

ℓ^2 -Invariants

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I. Introduction

Let X be a connected CW complex. Regular coverings $\{\bar{X}_N\}_{N \trianglelefteq G}$, where $G := \pi_1(X, x_0)$, give rise to actions $G/N \curvearrowright \bar{X}_N$ by deck transformations, simply transitive on the fibers of $p: \bar{X}_N \rightarrow X$, in particular $G \curvearrowright \tilde{X}$ on the universal covering.

We seek G -equivariant versions of the concepts considered so far (homology, Betti numbers, ...).

Consider:

$$\begin{array}{ccc} & & (\tilde{X}, y_0) \\ & \nearrow \exists! & \downarrow p \\ (D^n, 0) & \xrightarrow{Q^n} & (X, x_0). \end{array}$$

Thus, for each n -cell in X there are “ G -many” n -cells in $\tilde{X}$. Hence

$$C_n^{\text{cell}}(\tilde{X}; \mathbb{C}) := H_n(\tilde{X}^n, \tilde{X}^{n-1}; \mathbb{C}),$$

and the cellular chain complex has the form

$$\dots \longrightarrow (\mathbb{C}G)^{k_{n+1}} \longrightarrow (\mathbb{C}G)^{k_n} \longrightarrow (\mathbb{C}G)^{k_{n-1}} \longrightarrow \dots,$$

where $k_n = |I_n|$ is the number of n -cells in X . Since $\tilde{X}^n = p^{-1}(X^n)$ and

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{g} & \tilde{X} \\ & \searrow p & \swarrow p \\ & X & \end{array}$$

commutes, we have $p(g \cdot \tilde{X}^n) = p(\tilde{X}^n) = X^n$, that is, $g \cdot \tilde{X}^n \subseteq \tilde{X}^n$. For $g \in G$ we therefore obtain

$$g: (\tilde{X}^n, \tilde{X}^{n-1}) \xrightarrow{\cong} (g \cdot \tilde{X}^n, g \cdot \tilde{X}^{n-1}),$$

and hence, by functoriality, $G \curvearrowright C_*^{\text{cell}}(\tilde{X}; \mathbb{C})$. That is, $C_*^{\text{cell}}(\tilde{X}; \mathbb{C})$ is a chain complex of free $\mathbb{C}G$ -modules, since $d_n^{\text{cell}} \circ (g \cdot) = (g \cdot) \circ d_n^{\text{cell}}$ by naturality.

Idea. Define the n th equivariant Betti number by

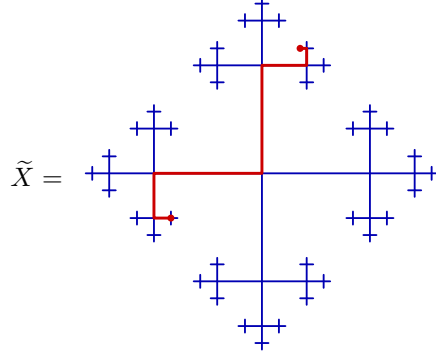
$$b_n^G(\tilde{X}) = \dim_{\mathbb{C}G} H_n(\tilde{X}; \mathbb{C}),$$

where “ $\dim_{\mathbb{C}G}$ ” is defined for subquotients of $(\mathbb{C}G)^k$ and satisfies the following two minimal requirements:

1. $\dim_{\mathbb{C}G} V \geq 0$ and $\dim_{\mathbb{C}G}(\mathbb{C}G) = 1$.
2. For $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$, we require $\dim_{\mathbb{C}G} V = \dim_{\mathbb{C}G} U + \dim_{\mathbb{C}G} W$.

Example. For $X = S^1 \vee S^1 = \bigcirc \bigcirc$, $G = F_2$, we have

$$C_*^{\text{cell}}(\tilde{X}; \mathbb{C}) = (0 \longrightarrow (\mathbb{C}F_2)^2 \xrightarrow{d_1} \mathbb{C}F_2 \longrightarrow 0).$$



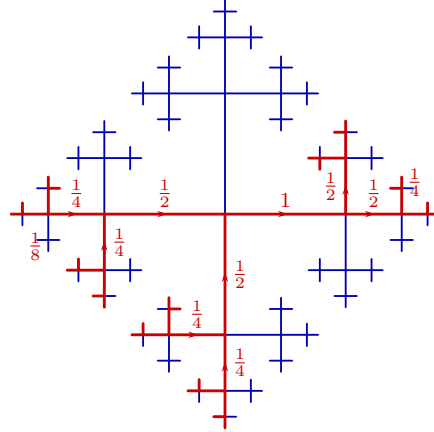
Since $\tilde{X}$ is a tree, every nontrivial finite 1-chain in $\tilde{X}$ has loose ends, that is, d_1 is injective! In particular,

$$0 \longrightarrow (\mathbb{C}F_2)^2 \xrightarrow{d_1} \mathbb{C}F_2 \longrightarrow \text{coker } d_1 \longrightarrow 0$$

is exact. By 1) and 2), we therefore have

$$1 = \dim_{\mathbb{C}F_2} \mathbb{C}F_2 = \underbrace{\dim_{\mathbb{C}F_2} (\mathbb{C}F_2)^2}_{=2} + \underbrace{\dim_{\mathbb{C}F_2} \text{coker } d_1}_{\geq 0},$$

a contradiction ($\mathbb{C}F_2$ is not Noetherian!). But:



So let

$$\ell^2 G = \left\{ \sum_{g \in G} c_g g : c_g \in \mathbb{C}, \sum_{g \in G} |c_g|^2 < \infty \right\},$$

and set

$$C_*^{(2)}(\tilde{X}) := \ell^2 G \otimes_{\mathbb{C}G} C_*^{\text{cell}}(\tilde{X}; \mathbb{C}).$$

Then $\ker d_1^{(2)} \neq 0$, as desired.

For the 1-chain shown in the figure, we have

$$\sum_{g \in G} |c_g|^2 = 1 + 2 \sum_{n=1}^{\infty} 2^n \left(\frac{1}{2^n} \right)^2 = 1 + 2 \sum_{n=1}^{\infty} \frac{1}{2^n} = 3 < \infty.$$

We set

$$H_n^{(2)}(\tilde{X}) := \frac{\ker d_n^{(2)}}{\operatorname{im} d_{n+1}^{(2)}} \cong (\operatorname{im} d_{n+1}^{(2)})^\perp \subseteq (\ell^2 G)^{k_n},$$

where the orthogonal complement is taken in $\ker d_n^{(2)}$. By construction, $H_n^{(2)}(\tilde{X})$ is thus a *Hilbert G -module*, that is, a closed, G -invariant subspace of $(\ell^2 G)^k$. For Hilbert G -modules, there is a dimension $\dim_{\mathcal{R}(G)}$ satisfying 1) and 2)!

Definition.

$$b_n^{(2)}(\tilde{X}) := \dim_{\mathcal{R}(G)} H_n^{(2)}(\tilde{X}) \in \mathbb{R}_{\geq 0}$$

is called the *n th ℓ^2 -Betti number*.

Applications

Conjecture (Kaplansky). Let G be torsion-free and R an integral domain. Then RG has no zero divisors:

$$a \cdot b = 0 \implies a = 0 \text{ or } b = 0.$$

Theorem. Let G be torsion-free. If $b_n^{(2)}(\bar{X}) \in \mathbb{Z}_{\geq 0}$ for every regular covering $\bar{X} \xrightarrow{p} X$ with a connected compact CW complex X and $\operatorname{Deck}(p) \leq G$, then Kaplansky's conjecture holds for G and $R = \mathbb{Q}$.

(“Atiyah conjecture $\implies$ Kaplansky zero-divisor conjecture”.)

Remark. The Kaplansky zero-divisor conjecture also follows from the Kaplansky unit conjecture: If $a \in (RG)^*$, then $a = r \cdot g$ with $r \in R^*$ and $g \in G$.

Proof. The zero-divisor conjecture is equivalent to $a^2 = 0 \implies a = 0$ (not entirely easy). Now let $a \in RG$ with $a^2 = 0$. Then $(1+a)(1-a) = 1 - a^2 = 1$, so $1+a = r \cdot g$ for some $r \in R^*$ and $g \in G$. Thus

$$0 = a^2 = (r \cdot g - 1)^2 = r^2 \cdot g^2 - 2r \cdot g + 1.$$

If $g \neq 1$, then also $g^2 \neq 1$ (G is torsion-free), contradicting linear independence. It follows that $g = 1$ and hence $r = 1$, since $0 = (r-1)^2 \in R$ and R is an integral domain, so $a = 0$. $\square$

But: [Gardam, 2021]: The unit conjecture is false! (Counterexample for $R = \mathbb{Z}/2$ and G virtually abelian.)

A topological space X is called *aspherical* if $\tilde{X} \simeq \{*\}$.

Conjecture (Hopf). Let M be a closed aspherical $2n$ -dimensional manifold. Then $(-1)^n \chi(M) \geq 0$.

Conjecture (Singer). If M is a closed aspherical m -dimensional manifold with $b_n^{(2)}(\tilde{M}) > 0$, then $m = 2n$.

Theorem. Singer $\implies$ Hopf.

Proof.

$$(-1)^n \chi(M) = (-1)^n \sum_{k=0}^{2n} (-1)^k b_k^{(2)}(\tilde{M}) = (-1)^n (-1)^n b_n^{(2)}(\tilde{M}) \geq 0. \quad \square$$

Relationship between $b_n(X)$ and $b_n^{(2)}(\widetilde{X})$?

Example. $b_n(T^k) = \binom{k}{n}$, but $b_n^{(2)}(\widetilde{T^k}) = 0$. Thus there is no immediate dependence, but:

$$b_n^{(2)}(\overline{X}) = \frac{b_n(\overline{X})}{d}, \quad \text{if } \overline{X} \rightarrow X \text{ is regular and } d\text{-sheeted, } d < \infty.$$

Idea. If “ $\overline{X}_i \xrightarrow{i \rightarrow \infty} \widetilde{X}$ ” for a sequence of d_i -sheeted regular coverings

$$\cdots \longrightarrow \overline{X}_i \longrightarrow \overline{X}_{i-1} \longrightarrow \cdots \longrightarrow X,$$

do we then have

$$\lim_{i \rightarrow \infty} \frac{b_n(\overline{X}_i)}{d_i} = b_n^{(2)}(\widetilde{X})?$$

More precisely, we set

$$\overline{X}_i \xrightarrow{i \rightarrow \infty} \widetilde{X} \quad :\Longleftrightarrow \quad \bigcap_{i=1}^{\infty} G_i = \{1\}$$

for the associated subgroups $G_i := \text{im}(\pi_1 \overline{X}_i \rightarrow \pi_1 X) \trianglelefteq \pi_1 X$.

