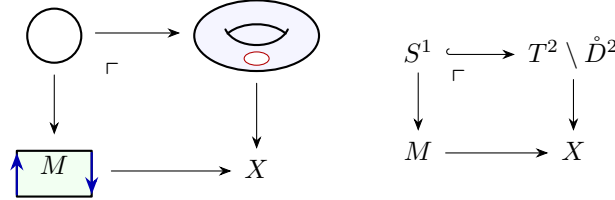
These abelian groups are pairwise nonisomorphic.

$\leadsto$ This completes the proof of the classification theorem for surfaces (see Chapter II). $\square$

$\leadsto$ The classification up to homeomorphism agrees with the classification up to homotopy equivalence. $\square$

$\leadsto$ Note that there are compact connected 3-manifolds M and N with $M \simeq N$ but $M \not\cong N$ ($\rightarrow$ lens spaces).

- Let X be the space obtained by attaching a Möbius band to a torus with an open disk removed.



Let U_1 and U_2 be the neighborhoods of $T^2 \setminus \mathring{D}^2$ and M in X obtained by thickening each by a collar. Then the diagram

$$\begin{array}{ccc} \pi_1(U_1 \cap U_2, \bullet) & \xrightarrow{\quad} & \pi_1(U_1, \bullet) \\ \downarrow & \lrcorner & \downarrow \\ \pi_1(U_2, \bullet) & \xrightarrow{\quad} & \pi_1(X, \bullet) \end{array}$$

becomes

$$\begin{array}{ccc} \langle t \rangle & \xrightarrow[t \mapsto aba^{-1}b^{-1}]{\quad} & \langle a, b \rangle \\ t \mapsto s^2 \downarrow & \lrcorner & \downarrow \\ \langle s \rangle & \xrightarrow{\quad} & \langle a, b, s \mid aba^{-1}b^{-1}s^{-2} \rangle. \end{array}$$

Since $\pi_1(X, \bullet)_{\text{ab}} \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$, it follows that $X \cong N_3$.

VI.4. Cofibrations

Definition VI.4.1. A continuous map $i: A \rightarrow X$ has the *homotopy extension property* with respect to a space Y if

$$\begin{array}{ccc} A & \xrightarrow{i_0^A} & A \times I \\ i \downarrow & & \downarrow i \times \text{id}_I \\ X & \xrightarrow{i_0^X} & X \times I \end{array} \quad \begin{array}{c} \searrow H \\ \exists H' \text{ (dashed)} \\ \downarrow f \end{array} \quad Y.$$

In words: for every homotopy $H: A \times I \rightarrow Y$ and every *initial condition* $f: X \rightarrow Y$ with $f(i(a)) = H(a, 0)$ for all $a \in A$, there is an *extension* $H': X \times I \rightarrow Y$ such that $H'(x, 0) = f(x)$ for all $x \in X$ and $H'(i(a), t) = H(a, t)$ for all $a \in A$ and $t \in I$. Thus we require only the existence, *not* the uniqueness, of H' .

Definition VI.4.2. A continuous map $i: A \rightarrow X$ is called a *cofibration* if it has the homotopy extension property with respect to every space Y .

Remark. The left-hand diagram expresses the homotopy extension property; the right-hand diagram expresses the dual *homotopy lifting property*.

$$\begin{array}{ccc}
 Y & \xleftarrow{f} & X \\
 \text{ev}_0 \uparrow & \swarrow & \uparrow i \\
 Y^I & \xleftarrow{H} & A
 \end{array}
 \qquad
 \begin{array}{ccc}
 Y & \xrightarrow{f} & E \\
 i_0^Y \downarrow & \swarrow & \downarrow p \\
 Y \times I & \xrightarrow{H} & B
 \end{array}$$

A map $p: E \rightarrow B$ with the homotopy lifting property for all Y is called a *fibration*. The most important examples are fiber bundles.

We will show that if $i: A \rightarrow X$ has the homotopy extension property with respect to its own mapping cylinder M_i , then i is a cofibration. Consider

$$\begin{array}{ccc}
 A & \xrightarrow{i_0^A} & A \times I \\
 i \downarrow & \ulcorner & \downarrow \bar{i} \\
 X & \xrightarrow{j} & M_i
 \end{array}
 \begin{array}{c}
 \searrow i \times \text{id} \\
 \text{---} s \text{---} \\
 \searrow i_0^X
 \end{array}
 \begin{array}{c}
 \\
 \\
 X \times I
 \end{array}$$

If i is an embedding, then s is a continuous bijection onto its image $X \times \{0\} \cup A \times I$. If i is the inclusion of a closed subspace, then s is a homeomorphism onto its image, hence also an embedding. By part (iii) of the following proposition, this also holds when i is a cofibration.

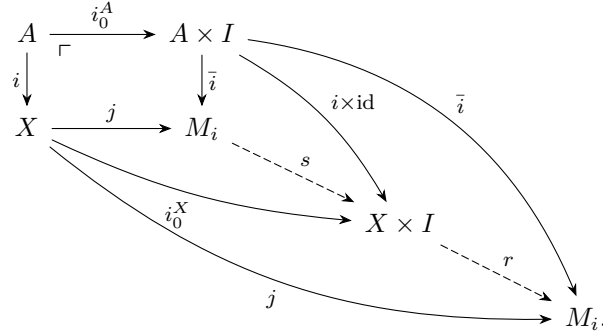
Proposition VI.4.3. Let $i: A \rightarrow X$ be continuous. Then the following are equivalent:

- (i) i is a cofibration.
- (ii) i has the homotopy extension property with respect to M_i .
- (iii) There is a retraction $r: X \times I \rightarrow M_i$ such that $r \circ s = \text{id}_{M_i}$.

Proof. (i) $\Rightarrow$ (ii): Trivial. (ii) $\Rightarrow$ (iii): The homotopy extension property with respect to M_i gives

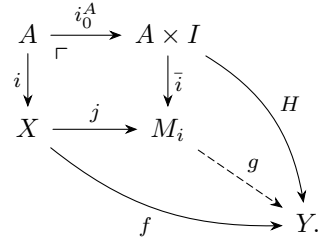
$$\begin{array}{ccc}
 A & \xrightarrow{i_0^A} & A \times I \\
 i \downarrow & & \downarrow i \times \text{id} \\
 X & \xrightarrow{i_0^X} & X \times I
 \end{array}
 \begin{array}{c}
 \searrow \bar{i} \\
 \text{---} r \text{---} \\
 \searrow j
 \end{array}
 \begin{array}{c}
 \\
 \\
 M_i
 \end{array}$$

Combining the defining diagrams for r and s gives

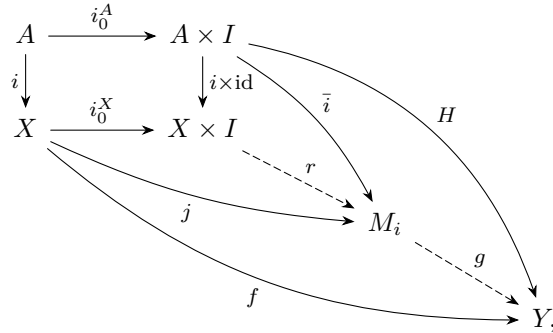


The uniqueness clause in the universal property of the pushout implies that $r \circ s = \text{id}_{M_i}$.

(iii) $\Rightarrow$ (i): Suppose we are given a homotopy extension problem $H: A \times I \rightarrow Y$, $f: X \rightarrow Y$, $f \circ i = H \circ i_0^A$. We obtain



Combining this with the defining diagram for r gives



so $g \circ r$ is an extension of H with initial condition f . $\square$

Theorem VI.4.4. A cofibration $i: A \rightarrow X$ is an embedding. If X is Hausdorff, then $A \subseteq X$ is closed.

Proof. Let $i_1^X: X \rightarrow X \times I$, $x \mapsto (x, 1)$. The final map $\bar{i}_1: A \rightarrow M_i$ of the homotopy $\bar{i}: A \times I \rightarrow M_i$ satisfies

$$\bar{i}_1(a) = \bar{i}(a, 1) = r(i(a), 1) = r \circ i_1^X \circ i(a) \quad \text{for } a \in A. \quad (*)$$

Claim: $\bar{i}_1$ is an embedding.