Theorem (Lück, 1994). Let $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$ with $G_i \trianglelefteq G$, $[G : G_i] < \infty$ and $\bigcap_{i=1}^{\infty} G_i = \{1\}$, and let $\overline{X}_i \rightarrow X$ be the regular covering corresponding to G_i . Then, for all $n \geq 0$,

$$\lim_{i \rightarrow \infty} \frac{b_n(\overline{X}_i)}{[G : G_i]} = b_n^{(2)}(\widetilde{X}).$$

Thus $b_n^{(2)}(\widetilde{X})$ detects *free homology growth* in towers of coverings of X .

Remark. The ℓ^2 -torsion “ $\rho^{(2)}(\widetilde{X})$ ” detects (conjecturally!) torsion growth in homology.

II. Hilbert modules and von Neumann dimension

II.1. Hilbert spaces

Definition II.1.1. A *pre-Hilbert space* is a $\mathbb{C}$ -vector space V with a map $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$ such that, for $x, y, z \in V$ and $\alpha \in \mathbb{C}$:

- (i) $\langle x, x \rangle \geq 0$, with equality if and only if $x = 0$;
- (ii) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$;
- (iii) $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$;
- (iv) $\langle y, x \rangle = \overline{\langle x, y \rangle}$.

Remark.

$$\langle \alpha x, y \rangle = \overline{\langle y, \alpha x \rangle} = \overline{\alpha \langle y, x \rangle} = \overline{\alpha} \langle x, y \rangle.$$

Examples.

- $\mathbb{C}^n$ with $\langle x, y \rangle = \sum_{i=1}^n \overline{x_i} y_i$.
- $C([a, b]; \mathbb{C})$ with $\langle f, g \rangle = \int_a^b \overline{f(x)} g(x) dx$.

Now let $(V, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space. For $x, y \in V$, x is called *orthogonal* to y if $\langle x, y \rangle = 0$, and $\{x_i : i \in I\} \subseteq V$ is called *orthonormal* if

$$\langle x_i, x_j \rangle = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Set $\|x\| := \sqrt{\langle x, x \rangle}$.

Lemma II.1.2. Let $\{x_1, \dots, x_n\} \subseteq V$ be orthonormal and $x \in V$. Then

$$\|x\|^2 = \sum_{i=1}^n |\langle x_i, x \rangle|^2 + \left\| x - \sum_{i=1}^n \langle x_i, x \rangle x_i \right\|^2.$$

Proof.

$$x = \sum_{i=1}^n \langle x_i, x \rangle x_i + \left(x - \sum_{i=1}^n \langle x_i, x \rangle x_i \right)$$

is an orthogonal decomposition. □

In particular, $\|x\|^2 \geq \sum_{i=1}^n |\langle x_i, x \rangle|^2$ (*Bessel's inequality*).

Corollary II.1.3 (Cauchy–Schwarz inequality). For all $x, y \in V$, we have $|\langle x, y \rangle| \leq \|x\| \|y\|$.

Proof. Clear if $y = 0$. Otherwise, Bessel's inequality gives

$$\|x\|^2 \geq \left| \left\langle \frac{y}{\|y\|}, x \right\rangle \right|^2 = \frac{|\langle x, y \rangle|^2}{\|y\|^2}. \quad \square$$

Lemma II.1.4. $(V, \|\cdot\|)$ is a normed space.

Proof. Triangle inequality: For $x, y \in V$, we have

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle = \langle x, x \rangle + 2\operatorname{Re}\langle x, y \rangle + \langle y, y \rangle \\ &\leq \langle x, x \rangle + 2|\langle x, y \rangle| + \langle y, y \rangle \\ &\leq \|x\|^2 + 2\|x\| \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2. \end{aligned}$$

$\square$

Thus V is a metric space with $d(x, y) := \|y - x\|$.

Definition II.1.5. A *Hilbert space* is a complete pre-Hilbert space.

Examples.

- $(\mathbb{C}^n, \langle \cdot, \cdot \rangle)$.
- $(\bigoplus_{k=1}^{\infty} \mathbb{C}e_k, \langle \cdot, \cdot \rangle)$ is not a Hilbert space, since the Cauchy sequence $(\sum_{k=1}^n e_k/k^2)_n$ does not converge. Its completion is

$$\ell^2 = \left\{ (a_n) : a_n \in \mathbb{C}, \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$$

with $\langle (a_n), (b_n) \rangle = \sum_{n=1}^{\infty} \overline{a_n} b_n$. The series converges absolutely, since

$$|\overline{a_n} b_n| = |a_n| |b_n| \leq \frac{|a_n|^2 + |b_n|^2}{2}.$$

- Completion of $C[a, b]$: Set

$$\mathcal{L}^2[a, b] = \left\{ f : [a, b] \rightarrow \mathbb{C} : f \text{ measurable, } \int_a^b |f(x)|^2 d\lambda < \infty \right\},$$

where λ is Lebesgue measure, and $\langle f, g \rangle = \int_a^b \overline{f} g d\lambda$. Then

$$L^2[a, b] := \mathcal{L}^2[a, b] / \{f \in \mathcal{L}^2[a, b] : \|f\| = 0\}$$

is a Hilbert space with the induced inner product $\langle \cdot, \cdot \rangle$.

Constructions.

1. If H is a Hilbert space, then every closed subspace $K \subseteq H$ is also a Hilbert space with the restriction of $\langle \cdot, \cdot \rangle$.

2. If $K \subseteq H$ is a closed subspace, then H/K is a Hilbert space via $K^\perp \xrightarrow{\cong} H/K$ (homework!) with the *orthogonal complement*

$$K^\perp = \{x \in H : \langle x, y \rangle = 0 \text{ for all } y \in K\}.$$

3. For Hilbert spaces $\{H_i\}_{i=1}^\infty$, set

$$\widehat{\bigoplus_{i=1}^\infty H_i} := \left\{ (x_i)_{i=1}^\infty : x_i \in H_i, \sum_{i=1}^\infty \|x_i\|_{H_i}^2 < \infty \right\}.$$

4. For Hilbert spaces H_1, H_2 and $x_1 \otimes x_2, y_1 \otimes y_2 \in H_1 \otimes_{\mathbb{C}} H_2$, set

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle := \langle x_1, y_1 \rangle_{H_1} \cdot \langle x_2, y_2 \rangle_{H_2}.$$

Extending and completing gives $H_1 \widehat{\otimes} H_2$. For example,

$$L^2(X_1, \mu_1) \widehat{\otimes} L^2(X_2, \mu_2) \cong L^2(X_1 \times X_2, \mu_1 \otimes \mu_2)$$

for measure spaces with countably generated σ -algebras.

Definition II.1.6. Let H be a Hilbert space. An *orthonormal basis* is a maximal orthonormal subset of H . (Existence by Zorn's lemma.)

Theorem II.1.7. Let H be a Hilbert space with orthonormal basis $\{x_i\}_{i \in I}$ and $x \in H$. Then

$$x = \sum_{i \in I} \langle x_i, x \rangle x_i, \quad \|x\|^2 = \sum_{i \in I} |\langle x_i, x \rangle|^2.$$

By Bessel's inequality, for $\delta > 0$ there are only finitely many x_i with $|\langle x_i, x \rangle| > \delta$, so there are only countably many $i \in I$ with $\langle x_i, x \rangle \neq 0$.

Proof. Let $x_{i_1}, x_{i_2}, \dots$ be an enumeration of the basis vectors with $\langle x_i, x \rangle \neq 0$, and set $y_n = \sum_{k=1}^n \langle x_{i_k}, x \rangle x_{i_k}$. For $n > m$, we have

$$\|y_n - y_m\|^2 = \left\| \sum_{k=m+1}^n \langle x_{i_k}, x \rangle x_{i_k} \right\|^2 = \sum_{k=m+1}^n |\langle x_{i_k}, x \rangle|^2,$$

so (y_n) is a Cauchy sequence, since $\sum_{k=1}^\infty |\langle x_{i_k}, x \rangle|^2 < \infty$ (the partial sums are monotone and bounded by Bessel's inequality). Let $y = \lim_{n \rightarrow \infty} y_n$. Since

$$\begin{aligned} \langle x_{i_j}, x - y \rangle &= \lim_{n \rightarrow \infty} \left\langle x_{i_j}, x - \sum_{k=1}^n \langle x_{i_k}, x \rangle x_{i_k} \right\rangle \\ &= \langle x_{i_j}, x \rangle - \langle x_{i_j}, x \rangle = 0 \end{aligned}$$

and $\langle x_i, x - y \rangle = 0$ for $i \in I$ with $i \neq i_j$ for all j , the maximality of $\{x_i\}_{i \in I}$ implies $x - y = 0$. Finally, [Lemma II.1.2](#) gives

$$0 = \lim_{n \rightarrow \infty} \left\| x - \sum_{k=1}^n \langle x_{i_k}, x \rangle x_{i_k} \right\|^2 = \|x\|^2 - \sum_{k=1}^\infty |\langle x_{i_k}, x \rangle|^2. \quad \square$$

From now on, we consider only *separable* Hilbert spaces, that is, I can be chosen to be countable.

A $\mathbb{C}$ -linear map $A: H_1 \rightarrow H_2$ between Hilbert spaces is also called an *operator*. Note: If A is continuous, there is $\delta > 0$ such that $\|Ax\| < 1$ for $\|x\| < \delta$. Set $C := \frac{2}{\delta}$. Then, for $y \neq 0$,

$$\|Ay\| = \left\| A \left(\frac{\delta}{2} \frac{y}{\|y\|} \right) \right\| \frac{2}{\delta} \|y\| \leq C \|y\|,$$

that is, A is continuous $\iff A$ is bounded.

- Let bounded operators be the morphisms in $\mathbf{Hil}$.
- A is called an *isometry* if $\langle Ax, Ay \rangle = \langle x, y \rangle$, and *unitary* if A is also surjective.
- If there is a unitary operator $u: H_1 \rightarrow H_2$, then H_1 and H_2 are called *isomorphic*.

Theorem II.1.8. A separable Hilbert space H is isomorphic to $\mathbb{C}^n$ or ℓ^2 .

Proof. Choose an orthonormal basis $\{x_i\}_{i \in \mathbb{N}}$ of H in the infinite-dimensional case. Then

$$\begin{aligned} H &\longrightarrow \ell^2, \\ x &\longmapsto (\langle x_1, x \rangle, \langle x_2, x \rangle, \dots) \end{aligned}$$

preserves the norm by Theorem II.1.7, so it is isometric by polarization (homework!) and is surjective because, for $(a_i) \in \ell^2$, the series $\sum_{i=1}^{\infty} a_i x_i$ converges, as we have seen. The finite-dimensional case similarly gives $\mathbb{C}^n$. $\square$

Example. $L^2[-\pi, \pi] \cong \ell^2 \mathbb{Z}$ (“Fourier transform”). The set

$$\left\{ f_n(x) = \frac{e^{inx}}{\sqrt{2\pi}} \right\}_{n=-\infty}^{\infty}$$

is an orthonormal basis! Indeed:

$$\langle f_n, f_m \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)x} dx = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

Maximality is difficult.