Proof: $\bar{i}_1$ is a continuous bijection onto its image. We must show that if $B \subseteq A$ is closed, then so is $\bar{i}_1(B) \subseteq M_i$. By the construction of M_i , the map $j \sqcup \bar{i}: X \sqcup A \times I \rightarrow M_i$ is a quotient map, and $(j \sqcup \bar{i})^{-1}(\bar{i}_1(B)) = B \times \{1\}$ is closed in $X \sqcup A \times I$. Hence $\bar{i}_1(B) \subseteq M_i$ is closed. $\square$

Equation $(*)$ first shows that i is injective. Using $\bar{i}_1^{-1}: \text{im}(\bar{i}_1) \rightarrow A$, we obtain the continuous inverse $\bar{i}_1^{-1} \circ (r \circ i_1^X)|_{i(A)}: i(A) \rightarrow A$. Thus i is a homeomorphism onto its image. For the

second part of the theorem, applying s to $(*)$ shows that the maps $s \circ r \circ i_1^X$ and i_1^X agree on $i(A)$. Conversely, if $x \in X$ satisfies $s \circ r \circ i_1^X(x) = i_1^X(x)$, then $(x, 1) \in \text{im}(s)$. Since the second coordinate is 1, we have $(x, 1) \in \text{im}(s \circ i) = \text{im}(i \times \text{id})$, so $x \in i(A)$. Thus

$$A \xrightarrow{i} X \xrightleftharpoons[i_1^X]{s \circ r \circ i_1^X} X \times I$$

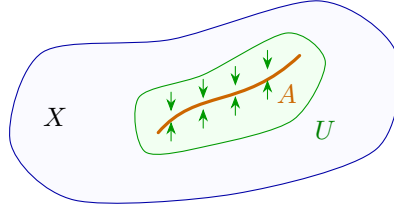
is an *equalizer* (see Problem Sheet 13, Problem 1), and therefore $i(A) \subseteq X$ is closed if X is Hausdorff (Problem Sheet 3, Problem 2). $\square$

From now on, we therefore regard cofibrations as pairs of spaces (X, A) and can usually assume that $A \subseteq X$ is closed.

Theorem VI.4.5. Let (X, A) be a pair of spaces. Then the following are equivalent:

- (i) The pair (X, A) is a closed cofibration.
- (ii) There is a retraction $R: X \times I \rightarrow X \times \{0\} \cup A \times I$, and $A \subseteq X$ is closed.
- (iii) There are a continuous map $u: X \rightarrow I$ and a homotopy $H: X \times I \rightarrow X$ such that
 - (1) $u^{-1}(0) = A$,
 - (2) $H(x, 0) = x$ for all $x \in X$,
 - (3) $H(a, t) = a$ for all $a \in A$ and $t \in I$,
 - (4) $H(x, 1) \in A$ for all $x \in X$ with $u(x) < 1$.

By (iii), the restriction $H|_{U \times I}$ deforms the open neighborhood $U := u^{-1}([0, 1))$ into $A = u^{-1}(0)$ within X , while fixing A pointwise. A closed cofibration (X, A) is also called a *neighborhood deformation retract pair*.



Proof. (i) $\Rightarrow$ (ii). The homotopy extension property gives:

$$\begin{array}{ccc} A & \longrightarrow & A \times I \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times I \\ & \searrow & \downarrow \exists R \\ & & X \times \{0\} \cup A \times I \end{array}$$

(ii) $\Rightarrow$ (i). Let $H: A \times I \rightarrow Y$, $f: X \rightarrow Y$, $H(a, 0) = f(a)$ be a homotopy extension problem. We have already seen that, since $A \subseteq X$ is closed, $s: M_i \xrightarrow{\cong} X \times \{0\} \cup A \times I$, so the square on

the left is a pushout. Then $H' = F \circ R$ solves the homotopy extension problem.

$$\begin{array}{ccc}
 A & \xrightarrow{i_0^A} & A \times I \\
 \downarrow i & \lrcorner & \downarrow i \times \text{id} \\
 X & \xrightarrow{i_0^X} & X \times \{0\} \cup A \times I \\
 & \searrow f & \swarrow \exists! F \\
 & & Y
 \end{array}$$

(A curved arrow labeled H goes from $A \times I$ to Y , and a dashed arrow labeled $\exists! F$ goes from $X \times \{0\} \cup A \times I$ to Y .)

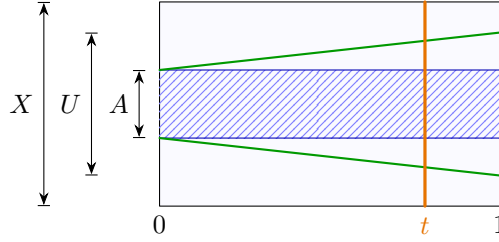
(ii) $\Rightarrow$ (iii). Define $H: X \times I \rightarrow X$ by $H = \text{pr}_X \circ R$. (2) $\checkmark$, (3) $\checkmark$. Define $u: X \rightarrow I$ by

$$u(x) := \max_{t \in I} |t - \text{pr}_I(R(x, t))|.$$

Then $A \subseteq u^{-1}(0)$. Conversely, if $u(x) = 0$, then $\text{pr}_X(R(x, (0, 1])) \subseteq A$. Since A is closed, it follows that $\text{pr}_X(R(x, 0)) \in A$, so $u^{-1}(0) \subseteq A$. (1) $\checkmark$. Now suppose that $u(x) < 1$. In particular, $|1 - \text{pr}_I(R(x, 1))| < 1$, so $R(x, 1) = (a, t)$ for some $a \in A$ and $t \in I$. Hence $H(x, 1) = \text{pr}_X(R(x, 1)) = a \in A$, giving (4) $\checkmark$. The continuity of u is elementary.

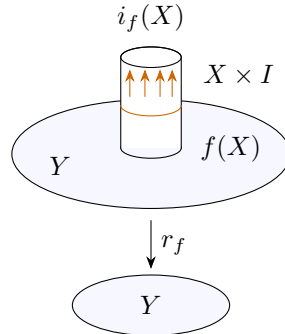
(iii) $\Rightarrow$ (ii). Define $R: X \times I \rightarrow X \times \{0\} \cup A \times I$ by

$$R(x, t) = \begin{cases} (H(x, \frac{t}{u(x)}), 0) & \text{for } u(x) > t, \\ (H(x, 1), t - u(x)) & \text{for } u(x) \leq t. \end{cases}$$



Again, continuity is elementary. □

Example. For $f: X \rightarrow Y$, the inclusion $i_f: X \rightarrow M_f$ is a closed cofibration. Thus every arrow in **Top** factors as a composite $f = r_f \circ i_f$ of a cofibration and a homotopy equivalence (dually, of a homotopy equivalence and a fibration $\rightarrow$ model categories).



In particular, for every space X , the pair (CX, X) is a neighborhood deformation retract pair, since $X \subseteq CX$ is the inclusion $i_f: X \hookrightarrow CX$ for $f: X \rightarrow \bullet$. Taking $X = S^{n-1}$ shows that (D^n, S^{n-1}) is a neighborhood deformation retract pair.

Theorem VI.4.6. Let $i: A \rightarrow X$ be a cofibration and let $f: A \rightarrow Y$ have the factorization

$$\begin{array}{ccc} & M_f & \\ i_f \nearrow & & \searrow r_f \\ A & \xrightarrow{f} & Y. \end{array}$$