For $f \in L^2[-\pi, \pi]$, the expansion

$$f = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

is called the *Fourier expansion* of f , with *Fourier coefficients* $c_n = \langle f_n, f \rangle$. From Theorem II.1.7, we obtain *Parseval’s identity*

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx.$$

II.2. Operators and operator algebras

Set $B(H, K) = \text{Hom}_{\mathbf{Hil}}(H, K)$.

Theorem II.2.1. If $T \in B(H, K)$ is bijective, then T is invertible.

Equivalently, using

$$\begin{array}{ccc} H & \xrightarrow{T} & K \\ \downarrow & \nearrow \bar{T} & \\ H/\ker T & & \end{array}$$

we can state:

Theorem II.2.2. If $T \in B(H, K)$ is surjective, then T is open.

For $T \in B(H, K)$,

$$\|T\| = \sup_{\|x\|_H=1} \|Tx\|_K$$

is called the *operator norm*. It gives the *norm topology* on $B(H, K)$ (complete, that is, $B(H, K)$ is a *Banach space*).

Set

$$B(H) := B(H, H), \quad U(H) = \{u \in B(H) \mid u \text{ unitary}\}, \quad H^* := B(H, \mathbb{C}).$$

The norm topology on $B(H)$ is too fine for many purposes, hence:

Definition II.2.3. The *strong operator topology* on $B(H)$ is the initial topology of the evaluation maps

$$\{E_x: B(H) \rightarrow H, T \mapsto Tx \mid x \in H\}.$$

Definition II.2.4. The *weak operator topology* on $B(H)$ is the initial topology of the “matrix coefficients”

$$\{E_{x,y}: B(H) \rightarrow \mathbb{C}, T \mapsto \langle x, Ty \rangle \mid x, y \in H\}.$$

If H is infinite-dimensional, these topologies are not first countable. We therefore need *nets*!

A net in a topological space X is a map

$$I \longrightarrow X, \quad i \longmapsto x_i,$$

with a directed set $(I, \leq)$, and

$$\lim_{i \in I} x_i = x \iff \begin{array}{l} \text{for every neighborhood } U \text{ of } x, \text{ there is } i \in I \\ \text{such that } x_j \in U \text{ for } j \geq i. \end{array}$$

This gives general characterizations of “closed” and “continuous.”

Lemma II.2.5 (Riesz lemma). For $T \in H^*$, there is a unique $y_T \in H$ with $T(x) = \langle y_T, x \rangle$. We have $\|T\| = \|y_T\|_H$.

Proof. For $T = 0$, take $y_T = 0$. For $T \neq 0$, let $z \in H \setminus \ker T$. Since T is continuous, $\ker T$ is closed, so there is (homework!) a unique $u \in \ker T$ at minimal distance from z . Set

$$w = z - u \in (\ker T)^\perp \quad \text{and} \quad y_T := \frac{\overline{Tw}}{\|w\|^2} w.$$

The rest is an exercise. □

For $T \in B(H, K)$, we define the *adjoint operator* $T^* \in B(K, H)$ as follows: For $x \in K$, let $\ell(T, x) \in H^*$ be defined by

$$\ell(T, x)(u) = \langle x, Tu \rangle_K.$$

Set $T^*x = y_{\ell(T, x)}$; thus $\langle T^*x, y \rangle = \langle x, Ty \rangle$ by construction. We have:

- $\ker T^* = (\operatorname{im} T)^\perp$, $(\ker T^*)^\perp = \overline{\operatorname{im} T}$.
- If T is invertible, so is T^* , and $(T^*)^{-1} = (T^{-1})^*$.
- $u \in B(H, K)$ is unitary $\iff u^*u = \operatorname{id}_H$, $uu^* = \operatorname{id}_K$.
- $u \in B(H, K)$ is an *isometry* $\iff u^*u = \operatorname{id}_H$.
- $u \in B(H, K)$ is a *partial isometry* $\iff u^*u = P$ is a projection.
- $P \in B(H)$ is a *projection* $\iff P = P^* = P^2$.
- $A \in B(H)$ is *positive* $\iff \langle Ax, x \rangle \geq 0$ for all $x \in H$. Here we set $A \leq B \iff B - A$ is positive.
- $T \in B(H)$ is *self-adjoint* $\iff T^* = T$. (Homework: T positive $\implies T$ self-adjoint.)
- $T \in B(H)$ is *normal* $\iff T^*T = TT^*$.

$B(H)$ is a **-algebra*: $T^{**} = T$, $(TS)^* = S^*T^*$.

The *commutant* of $M \subseteq B(H)$ is

$$M' = \{T \in B(H) \mid ST = TS \text{ for all } S \in M\}.$$

The *bicommutant* is $M'' := (M')'$.

Theorem II.2.6 (von Neumann). Let M be a unital ($\operatorname{id}_H \in M$) *-subalgebra of $B(H)$. Then the following are equivalent:

- (i) M is weakly closed.
- (ii) M is strongly closed.
- (iii) $M = M''$.

Proof. (i) $\implies$ (ii): clear.

(iii) $\implies$ (i): We show more: Commutants are always weakly closed. Let $N \subseteq B(H)$, $N' \subseteq B(H)$, $T_0 \in B(H) \setminus N'$, $S \in N$, $x \in H$ with

$$ST_0x - T_0Sx = y \neq 0.$$

Then there is a weakly open neighborhood U of T_0 with

$$\langle y, (ST - TS)x \rangle \neq 0 \quad \text{for } T \in U,$$

that is, $U \subseteq B(H) \setminus N'$.

(ii) $\implies$ (iii): $M \subseteq M''$ is tautological. We must show $M \supseteq M''$. Let $T \in M''$. We must show: For every neighborhood $U \subseteq B(H)$ of T , we have $U \cap M \neq \emptyset$. A neighborhood subbasis of T is

$$E_x^{-1}(B_\varepsilon(Tx)) = \{S \in B(H) \mid \|Sx - Tx\| < \varepsilon\} \quad (x \in H, \varepsilon > 0),$$

so

$$U(T; x_1, \dots, x_n; \varepsilon) = \{S \in B(H) \mid \|(S - T)x_i\| < \varepsilon \text{ for all } i\}$$

is a neighborhood basis of T in the strong operator topology. We thus seek $S \in M$ in this neighborhood. Set $x = (x_1, \dots, x_n) \in H^n$.

Through the diagonal action $M \curvearrowright H^n$, we obtain

$$M \subseteq B(H^n) = \text{Mat}(n, n; B(H))$$

as constant diagonal matrices. With respect to this embedding, we have

$$M' = \text{Mat}(n, n; M'),$$

and M'' consists of constant diagonal matrices with entries in $M'' \subseteq B(H)$. Let P be the projection onto $K = \overline{Mx}$.

Claim: $P \in M'$. Proof: Since K is M -invariant by construction, so is $K^\perp$, because $M = M^*$. If we decompose an arbitrary $y \in H^n$ as $y = y_K + y_{K^\perp}$, then, for $A \in M$,

$$PAy = P(Ay_K + Ay_{K^\perp}) = Ay_K = APy,$$

that is, $P \in M'$. Since $\text{id}_H \in M$, we have $x \in Mx$ and therefore

$$Tx = TPx = PTx \in K = \overline{Mx}.$$

Thus there is $S \in M$ with $\|Tx - Sx\| < \varepsilon$, so in particular $\|(S - T)x_i\| < \varepsilon$ for all $i = 1, \dots, n$. $\square$

Definition II.2.7. A unital $*$ -subalgebra of $B(H)$ satisfying one (and hence all) of conditions (i), (ii), (iii) is called a *von Neumann algebra*.

Remark. A norm-closed $*$ -subalgebra of $B(H)$ is called a *C^* -algebra*.

Examples.

- $\mathbb{C} \cdot \text{id}_H$ and $B(H)$. We have $(\mathbb{C} \cdot \text{id}_H)' = B(H)$.
- If M is a von Neumann algebra, so is M' , because, for $T \in M'$ and $S \in M$,

$$(T^*S - ST^*)^* = S^*T - TS^* = 0,$$

that is, M' is a unital $*$ -subalgebra, and $M' = (M'')' = (M')''$.

Let G (here and henceforth) be a discrete countable group. Then

$$\begin{aligned} \lambda: G &\longrightarrow U(\ell^2 G), & \rho: G &\longrightarrow U(\ell^2 G), \\ g &\longmapsto (h \mapsto gh), & g &\longmapsto (h \mapsto hg^{-1}) \end{aligned}$$

are called the *left* and *right regular representations* of G . These give the algebra representations (of the same names)

$$\lambda: \mathbb{C}G \longrightarrow B(\ell^2 G) \quad \text{and} \quad \rho: \mathbb{C}G \longrightarrow B(\ell^2 G).$$

Definition II.2.8. The *group von Neumann algebra* of G is

$$\mathcal{R}(G) := \rho(\mathbb{C}G)'' \subseteq B(\ell^2 G).$$

Remark. More precisely, $\mathcal{R}(G)$ is the *right* group von Neumann algebra and

$$\mathcal{L}(G) = \lambda(\mathbb{C}G)''$$

is the *left* group von Neumann algebra. We have $\mathcal{R}(G)' = \mathcal{L}(G)$ (homework!).

Let $B(\ell^2 G)^\lambda$ and $B(\ell^2 G)^\rho$ be the von Neumann algebras of λ - and ρ -equivariant operators, respectively. Clearly,

$$\rho(\mathbb{C}G) \subseteq B(\ell^2 G)^\lambda \quad \text{and} \quad \lambda(\mathbb{C}G) \subseteq B(\ell^2 G)^\rho.$$

Theorem II.2.9. We have $\mathcal{R}(G) = B(\ell^2 G)^\lambda$ and $\mathcal{L}(G) = B(\ell^2 G)^\rho$.

Proof. By symmetry, it suffices to show $\mathcal{R}(G) = B(\ell^2 G)^\lambda$. The inclusion “ $\subseteq$ ” is clear because $B(\ell^2 G)^\lambda$ is strongly closed.

“ $\supseteq$ ”: Let $T \in B(\ell^2 G)^\lambda$ and $S \in \mathcal{R}(G)' = \mathcal{L}(G) = \lambda(\mathbb{C}G)''$, that is,

$$S = \text{s-}\lim_{i \in I} S_i \quad \text{with } S_i \in \lambda(\mathbb{C}G).$$

Then $TS = ST$, so $T \in \mathcal{R}(G)'' = \mathcal{R}(G)$. □

Examples.

- For $|G| < \infty$, we have $\mathcal{R}(G) = \mathbb{C}G$, in particular $\mathcal{R}(\{1\}) = \mathbb{C}$.
- $G = \mathbb{Z} = \langle 1 \rangle$: The generator acts on the orthonormal basis $\mathbb{Z} \subseteq \ell^2 \mathbb{Z}$ by $k \mapsto k + 1$, and, under the Fourier transform, on $L^2[-\pi, \pi]$ by

$$\frac{e^{ikx}}{\sqrt{2\pi}} \mapsto \frac{e^{i(k+1)x}}{\sqrt{2\pi}} = e^{ix} \cdot \frac{e^{ikx}}{\sqrt{2\pi}}.$$

Set $z = e^{ix}$. Then $\mathbb{C}\mathbb{Z} \cong \mathbb{C}[z, z^{-1}]$ acts by multiplication on $L^2[-\pi, \pi]$. For $T \in \mathcal{R}(G) = B(\ell^2 G)^\lambda$ and $f \in L^2[-\pi, \pi]$, we have $f = \lim_{k \rightarrow \infty} p_k$ with $p_k \in \mathbb{C}[z, z^{-1}]$,

so

$$Tf = \lim_{k \rightarrow \infty} T(p_k \cdot 1) = \lim_{k \rightarrow \infty} p_k T(1) = T(1) \cdot f.$$

Claim: $T(1) \in L^\infty[-\pi, \pi]$. Proof: We show $\|T(1)\|_\infty = \|T\|$. Suppose there were $\varepsilon > 0$ such that

$$A = \{|T(1)| \geq \|T\| + \varepsilon\}$$

has positive λ -measure. Set $f_A := \chi_A / \sqrt{\lambda(A)}$, that is, $\|f_A\| = 1$. Then

$$\|T\|^2 \geq \|Tf_A\|^2 = \int_{-\pi}^{\pi} |T(1)|^2 \frac{\chi_A}{\lambda(A)} d\lambda \geq (\|T\| + \varepsilon)^2,$$

a contradiction.

Conversely, multiplication by any function $g \in L^\infty[-\pi, \pi]$ defines an operator $T \in B(\ell^2 G)^\lambda$. We have thus shown:

Theorem II.2.10. $\mathcal{R}(\mathbb{Z}) \cong L^\infty[-\pi, \pi] \cong L^\infty(S^1)$.

Remark. Similarly, we see that $\mathcal{R}(\mathbb{Z}^n) \cong L^\infty(T^n)$.

More generally: Every commutative von Neumann algebra is isomorphic to $L^\infty(X, \mu)$ (standard Borel space), and every commutative unital C^* -algebra to $C(X)$ (X compact and Hausdorff).

Definition II.2.11. The *enlarged group von Neumann algebra* is

$$B((\ell^2 G)^n)^\lambda = M_n(\lambda(\mathbb{C}G)') = M_n(\mathcal{R}(G)).$$

(Here G acts diagonally through λ .)

Open question. $\mathcal{R}(F_n) \cong \mathcal{R}(F_m) \stackrel{?}{\implies} n = m$.

II.3. Trace and dimension

Definition II.3.1. The *von Neumann trace* is

$$\mathrm{tr}_{\mathcal{R}(G)}: \mathcal{R}(G) \longrightarrow \mathbb{C}, \quad T \longmapsto \langle e, Te \rangle,$$

where $e \in \ell^2 G$ is the identity element.

Lemma II.3.2. $\mathrm{tr}_{\mathcal{R}(G)}(ST) = \mathrm{tr}_{\mathcal{R}(G)}(TS)$.

Proof. For $g, h \in G$,

$$\begin{aligned} \mathrm{tr}_{\mathcal{R}(G)}((\cdot g)(\cdot h)) &= \langle e, gh \rangle = \begin{cases} 1, & g = h^{-1}, \\ 0, & \text{otherwise} \end{cases} \\ &= \langle e, hg \rangle = \mathrm{tr}_{\mathcal{R}(G)}((\cdot h)(\cdot g)). \end{aligned}$$

Thus the claim follows for $S, T \in \rho(\mathbb{C}G)$, and hence, by weak continuity, also for $S, T \in \mathcal{R}(G)$. $\square$

The same holds for

$$\mathrm{tr}_{\mathcal{R}(G)}: M_n(\mathcal{R}(G)) \longrightarrow \mathbb{C}, \quad A \longmapsto \sum_{i=1}^n \mathrm{tr}_{\mathcal{R}(G)} A_{ii},$$

because

$$\mathrm{tr}(AB) = \mathrm{tr}((AB)^\top) = \mathrm{tr}(B^\top A^\top) = \mathrm{tr}(BA).$$

Example. In $\mathcal{R}(\mathbb{Z}) \cong L^\infty[-\pi, \pi]$, we have

$$\mathrm{tr} f = \frac{1}{2\pi} \int_{-\pi}^{\pi} f d\lambda.$$

If $A \subseteq [-\pi, \pi]$ is measurable, then $\chi_A = \overline{\chi_A} = \chi_A^2$, that is, $\chi_A \in L^\infty[-\pi, \pi]$ is a projection, and $\mathrm{tr} \chi_A = \lambda(A)/(2\pi) \in [0, 1]$ can be arbitrary. Hence there is $p \in M_n(L^\infty[-\pi, \pi])$ with arbitrary $\mathrm{tr} p \in [0, n]$. Note that $\mathrm{tr} p \in \{0, 1, \dots, n\}$ for a projection $p \in M_n(\mathbb{C})$.

Let H be a Hilbert space with a unitary action $G \curvearrowright H$ and $T \in B(H)^G$. We want to define “ $\mathrm{tr}_{\mathcal{R}(G)} T$.” To do so, we assume that we have an isometric G -embedding

$$i: H \hookrightarrow (\ell^2 G)^n$$

(with respect to the diagonal action $G \curvearrowright (\ell^2 G)^n$ through λ). Here $\mathrm{pr}_{i(H)}$ denotes the orthogonal projection onto $i(H)$, followed by i^{-1} .

Proposition II.3.3. The linear functional

$$B(H)^G \longrightarrow \mathbb{C}, \quad T \longmapsto \operatorname{tr}_{\mathcal{R}(G)}(i \circ T \circ \operatorname{pr}_{i(H)})$$

is independent of the choice of i .

Proof. Let $j: H \hookrightarrow (\ell^2 G)^m$ be another embedding; without loss of generality, $n = m$. By Theorem II.2.1, we obtain a unitary transformation

$$j \circ i^{-1}: i(H) \longrightarrow j(H),$$

which we extend to a partial isometry u on $(\ell^2 G)^n$. Since $j = u \circ i$, taking adjoints gives $\operatorname{pr}_{j(H)} = \operatorname{pr}_{i(H)} \circ u^*$. Thus, if $P_{i(H)}$ denotes the orthogonal projection onto $i(H)$ as an endomorphism of $(\ell^2 G)^n$, we have

$$\begin{aligned} \operatorname{tr}(j \circ T \circ \operatorname{pr}_{j(H)}) &= \operatorname{tr}(u \circ i \circ T \circ \operatorname{pr}_{i(H)} \circ u^*) \\ &= \operatorname{tr}(i \circ T \circ \operatorname{pr}_{i(H)} \circ u^* \circ u) \\ &= \operatorname{tr}(i \circ T \circ \operatorname{pr}_{i(H)} \circ P_{i(H)}) \\ &= \operatorname{tr}(i \circ T \circ \operatorname{pr}_{i(H)}). \end{aligned}$$

□

Definition II.3.4. A Hilbert $\mathcal{L}(G)$ -module (*Hilbert G -module*, *Hilbert module*) is a Hilbert space H with a unitary action $G \curvearrowright H$ such that there is an isometric G -embedding $H \hookrightarrow (\ell^2 G)^n$ for some n .

(For $T \in \mathcal{L}(G)$, let $T = \text{s-lim}_{i \in I} T_i$ with $T_i \in \lambda(\mathbb{C}G)$, and, for $x \in H$, define $T \cdot x = \lim_{i \in I} (T_i \cdot x)$.)

- The category $\mathcal{L}(G)\text{-Mod}$ has bounded G -operators as morphisms.
- H is called *free* if $i: H \longrightarrow (\ell^2 G)^n$ can be chosen to be unitary.
- By Proposition II.3.3, we obtain $\operatorname{tr}_{\mathcal{R}(G)}$ on

$$\operatorname{End}_{\mathcal{L}(G)}(H) = B(H)^G.$$

Theorem II.3.5. Let H be a Hilbert $\mathcal{L}(G)$ -module.

- (i) $\operatorname{tr}_{\mathcal{R}(G)}$ is $\mathbb{C}$ -linear.
- (ii) $\operatorname{tr}_{\mathcal{R}(G)}$ is weakly continuous.
- (iii) For $S, T \in B(H)^G$, we have

$$\operatorname{tr}_{\mathcal{R}(G)}(ST) = \operatorname{tr}_{\mathcal{R}(G)}(TS).$$

- (iv) For $T \in B(H)^G$, we have

$$\operatorname{tr}_{\mathcal{R}(G)}(T^*T) = 0 \iff T = 0.$$

- (v) For $S, T \in B(H)^G$ with $S \leq T$, we have

$$\operatorname{tr}_{\mathcal{R}(G)}(S) \leq \operatorname{tr}_{\mathcal{R}(G)}(T).$$

In $\mathcal{L}(G)\text{-Mod}$, we have the following constructions:

- A *submodule* $K \leq H$ is a closed G -invariant subspace.

- If $K \leq H$ is a submodule, then $H/K \cong K^\perp$ is a *factor module* or *quotient module*.
- *Direct sum* $H_1 \oplus H_2$.
- The *tensor product* $H_1 \otimes H_2$ is a $G_1 \times G_2$ -Hilbert module:

$$i_j: H_j \longrightarrow (\ell^2 G_j)^{n_j}, \quad j = 1, 2,$$

gives

$$i_1 \otimes i_2: H_1 \otimes H_2 \longrightarrow (\ell^2 G_1)^{n_1} \otimes (\ell^2 G_2)^{n_2} \cong (\ell^2(G_1 \times G_2))^{n_1 n_2}.$$

- For $G_0 \leq G$ with $[G : G_0] < \infty$, we have

$$\text{res}_{G_0}^G: \mathcal{L}(G)\text{-}\mathbf{Mod} \longrightarrow \mathcal{L}(G_0)\text{-}\mathbf{Mod}$$

(restriction); use

$$\text{res}_{G_0}^G \ell^2 G \cong (\ell^2 G_0)^{[G:G_0]}.$$

Theorem II.3.6.