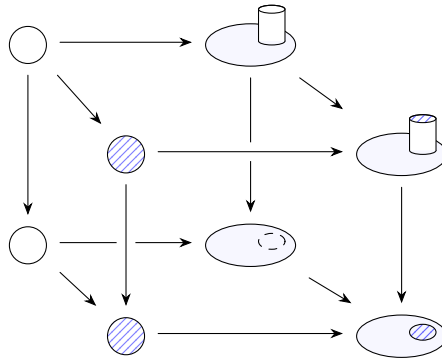
Consider the two pushouts

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ i \downarrow \lrcorner & & \downarrow j \\ X & \xrightarrow{\bar{f}} & Z \end{array} \quad \begin{array}{ccc} A & \xrightarrow{i_f} & M_f \\ i \downarrow \lrcorner & & \downarrow j_f \\ X & \xrightarrow{\bar{i}_f} & \bar{Z}. \end{array}$$

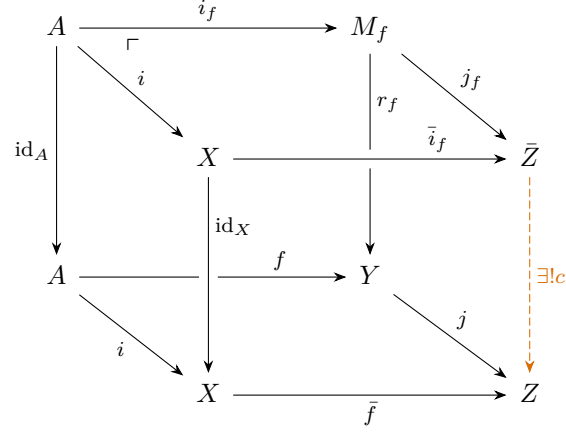
Then there is a unique homotopy equivalence $c: \bar{Z} \xrightarrow{\simeq} Z$ such that

$$\begin{array}{ccccc} A & \xrightarrow{i_f} & M_f & & \\ \text{id}_A \downarrow & \searrow i & \downarrow r_f & \searrow j_f & \\ & X & \xrightarrow{\bar{i}_f} & \bar{Z} & \\ & \downarrow \text{id}_X & \downarrow & \downarrow c & \\ A & \xrightarrow{f} & Y & \searrow j & \\ i \searrow & \downarrow & & & Z \\ & X & \xrightarrow{\bar{f}} & & \end{array}$$

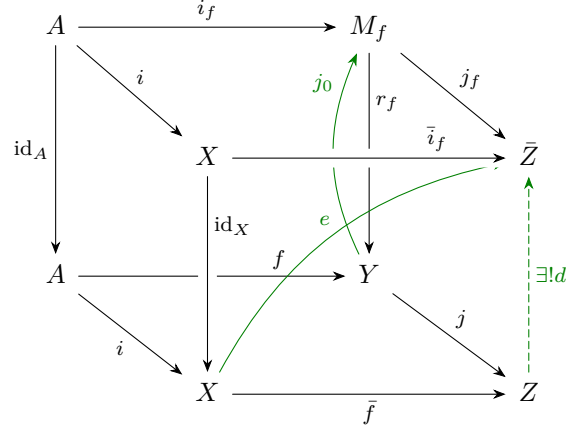
commutes.



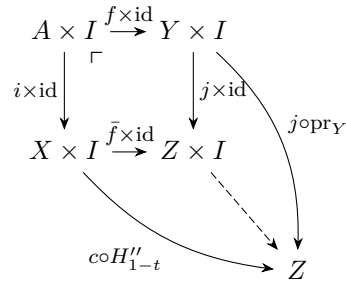
Proof. We obtain c from the upper pushout, as shown below. It remains to show that c is a homotopy equivalence, that is, to find a homotopy inverse d .



The idea is shown below. The problem is that the maps $\bar{i}_f$ and $j_f \circ j_0$ do not form a cocone on $X \xleftarrow{i} A \xrightarrow{f} Y$.