(i) Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & K & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow r & & \downarrow s & & \downarrow t & & \\ 0 & \longrightarrow & H & \longrightarrow & K & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

be a diagram in $\mathcal{L}(G)\text{-}\mathbf{Mod}$. Then $\text{tr } s = \text{tr } r + \text{tr } t$.

(ii) For $s \in B(H_1)^{G_1}$, $t \in B(H_2)^{G_2}$, we have

$$\text{tr}(s \otimes t) = \text{tr}(s) \cdot \text{tr}(t).$$

(iii) For $s \in B(H)^G$ and $[G : G_0] < \infty$, we have

$$\text{tr}(\text{res}_{G_0}^G(s)) = [G : G_0] \text{tr}(s).$$

Proof. For (i), use $u: H \oplus L \xrightarrow{\cong} K$ and

$$u^* s u \approx \begin{pmatrix} r & * \\ 0 & t \end{pmatrix}.$$

□

Definition II.3.7. The *von Neumann dimension* of a Hilbert G -module H is

$$\dim_{\mathcal{R}(G)}(H) = \text{tr}_{\mathcal{R}(G)}(\text{id}_H).$$

Examples.

- $\dim_{\mathcal{R}(\{1\})}(H) = \dim_{\mathbb{C}} H$.
- $|G| < \infty$:

$$\dim_{\mathcal{R}(G)} H = \frac{\dim_{\mathcal{R}(\{1\})} H}{|G|} = \frac{\dim_{\mathbb{C}} H}{|G|},$$

independently of $G \curvearrowright H$!

- $G = \mathbb{Z}$: For a measurable set $A \subseteq [-\pi, \pi]$, we have $L^2 A \subseteq L^2[-\pi, \pi] \cong \ell^2 \mathbb{Z}$ and

$$\begin{aligned} \dim_{L^\infty[-\pi, \pi]}(L^2 A) &= \operatorname{tr}_{L^\infty[-\pi, \pi]}(\operatorname{id}_A) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_A d\lambda = \frac{\lambda(A)}{2\pi} \in [0, 1]. \end{aligned}$$

By II.3.6(i),

$$\dim_{L^\infty[-\pi, \pi]}((L^2 A)^n) = n \cdot \frac{\lambda(A)}{2\pi} \in [0, n].$$

Definition II.3.8. $H \xrightarrow{i} K \xrightarrow{p} L$ in $\mathcal{L}(G)\text{-Mod}$ is called *weakly exact* if $\ker p = \overline{\operatorname{im} i}$. A morphism $s: H \rightarrow K$ is called a *weak isomorphism* if

$$0 \longrightarrow H \xrightarrow{s} K \longrightarrow 0$$

is weakly exact.

Theorem II.3.9.

- (i) $\dim_{\mathcal{R}(G)} \ell^2 G = 1$.
- (ii) If $0 \rightarrow H \rightarrow K \rightarrow L \rightarrow 0$ is weakly exact, then

$$\dim_{\mathcal{R}(G)} K = \dim_{\mathcal{R}(G)} H + \dim_{\mathcal{R}(G)} L.$$

(iii)

$$\dim_{\mathcal{R}(G_1 \times G_2)}(H_1 \otimes H_2) = (\dim_{\mathcal{R}(G_1)} H_1)(\dim_{\mathcal{R}(G_2)} H_2).$$

(iv)

$$\dim_{\mathcal{R}(G_0)} \operatorname{res}_{G_0}^G H = [G : G_0] \dim_{\mathcal{R}(G)} H.$$

(v) Regularity.

This completes the connection with the introduction.

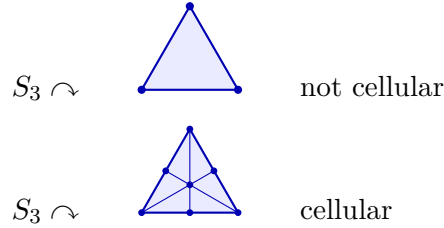
III. ℓ^2 -Betti numbers of CW complexes

III.1. G -CW complexes

Definition III.1.1. Let X be a CW complex. An action $G \curvearrowright X$ is called *cellular* if, for every open cell $E \subseteq X$ and every $g \in G$, the following hold:

- (i) gE is again an open cell ($\Leftrightarrow g: X \rightarrow X$ is cellular).
- (ii) If $gE \cap E \neq \emptyset$, then $gx = x$ for all $x \in E$.

Examples.



Choose pushouts for X ; then

$$X^n \setminus X^{n-1} \cong \coprod_{j \in J_n} \mathring{D}^n.$$

By (i), we obtain $G \curvearrowright J_n$. Set $I_n = J_n/G$. Then

$$X^n \setminus X^{n-1} \cong \coprod_{i \in I_n} G/H_i \times \mathring{D}^n$$

(a G -homeomorphism by (ii)). We call $G/H \times D^n$ an n -dimensional G -cell.

Theorem III.1.2. Let X be a CW complex with $G \curvearrowright X$. The following statements are equivalent:

- (i) G acts cellularly.
- (ii) The skeleta X^n are G -invariant, and there are pushouts in $G\text{-Top}$:

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \longrightarrow & X^{n-1} \\ \downarrow & \lrcorner & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \longrightarrow & X^n. \end{array}$$

Proof. (ii) $\Rightarrow$ (i) is clear. For (i) $\Rightarrow$ (ii), see the discussion above. We have pushouts

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \longrightarrow & X^{n-1} \\ \downarrow & \lrcorner & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \longrightarrow & X^n \end{array}$$

in **Top**. We must show that these are also pushouts in $G\text{-}\mathbf{Top}$.

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \longrightarrow & X^{n-1} \\ \downarrow & \lrcorner & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \longrightarrow & X^n \end{array} \quad \begin{array}{c} \searrow \\ \downarrow \\ \searrow \end{array} \quad \begin{array}{c} \\ \\ \exists! u \end{array} \quad \begin{array}{c} \\ \\ Z \end{array}$$

Since every $x \in X^n$ lifts to $\coprod_{i \in I_n} G/H_i \times D^n$ or X^{n-1} , u is also G -equivariant. $\square$

Definition III.1.3. A G -CW complex is a CW complex X with a cellular action $G \curvearrowright X$. It

- is of *finite type* if, for each n , it has only finitely many n -dimensional G -cells;
- is *finite* if it has only finitely many G -cells;
- is called *proper* if all the H_i in (ii) are finite;
- is called *free* if all the H_i are trivial.

Remarks.

- X is finite $\iff G \backslash X$ is compact ($\iff X$ is cocompact).
- $N \trianglelefteq G \implies N \backslash X$ is a G/N -CW complex.
- For $G_0 \leq G$, we obtain

$$\text{res}_{G_0}^G : G\text{-CW} \longrightarrow G_0\text{-CW}.$$

Example. Let X be a connected CW complex and $p: Y \rightarrow X$ a regular covering. Then Y is a G -CW complex with $G = \text{Deck}(p)$. Conversely, if Y is a connected free G -CW complex, then $p: Y \rightarrow G \backslash Y$ is a regular covering with $\text{Deck}(p) \cong G$.

III.2. The ℓ^2 -completion of the cellular chain complex

Let X be a G -CW complex. Every $g \in G$ defines a cellular map $g: X \xrightarrow{\cong} X$, and we obtain functorially

$$g: C_*^{\text{CW}}(X; \mathbb{Z}) \xrightarrow{\cong} C_*^{\text{CW}}(X; \mathbb{Z}),$$

that is, $C_*^{\text{CW}}(X; \mathbb{Z})$ is canonically a chain complex of left $\mathbb{Z}G$ -modules, and $C_*^{\text{CW}}(\cdot; \mathbb{Z})$ defines a functor

$$G\text{-}\mathbf{CW} \longrightarrow \mathbb{Z}G\text{-}\mathbf{chain}.$$

Definition III.2.1. The (cellular) ℓ^2 -chain complex of a G -CW complex X is the $\mathbb{C}G$ -chain complex

$$C_*^{(2)}(X) := \ell^2 G \otimes_{\mathbb{Z}G} C_*^{\text{cell}}(X; \mathbb{Z}).$$

Let $G\text{-}\mathbf{CW}_{\text{f.t.}}^{\text{prop.}}$ be the full subcategory of proper G -CW complexes of finite type.

Theorem III.2.2. The ℓ^2 -chain complex defines a functor

$$C_*^{(2)}(\cdot) : G\text{-}\mathbf{CW}_{\text{f.t.}}^{\text{prop.}} \longrightarrow \mathcal{L}(G)\text{-}\mathbf{Mod}.$$

Proof. We consider only a free G -CW complex X of finite type. Then, for all n ,

$$C_n^{\text{cell}}(X; \mathbb{Z}) \cong (\mathbb{Z}G)^{k_n}$$

(by choosing an n -cell in each G -orbit of n -cells). Thus

$$C_n^{(2)}(X) \cong (\ell^2 G)^{k_n},$$

and this algebraic isomorphism makes $C_n^{(2)}(X)$ a free Hilbert $\mathcal{L}(G)$ -module (independently of the choice, since any two isomorphisms are unitarily conjugate). The differentials $d_n^{(2)}$ are represented by right multiplication by matrices in $\text{Mat}(k_n, k_{n-1}; \mathbb{Z}G)$ and are therefore bounded G -operators. The same holds for

$$f_n^{(2)} : C_n^{(2)}(X) \longrightarrow C_n^{(2)}(Y),$$

induced by $f : X \rightarrow Y$. □

Thus, for a free G -CW complex with k_n n -dimensional G -cells,

$$C_*^{(2)}(X) \cong \left(\cdots \longrightarrow (\ell^2 G)^{k_{n+1}} \xrightarrow{\cdot \text{INC}_{n+1}^G} (\ell^2 G)^{k_n} \xrightarrow{\cdot \text{INC}_n^G} (\ell^2 G)^{k_{n-1}} \longrightarrow \cdots \right)$$

with $\text{INC}_n^G \in \text{Mat}(k_n, k_{n-1}; \mathbb{Z}G)$.

III.3. ℓ^2 -Betti numbers and how to compute them

Definition III.3.1. Let X be a proper G -CW complex of finite type, and let $(C_*^{(2)}(X), d_*^{(2)})$ be the associated ℓ^2 -chain complex. Then the n th (reduced) ℓ^2 -homology of X is given by

$$H_n^{(2)}(X) = \ker d_n^{(2)} / \overline{\text{im } d_{n+1}^{(2)}}.$$

Remark. By Theorem III.2.2 and our discussion following Theorem II.3.5, $H_n^{(2)}(X)$ is a Hilbert $\mathcal{L}(G)$ -module.

Definition III.3.2. Let X be a proper G -CW complex of finite type. The n th ℓ^2 -Betti number of X is

$$b_n^{(2)}(X) = \dim_{\mathcal{R}(G)} H_n^{(2)}(X) \in \mathbb{R}_{\geq 0}.$$

- We also write $b_n^{(2)}(G \curvearrowright X)$.
- For a regular covering $p: Y \rightarrow X$, we set

$$b_n^{(2)}(Y) := b_n^{(2)}(\text{Deck}(p) \curvearrowright Y).$$

- In particular,

$$b_n^{(2)}(\tilde{X}) = b_n^{(2)}(\pi_1 X \curvearrowright \tilde{X})$$

for a connected finite CW complex X .

Examples. **I)** If $|G| < \infty$, every G -CW complex X is proper, and X is of finite type if and only if X^n is compact for all $n \geq 0$. We obtain

$$\begin{aligned} C_*^{(2)}(X) &= \ell^2 G \otimes_{\mathbb{Z}G} C_*^{\text{CW}}(X) \cong \mathbb{C}G \otimes_{\mathbb{Z}G} C_*^{\text{CW}}(X) \\ &\cong (\mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}G) \otimes_{\mathbb{Z}G} C_*^{\text{CW}}(X) \\ &\cong \mathbb{C} \otimes_{\mathbb{Z}} (\mathbb{Z}G \otimes_{\mathbb{Z}G} C_*^{\text{CW}}(X)) \\ &\cong \mathbb{C} \otimes_{\mathbb{Z}} C_*^{\text{CW}}(X) \cong C_*^{\text{CW}}(X; \mathbb{C}), \end{aligned}$$

so $H_n^{(2)}(X) = H_n(X; \mathbb{C})$, and Theorem II.3.9(iv) gives

$$b_n^{(2)}(X) = \dim_{\mathcal{R}(G)} H_n(X; \mathbb{C}) = \frac{\dim_{\mathbb{C}} \text{res}_{\{1\}}^G H_n(X; \mathbb{C})}{|G|} = \frac{b_n(X)}{|G|}.$$

In particular, $b_n^{(2)}(X) = b_n(X)$ if $G = \{1\}$.

II) Let $S^1 = \bigcirc$ and let $X = \tilde{S}^1$ be the $\mathbb{Z}$ -CW complex



$$\begin{array}{ccccccc} C_*^{(2)}(X) \cong \dots & \longrightarrow & 0 & \longrightarrow & \ell^2(\mathbb{Z}) & \xrightarrow{d_1^{(2)}} & \ell^2(\mathbb{Z}) \\ & & & & \cong \downarrow \text{Fourier} & & \downarrow \cong \\ & & & & L^2 S^1 & \xrightarrow{\cdot(z-1)} & L^2 S^1. \end{array}$$

Thus every $f \in \ker d_1^{(2)}$ must have constant Fourier coefficients. The ℓ^2 condition implies $f = 0$, that is, $\ker d_1^{(2)} = \{0\}$. Moreover, $(d_1^{(2)})^* = \cdot(z^{-1} - 1)$, so we also have

$$0 = \ker(d_1^{(2)})^* = (\text{im } d_1^{(2)})^\perp,$$

and hence $\overline{\text{im } d_1^{(2)}} = C_0^{(2)}(X)$. It follows that $H_n^{(2)}(X) = 0$ for all $n \geq 0$, so $b_n^{(2)}(X) = 0$ for all $n \geq 0$ (X is ℓ^2 -acyclic).

III) Let

$$\begin{array}{ccc} X^0 & \xrightarrow{\quad} & X \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{\quad} & Y \end{array}$$

with X as in II, so



with a $G = \mathbb{Z} \times \mathbb{Z}/2$ -action.

We have

$$C_*^{(2)}(Y) \cong \left(\cdots \longrightarrow 0 \longrightarrow \ell^2(\mathbb{Z} \times \mathbb{Z}/2) \xrightarrow{d_1^{(2)}} \ell^2\left(\frac{\mathbb{Z} \times \mathbb{Z}/2}{\{0\} \times \mathbb{Z}/2}\right) \right).$$

Since $X \subseteq Y$ and $X^0 = Y^0$, we again obtain $\overline{\text{im } d_1^{(2)}} = C_0^{(2)}(Y)$, so $b_0^{(2)}(Y) = 0$. Thus

$$0 \longrightarrow \ker d_1^{(2)} \longrightarrow \ell^2(\mathbb{Z} \times \mathbb{Z}/2) \xrightarrow{d_1^{(2)}} \ell^2\left(\frac{\mathbb{Z} \times \mathbb{Z}/2}{\{0\} \times \mathbb{Z}/2}\right) \longrightarrow 0$$

is weakly exact. Moreover, by II.3.9(iv),

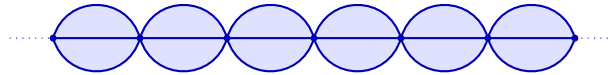
$$\dim_{\mathcal{R}(G)} \ell^2\left(\frac{\mathbb{Z} \times \mathbb{Z}/2}{\{0\} \times \mathbb{Z}/2}\right) = \frac{1}{2} \dim_{\mathcal{R}(\mathbb{Z})} \ell^2 \mathbb{Z} = \frac{1}{2},$$

that is,

$$b_1^{(2)}(Y) = \dim_{\mathcal{R}(G)} \ell^2 G - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}.$$

There are no higher-dimensional G -cells, so $b_n^{(2)}(Y) = 0$ for $n \geq 2$.

IV) Now let $Z =$



Then

$$\begin{aligned} C_*^{(2)}(Z) = \left(\cdots \longrightarrow 0 \longrightarrow \ell^2(\mathbb{Z} \times \mathbb{Z}/2) \xrightarrow{d_2^{(2)}} \right. \\ \left. \ell^2(\mathbb{Z} \times \mathbb{Z}/2) \oplus \ell^2\left(\frac{\mathbb{Z} \times \mathbb{Z}/2}{\{0\} \times \mathbb{Z}/2}\right) \xrightarrow{d_1^{(2)}} \ell^2\left(\frac{\mathbb{Z} \times \mathbb{Z}/2}{\{0\} \times \mathbb{Z}/2}\right) \right). \end{aligned} \quad (*)$$

Again, $d_1^{(2)}$ has dense image, and $d_2^{(2)}$ is injective (the identity in the first component). Since

$$\begin{aligned} \dim_{\mathcal{R}(G)} \left(\ell^2(\mathbb{Z} \times \mathbb{Z}/2) \oplus \ell^2((\mathbb{Z} \times \mathbb{Z}/2)/(\{0\} \times \mathbb{Z}/2)) \right) &= \frac{3}{2} = 1 + \frac{1}{2} \\ &= \dim_{\mathcal{R}(G)} \ell^2(\mathbb{Z} \times \mathbb{Z}/2) + \dim_{\mathcal{R}(G)} \ell^2((\mathbb{Z} \times \mathbb{Z}/2)/(\{0\} \times \mathbb{Z}/2)), \end{aligned}$$

(*) is weakly exact, so Z is ℓ^2 -acyclic.

Instead of arguing with the ℓ^2 -chain complex, one uses:

Theorem III.3.3 (Computing ℓ^2 -Betti numbers).

- (i) Let $f: X \xrightarrow{\sim} Y$ be a G -homotopy equivalence of proper G -CW complexes of finite type. Then

$$b_n^{(2)}(X) = b_n^{(2)}(Y) \quad \text{for all } n \geq 0.$$

- (ii) Let $X \neq \emptyset$ be a connected proper G -CW complex of finite type. Then

$$b_0^{(2)}(X) = \frac{1}{|G|} \quad \left(\text{with } \frac{1}{\infty} = 0 \right).$$

- (iii) Let X_1, X_2 be proper G_1 - and G_2 -CW complexes of finite type, respectively. Then $X_1 \times X_2$ is a proper $G_1 \times G_2$ -CW complex of finite type, and

$$b_n^{(2)}(X_1 \times X_2) = \sum_{p+q=n} b_p^{(2)}(X_1) b_q^{(2)}(X_2).$$

- (iv) Let X be a proper G -CW complex of finite type and $G_0 \leq G$ with $[G : G_0] < \infty$. Then $\text{res}_{G_0}^G X$ is a proper G_0 -CW complex of finite type, and

$$b_n^{(2)}(\text{res}_{G_0}^G X) = [G : G_0] b_n^{(2)}(X) \quad \text{for } n \geq 0.$$

Definition III.3.4. Let X be a finite proper G -CW complex. Its ℓ^2 -Euler characteristic is

$$\chi^{(2)}(X) := \sum_{n \geq 0} (-1)^n b_n^{(2)}(X) \in \mathbb{R}.$$

The ℓ^2 -Euler characteristic $\chi^{(2)}$ is a G -homotopy equivalence invariant (III.3.3(i)).

Theorem III.3.5 (ℓ^2 -Euler–Poincaré formula). Let X be a finite proper G -CW complex, and let $\{H_i\}_{i \in I_n}$ be the family of stabilizers of the equivariant n -cells of X . Then

$$\chi^{(2)}(X) = \sum_{n \geq 0} (-1)^n \sum_{i \in I_n} \frac{1}{|H_i|}.$$

Proof. See Topology 1, IV.2.8! Apply $\dim_{\mathcal{R}(G)}$ to

$$0 \longrightarrow \ker d_n^{(2)} \longrightarrow C_n^{(2)}(X) \xrightarrow{d_n^{(2)}} \overline{\text{im } d_n^{(2)}} \longrightarrow 0$$

and

$$0 \longrightarrow \overline{\text{im } d_{n+1}^{(2)}} \longrightarrow \ker d_n^{(2)} \longrightarrow H_n^{(2)}(X) \longrightarrow 0$$

(both weakly exact) and use

$$C_n^{(2)}(X) \cong \bigoplus_{i \in I_n} \ell^2(G/H_i).$$

It remains to show that $\dim_{\mathcal{R}(G)} \ell^2(G/H) = 1/|H|$. To this end, consider

$$\iota: \ell^2(G/H) \hookrightarrow \ell^2 G, \quad gH \mapsto \frac{1}{\sqrt{n}}(gh_1 + \cdots + gh_n)$$

with $H = \{h_1 = e, h_2, \dots, h_n\}$ (well-defined, isometric, and G -equivariant). Then

$$\text{pr}_{\iota(\ell^2(G/H))} = \frac{1}{n}(h_1 + \cdots + h_n)$$

and hence

$$\dim_{\mathcal{R}(G)} \ell^2(G/H) = \text{tr}_{\mathcal{R}(G)} \text{pr}_{\iota(\ell^2(G/H))} = \frac{1}{n}.$$

□

Corollary III.3.6. Let X be a finite free G -CW complex. Then

$$\chi^{(2)}(X) = \chi(G \setminus X).$$

In particular, $\chi(X) = \chi^{(2)}(\tilde{X})$ for a compact connected CW complex X .