However, $j_f \circ j_0 \circ f \simeq_{H'} \bar{i}_f \circ i$, where $H' = j_f \circ H \circ (i_f \times \text{id}_I)$ and $j_0 \circ r_f \simeq_H \text{id}_{M_f}$. By the homotopy extension property for i , the homotopy H'_{1-t} with initial condition $\bar{i}_f$ extends to $H'': X \times I \rightarrow \bar{Z}$. Set $e = H''_1$. Then $e \circ i = H'_0 = j_f \circ j_0 \circ f$. Thus the cocone consisting of e and $j_f \circ j_0$ induces a map $d: Z \rightarrow \bar{Z}$. By the proposition in V.5, we obtain the homotopies $c \circ d \simeq \text{id}_Z$ and $d \circ c \simeq \text{id}_{\bar{Z}}$ from the pushout diagrams



$$\begin{array}{ccc}
A \times I & \xrightarrow{i_f \times \text{id}} & M_f \times I \\
\downarrow i \times \text{id} & \lrcorner & \downarrow j_f \times \text{id} \\
X \times I & \xrightarrow{\bar{i}_f \times \text{id}} & \bar{Z} \times I \\
& & \searrow \text{dashed} \\
& & \bar{Z}
\end{array}
\quad
\begin{array}{c}
\downarrow j_f \circ H \\
\downarrow H''_{1-t}
\end{array}$$

□

For two maps $X \xleftarrow{f_1} A \xrightarrow{f_2} Y$, the pushout of $X \xleftarrow{f_1} A \xrightarrow{i_{f_2}} M_{f_2}$ has the symmetric description

$$\begin{array}{ccc}
A \times I & \xrightarrow{\bar{f}_2(a, 1-t)} & M_{f_2} \\
\bar{f}_1(a, t) \downarrow & \lrcorner & \downarrow \\
M_{f_1} & \longrightarrow & M_{f_1, f_2}
\end{array}$$

and M_{f_1, f_2} is called the *double mapping cylinder* of f_1 and f_2 . If

$$\begin{array}{ccc}
A & \xrightarrow{f_2} & Y \\
f_1 \downarrow & & \downarrow g_2 \\
X & \xrightarrow{g_1} & Z
\end{array}
\quad (*)$$

is *homotopy commutative*, that is, $g_1 \circ f_1 \simeq_H g_2 \circ f_2$, then g_1 and $H(a, t)$ induce the map $c_1: M_{f_1} \rightarrow Z$, while g_2 and $H(a, 1-t)$ induce the map $c_2: M_{f_2} \rightarrow Z$. Finally, c_1 and c_2 induce the *comparison map* $c: M_{f_1, f_2} \rightarrow Z$. If c is a homotopy equivalence, the square $(*)$ is called *homotopy cocartesian*, and the cocone $X \xrightarrow{g_1} Z \xleftarrow{g_2} Y$ is called a *homotopy pushout*.

Example. By the preceding theorem, a cocartesian square

$$\begin{array}{ccc}
A & \xrightarrow{f_2} & Y \\
f_1 \downarrow & \lrcorner & \downarrow g_2 \\
X & \xrightarrow{g_1} & Z
\end{array}$$

in **Top** is homotopy cocartesian if f_1 (or f_2) is a cofibration, since c is the comparison map for the constant homotopy $g_1 \circ f_1 \simeq g_2 \circ f_2$.

VI.5. Fundamental Groups of Adjunction Spaces

Theorem VI.5.1. (Seifert–van Kampen – pushout version) Let

$$\begin{array}{ccc}
A & \xrightarrow{f_2} & Y \\
f_1 \downarrow & \lrcorner & \downarrow g_2 \\
X & \xrightarrow{g_1} & Z
\end{array}$$

be a pushout of path-connected spaces, and let $a_0 \in A$. Set $x_0 = f_1(a_0)$, $y_0 = f_2(a_0)$, and $z_0 = g_1(f_1(a_0))$. If f_1 or f_2 is a cofibration, then the following square is a pushout of groups:

$$\begin{array}{ccc}
\pi_1(A, a_0) & \xrightarrow{\pi_1(f_2)} & \pi_1(Y, y_0) \\
\pi_1(f_1) \downarrow & \lrcorner & \downarrow \pi_1(g_2) \\
\pi_1(X, x_0) & \xrightarrow{\pi_1(g_1)} & \pi_1(Z, z_0)
\end{array}$$