This proves the implication “Singer $\Rightarrow$ Hopf” from Chapter I.

We revisit Examples II–IV:

- $b_0^{(2)}(X) = b_0^{(2)}(Y) = b_0^{(2)}(Z) = 0$ follows from III.3.3(ii).
- Since $0 = \chi(S^1) = \chi^{(2)}(X) = b_0^{(2)}(X) - b_1^{(2)}(X)$, we obtain $b_1^{(2)}(X) = 0$.
- By III.3.5, we have $\chi^{(2)}(Y) = \frac{1}{2} - 1 = -\frac{1}{2}$, so $b_1^{(2)}(Y) = \frac{1}{2}$.
- Since $Z \simeq_G X$, Z , like X , is ℓ^2 -acyclic.

Theorem III.3.7 (ℓ^2 -Poincaré duality). Suppose the topological space M carries both the structure of an orientable n -dimensional manifold and that of a finite free G -CW complex. Then

$$b_k^{(2)}(G \curvearrowright M) = b_{n-k}^{(2)}(G \curvearrowright M).$$

Example. Let $g \geq 1$.

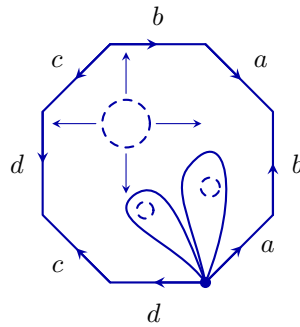
- We have $b_2^{(2)}(\tilde{\Sigma}_g) = b_0^{(2)}(\tilde{\Sigma}_g) = 0$ and

$$b_1^{(2)}(\tilde{\Sigma}_g) = -\chi^{(2)}(\tilde{\Sigma}_g) = -\chi(\Sigma_g) = -(2 - 2g) = 2g - 2.$$

- Let $\Sigma_{g,d}$ be the surface with $d > 0$ holes,

$$\Sigma_{g,d} = \Sigma_g \setminus \bigcup_{i=1}^d D_i^2.$$

Then $\Sigma_{g,d} \simeq \bigvee_{i=1}^{2g+d-1} S^1$.



Thus

$$\begin{aligned} 1 - (2g + d - 1) &= \chi(\Sigma_{g,d}) = \chi^{(2)}(\tilde{\Sigma}_{g,d}) \\ &= \underbrace{b_0^{(2)}(\tilde{\Sigma}_{g,d})}_{=0} - b_1^{(2)}(\tilde{\Sigma}_{g,d}) + \underbrace{b_2^{(2)}(\tilde{\Sigma}_{g,d})}_{=0}. \end{aligned}$$

Here $\pi_1 \Sigma_{g,d} = F_{2g+d-1}$, and $\Sigma_{g,d} \simeq \bigvee S^1$ has a model without 2-cells. That is,

$$b_1^{(2)}(\tilde{\Sigma}_{g,d}) = 2g + d - 2 = d + 2(g - 1).$$

III.4. Atiyah's question and Kaplansky's conjecture

Let

$$B^{(2)} = \{b \in \mathbb{R}_{\geq 0} : \text{There exists a finite free connected } G\text{-CW complex } X \text{ with } b_n^{(2)}(G \curvearrowright X) = b \text{ for some } n \geq 0\}.$$

Question (Atiyah, 1974). Is there a $b \in B^{(2)}$ with $b \notin \mathbb{Q}$?

Theorem (Austin, 2013). The set $B^{(2)}$ is uncountable.

Theorem (Grabowski, 2014; Pichot–Schick–Žuk, 2015). We have $B^{(2)} = \mathbb{R}_{\geq 0}$.

Let $\tilde{B}^{(2)} \subseteq B^{(2)}$ be the subset obtained by additionally assuming that $\pi_1 X = \{1\}$, so that $X \rightarrow G \backslash X$ is a universal covering. Since

$$\{\text{finite connected CW complexes}\} / \simeq$$

is countable, $\tilde{B}^{(2)}$ is also countable. However, $\tilde{B}^{(2)}$ contains all $x \in \mathbb{R}_{\geq 0}$ with a computable binary expansion (that is, $\overline{\mathbb{Q}} \cap \mathbb{R}_{\geq 0} \subseteq \tilde{B}^{(2)}$, $\pi, e \in \tilde{B}^{(2)}, \dots$).

We want to characterize $B^{(2)}$ and $\tilde{B}^{(2)}$ without recourse to cell complexes.

Proposition III.4.1. Let $(C_*^{(2)}, d_*^{(2)})$ be a chain complex of Hilbert modules. Set

$$\Delta_n^{(2)} := d_n^{(2)*} d_n^{(2)} + d_{n+1}^{(2)} d_{n+1}^{(2)*}.$$

Then

$$C_n^{(2)} = \ker \Delta_n^{(2)} \oplus \overline{\operatorname{im} d_{n+1}^{(2)}} \oplus \overline{\operatorname{im} d_n^{(2)*}},$$

and the canonical map

$$\ker \Delta_n^{(2)} \xrightarrow{\cong} H_n^{(2)}(C_*^{(2)})$$

is an isomorphism (ℓ^2 -Hodge-de Rham decomposition).

Proof. We have

$$C_n^{(2)} = \ker d_n^{(2)} \oplus (\ker d_n^{(2)})^\perp = \ker d_n^{(2)} \oplus \overline{\operatorname{im} d_n^{(2)*}}.$$

Moreover, it follows from

$$\langle \Delta_n^{(2)} x, x \rangle = \|d_n^{(2)} x\|^2 + \|d_{n+1}^{(2)*} x\|^2,$$

that

$$\begin{aligned} \ker \Delta_n^{(2)} &= \ker d_n^{(2)} \cap \ker d_{n+1}^{(2)*} \\ &= \ker d_n^{(2)} \cap (\operatorname{im} d_{n+1}^{(2)})^\perp. \end{aligned}$$

Thus

$$\ker d_n^{(2)} = \ker \Delta_n^{(2)} \oplus \overline{\operatorname{im} d_{n+1}^{(2)}},$$

which also yields the second assertion. □

Thus every $b \in B^{(2)}$ is given by

$$b = \dim_{\mathcal{R}(G)} \ker(\cdot A)$$

with $A \in \operatorname{Mat}(k, \ell; \mathbb{Z}G)$. Conversely:

For a ring $\mathbb{Z} \subseteq R \subseteq \mathbb{C}$, we set

$$B_R^{(2)}(G) = \left\{ \dim_{\mathcal{R}(G)} \ker(\cdot A) : A \in \text{Mat}(k, \ell; RG) \right\}.$$

Then Proposition III.4.2 gives

$$B^{(2)} = \bigcup_{\substack{G \text{ finitely} \\ \text{generated}}} B_{\mathbb{Z}}^{(2)}(G) \quad \text{and} \quad \tilde{B}^{(2)} = \bigcup_{\substack{G \text{ finitely} \\ \text{presented}}} B_{\mathbb{Z}}^{(2)}(G).$$

Grabowski and Pichot–Schick–Žuk considered *lamplighter groups*:

$$G = \mathbb{Z}/2 \wr \mathbb{Z} := \left(\bigoplus_{\mathbb{Z}} \mathbb{Z}/2 \right) \rtimes \mathbb{Z}.$$

- Connection to Turing machines.
- Unbounded torsion.

Conjecture III.4.3 (Atiyah, Lück–Schick). Let $\mathbb{Z} \subseteq R \subseteq \mathbb{C}$ be a ring and G a group whose finite subgroups have bounded order. Let $\text{lcm}(G)$ be the least common multiple of the orders that occur. Then

$$B_R^{(2)}(G) \subseteq \frac{1}{\text{lcm}(G)} \mathbb{Z}_{\geq 0}.$$

For torsion-free G , the conjecture thus asserts that

$$B_R^{(2)}(G) \subseteq \mathbb{Z}_{\geq 0}.$$

Conjecture III.4.4 (Kaplansky). Let $\mathbb{Z} \subseteq R \subseteq \mathbb{C}$ be a ring and G a torsion-free group. Then RG has no zero divisors.

Theorem III.4.5. If G is torsion-free and satisfies III.4.3 with coefficients in R , then III.4.4 also holds for RG .

Proof. Let $a, b \in RG$ with $a \cdot b = 0$. If $a = 0$, we are done. Otherwise, for $a \in RG \subseteq \ell^2 G$, we have

$$0 \neq a \in \ker(\ell^2 G \xrightarrow{\cdot b} \ell^2 G)$$

and hence

$$0 < \dim_{\mathcal{R}(G)} \ker(\cdot b) \leq 1.$$

By III.4.3, it follows that $\dim_{\mathcal{R}(G)} \ker(\cdot b) = 1$. Since $\ell^2 G = \ker(\cdot b) \oplus \ker(\cdot b)^\perp$, we obtain $\ker(\cdot b) = \ell^2 G$; in particular, $0 = e \cdot b = b$. $\square$

Recall from Chapter I:

Theorem. Let G be torsion-free. If $b_n^{(2)}(\bar{X}) \in \mathbb{Z}_{\geq 0}$ for every regular covering

$$\bar{X} \xrightarrow{p} X$$

with a connected compact cell complex X and $\text{Deck}(p) \leq G$, then the Kaplansky conjecture holds for G and $R = \mathbb{Q}$.

Proof. Let $a, b \in \mathbb{Q}G$ such that $a \cdot b = 0$. Since

$$a = \sum_{g \in G} a_g g \quad \text{and} \quad b = \sum_{g \in G} b_g g$$

are finite linear combinations, there exists a finitely generated $H \leq G$ with $a, b \in \mathbb{Q}H$. Proposition III.4.2 gives $B_{\mathbb{Z}}^{(2)}(H) \subseteq \mathbb{Z}_{\geq 0}$. But then $B_{\mathbb{Q}}^{(2)}(H) \subseteq \mathbb{Z}_{\geq 0}$ as well, since

$$\ker(\cdot A) = \ker(\cdot cA) \quad \text{for all } A \in \text{Mat}(k, \ell; \mathbb{Q}H) \text{ and } c \in \mathbb{Z}_{>0}.$$

The assertion follows from Theorem III.4.5. □

Remark. By Homework 3.5.1, the Kaplansky conjectures on nilpotent and idempotent elements then also hold for $\mathbb{Q}G$.