Proof. Set $\mathring{M}_{f_j} := M_{f_j} \setminus \text{im}(i_{f_j})$ for $j = 1, 2$. Then $\mathcal{O} = \{\mathring{M}_{f_1}, \mathring{M}_{f_2}, A \times (0, 1)\}$ is an open cover of M_{f_1, f_2} . All vertical arrows in

$$\begin{array}{ccccc}
 A \times (0, 1) & \xrightarrow{\quad} & \mathring{M}_{f_2} & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & \mathring{M}_{f_1} & \xrightarrow{\quad} & M_{f_1, f_2} \\
 & & \downarrow & & \downarrow \\
 A \times I & \xrightarrow{\quad} & M_{f_2} & & \\
 \downarrow \text{pr}_A & \searrow & \downarrow & \searrow & \\
 & & M_{f_1} & \xrightarrow{\quad} & M_{f_1, f_2} \\
 & & \downarrow & & \downarrow \\
 A & \xrightarrow{\quad} & Y & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & X & \xrightarrow{\quad} & Z
 \end{array}$$

are homotopy equivalences. Applying π_1 reduces the proof to the group version. $\square$

Examples.

•

$$\begin{array}{ccc}
 A & \xrightarrow{\quad} & \bullet \\
 i \downarrow \lrcorner & & \downarrow \\
 X & \longrightarrow & X/A
 \end{array}$$

If i is a cofibration, then

$$\pi_1(X/A, A/A) \cong \pi_1(X) / \langle\langle \text{im } \pi_1(i) \rangle\rangle.$$

•

$$\begin{array}{ccc}
 S^{n-1} & \xrightarrow{f} & Y \\
 i \downarrow \lrcorner & & \downarrow j \\
 D^n & \longrightarrow & Z
 \end{array}$$

Theorem VI.5.2.

(i) $n \geq 3$: $\pi_1(j): \pi_1(Y, \bullet) \xrightarrow{\cong} \pi_1(Z, \bullet),$

(ii) $n = 2$: $\overline{\pi_1(j)}: \pi_1(Y, \bullet) / \langle\langle [f] \rangle\rangle \xrightarrow{\cong} \pi_1(Z, \bullet).$

•

$$\begin{array}{ccc}
 S^1 & \xrightarrow[\lrcorner]{\prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1}} & \bigvee_{i=1}^{2g} S^1 \\
 \downarrow & & \downarrow \\
 D^2 & \longrightarrow & \Sigma_g
 \end{array}$$

$$\pi_1(\Sigma_g, \bullet) = \left\langle a_1, b_1, \dots, a_g, b_g \left| \prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1} \right. \right\rangle.$$

Conversely, let $G = \langle S \mid R \rangle$ and consider

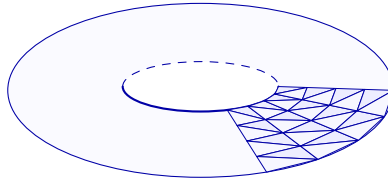
$$\begin{array}{ccc} \bigvee_{r \in R} S^1 & \xrightarrow{\quad \amalg^r \quad} & \bigvee_{s \in S} S^1 \\ \downarrow & \lrcorner & \downarrow \\ \bigvee_{r \in R} D^2 & \longrightarrow & X_G. \end{array}$$

Then $\pi_1(X_G, \bullet) \cong G$. The space X_G is called the *presentation complex*. It is a 2-dimensional *cell complex*, with the closure-finite property and the weak topology.

VI.6. What Comes Next?

- Homology: simplicial, singular, cellular

$$H_k(X) = k\text{-cycles} / k\text{-boundaries}$$



- $H_1(X; \mathbb{Z}) \cong \pi_1(X, \bullet)_{\text{ab}}$
- Degree of a map $f: S^d \rightarrow S^d \rightsquigarrow H_d(f)$:

$$\begin{array}{ccc} H_d(S^d) & \xrightarrow{H_d(f)} & H_d(S^d) \\ \cong \downarrow & & \downarrow \cong \\ \mathbb{Z} & \xrightarrow{\cdot \deg f} & \mathbb{Z} \end{array}$$

- Cell complexes and uniqueness of ordinary homology