Let $\mathcal{E}$ be the smallest class of groups that is closed under taking subgroups, quotients, extensions, and directed unions and such that $\mathcal{E}$ contains all finite and all abelian groups. A group $G \in \mathcal{E}$ is called *elementary amenable*.

Let $\mathcal{C}$ be the smallest class of groups that is closed under directed unions, contains all free groups, and contains all groups G in

$$1 \longrightarrow N \longrightarrow G \longrightarrow A \longrightarrow 1$$

with $N \in \mathcal{C}$ and $A \in \mathcal{E}$.

Theorem III.4.6 (Linnell, 1993). Let $G \in \mathcal{C}$ be such that the finite subgroups of G have bounded order. Then G satisfies the Atiyah conjecture (III.4.3) with coefficients in $R = \mathbb{C}$.

IV. ℓ^2 -Betti numbers of groups

IV.1. Classifying spaces for families

Definition IV.1.1. A *family of subgroups* of G is a set $\mathcal{F}$ of subgroups $H \leq G$ such that

- $H \in \mathcal{F} \implies g^{-1}Hg \in \mathcal{F}$ for all $g \in G$;
- $H, K \in \mathcal{F} \implies H \cap K \in \mathcal{F}$.

Examples.

$$\begin{aligned} \text{Triv} &= \{\{1\}\}, \\ \text{All} &= \{H \leq G\}, \\ \text{Fin} &= \{H \leq G : |H| < \infty\}, \\ \text{Vcyc} &= \{H \leq G : \text{There is a cyclic subgroup } K \leq H \text{ with } [H : K] < \infty\}. \end{aligned}$$

Recall. For a G -space X and $H \leq G$, we have

$$X^H = \{x \in X : hx = x \text{ for all } h \in H\}.$$

X is called *weakly contractible* if, for $n \geq 0$, every map $S^{n-1} \rightarrow X$ extends:

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\quad} & X \\ \downarrow & \nearrow \exists & \\ D^n & & \end{array}$$

This is equivalent to $X \neq \emptyset$, X being path-connected and simply connected, and $\pi_n(X, \cdot) = 0$ for $n \geq 2$.

Theorem IV.1.2. Let $\mathcal{F}$ be a family of subgroups of G , and let E be a G -CW complex with all stabilizers in $\mathcal{F}$. Then the following are equivalent:

- For every G -CW complex X with stabilizers in $\mathcal{F}$, there is a G -equivariant map $X \rightarrow E$ that is unique up to G -homotopy.
- For every $H \in \mathcal{F}$, the space E^H is weakly contractible.

Proof. Each $H \in \mathcal{F}$ defines a functor

$$\begin{aligned} G\text{-}\mathbf{Top} &\longrightarrow \mathbf{Top}, \\ X &\longmapsto X^H, \\ (f : X \rightarrow Y) &\longmapsto (f|_{X^H} : X^H \rightarrow Y^H). \end{aligned}$$

Its left adjoint functor is

$$\begin{aligned} \mathbf{Top} &\longrightarrow G\text{-}\mathbf{Top}, \\ X &\longmapsto G/H \times X, \\ (f : X \rightarrow Y) &\longmapsto (\mathrm{id} \times f : G/H \times X \rightarrow G/H \times Y). \end{aligned}$$

The Hom-set adjunction is given by

$$\begin{aligned} \mathrm{Hom}_{G\text{-}\mathbf{Top}}(G/H \times X, Y) &\cong \mathrm{Hom}_{\mathbf{Top}}(X, Y^H), \\ (f : G/H \times X &\rightarrow Y) \longmapsto (x \mapsto f(H, x)), \\ ((gH, x) \mapsto gh(x)) &\longleftarrow (h : X \rightarrow Y^H). \end{aligned}$$

Now (i) $\Rightarrow$ (ii). Let $H \in \mathcal{F}$. By (i), we obtain a G -map $f : G/H \rightarrow E$. In particular,

$$hf(H) = f(hH) = f(H) \quad \text{for } h \in H,$$

that is, $f(H) \in E^H$, so $E^H \neq \emptyset$. For $n \geq 1$, every $f : S^{n-1} \rightarrow E^H$ defines a point in

$$\mathrm{Hom}_{\mathbf{Top}}(S^{n-1}, E^H) \cong \mathrm{Hom}_{G\text{-}\mathbf{Top}}(G/H \times S^{n-1}, E).$$

This is a homeomorphism in the compact-open topology. The latter space is path-connected by (i). Thus we find a path from f to a constant map $S^{n-1} \rightarrow E^H$, that is, a null-homotopy of f in E^H or, equivalently, an extension of f to $D^n \rightarrow E^H$.

(ii) $\Rightarrow$ (i). Let X be a G -CW complex with stabilizers in $\mathcal{F}$. We need to construct a G -map $f : X \rightarrow E$, unique up to G -homotopy. We proceed by induction over X^n . For

$$X^0 \cong \coprod_{i \in I_0} G/H_i$$

we have $E^{H_i} \neq \emptyset$ by (ii). Choose $x_i \in E^{H_i}$. This determines $f_0 : X^0 \rightarrow E$ by $f_0(H_i) = x_i$. Given $f'_0 : X^0 \rightarrow E$, set $x'_i := f'_0(H_i)$. Then x_i, x'_i define a map

$$h'_i : G/H_i \times S^0 \longrightarrow E.$$

By (ii), the adjoint map $S^0 \rightarrow E^{H_i}$ extends to $D^1 \rightarrow E^{H_i}$, and taking adjoints gives

$$h_i : G/H_i \times D^1 \longrightarrow E.$$

Then $h = \coprod_{i \in I_0} h_i$ is a G -homotopy $f_0 \simeq f'_0$.

Now let $f_{n-1} : X^{n-1} \rightarrow E$ be given. Consider

$$\begin{array}{ccc}
\coprod_{i \in I_n} G/H_i \times S^{n-1} & \longrightarrow & X^{n-1} \\
\downarrow & \lrcorner & \downarrow \\
\coprod_{i \in I_n} G/H_i \times D^n & \longrightarrow & X^n
\end{array}
\begin{array}{c}
\searrow f_{n-1} \\
\downarrow \text{!} \\
E
\end{array}$$

The dashed map from the disjoint union of the cells exists by (ii). Uniqueness is proved as in the case $n = 0$. $\square$

Definition IV.1.3. A G -CW complex $E_{\mathcal{F}}G$ with stabilizers in $\mathcal{F}$ that satisfies (i) and (ii) is called a *classifying space* of G for $\mathcal{F}$.

Theorem IV.1.4. For every group G and every family $\mathcal{F}$ of subgroups of G , there is a classifying space $E_{\mathcal{F}}G$, unique up to G -homotopy equivalence.

Proof. Existence: Set

$$(E_{\mathcal{F}}G)^0 := \coprod_{H \in \mathcal{F}} G/H.$$

Suppose $(E_{\mathcal{F}}G)^{n-1}$ has already been constructed. For each $H \in \mathcal{F}$, choose a map

$$S^{n-1} \longrightarrow ((E_{\mathcal{F}}G)^{n-1})^H$$

in each homotopy class and attach $G/H \times D^n$ using the adjoint map to obtain $(E_{\mathcal{F}}G)^n$.

Uniqueness: Abstract nonsense. There are maps

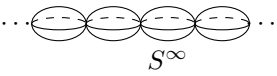
$$E_{\mathcal{F}}^1 G \xrightarrow{\varphi} E_{\mathcal{F}}^2 G, \quad E_{\mathcal{F}}^2 G \xrightarrow{\psi} E_{\mathcal{F}}^1 G, \quad \psi \circ \varphi \simeq \text{id}, \quad \varphi \circ \psi \simeq \text{id}.$$

□

Set

$$\underline{E}G := E_{\text{Fin}}G, \quad EG := E_{\text{Triv}}G, \quad BG := G \backslash EG.$$

Quiz I.

G	BG	EG	$\underline{E}G$
$\mathbb{Z}$	S^1	$\mathbb{R}$	$\mathbb{R}$
$\mathbb{Z}^k$	$(S^1)^k$	$\mathbb{R}^k$	$\mathbb{R}^k$
F_k	$\bigvee_{i=1}^k S^1$		
$\mathbb{Z}/2$	$\mathbb{RP}^\infty = \text{colim}_n \mathbb{RP}^n$	S^∞	$\bullet$
D_∞	$\mathbb{RP}^\infty \vee \mathbb{RP}^\infty$		$\mathbb{R}$

IV.2. ℓ^2 -Betti numbers of groups

Definition IV.2.1. A group G is of *type* F_∞ if there is a G -CW model for EG of finite type. It is of *type* F_∞ *with respect to* $\mathcal{F}$ if there is a G -CW model for $E_{\mathcal{F}}G$ of finite type.

Definition IV.2.2. If G is of type F_∞ (or of type F_∞ with respect to $\mathcal{F}$ with $\mathcal{F} \subseteq \text{Fin}$), we set, respectively,

$$b_n^{(2)}(G) := b_n^{(2)}(EG) \quad \text{or} \quad b_n^{(2)}(G, \mathcal{F}) := b_n^{(2)}(E_{\mathcal{F}}G).$$

Theorem IV.2.3 (without proof).

$$b_n^{(2)}(G) = b_n^{(2)}(G, \text{Fin}).$$

Thus we can set $b_n^{(2)}(G) := b_n^{(2)}(G, \text{Fin})$ if G is of type F_∞ with respect to Fin , even if G is not of type F_∞ .

Quiz II.

G	$\mathbb{Z}$	$\mathbb{Z}^k$	F_k	$\mathbb{Z}/2$	D_∞
$b_n^{(2)}(G)$	0	0	$\begin{cases} k-1 & n=1, \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} \frac{1}{2} & n=0, \\ 0 & \text{otherwise} \end{cases}$	0

Theorem IV.2.4. We consider groups of type F_∞ .

(i)

$$b_n^{(2)}(G_1 \times G_2) = \sum_{p+q=n} b_p^{(2)}(G_1) b_q^{(2)}(G_2).$$

(ii)

$$b_n^{(2)}(G_1 * G_2) = b_n^{(2)}(G_1) + b_n^{(2)}(G_2) \quad (n \geq 2).$$

If G_1 and G_2 are nontrivial, then

$$b_1^{(2)}(G_1 * G_2) = 1 + b_1^{(2)}(G_1) - \frac{1}{|G_1|} + b_1^{(2)}(G_2) - \frac{1}{|G_2|},$$

and $b_0^{(2)}(G_1 * G_2) = 0$.

(iii) For $G_0 \leq G$ with $[G : G_0] < \infty$, we have

$$b_n^{(2)}(G_0) = [G : G_0] b_n^{(2)}(G).$$

Proof. (i) $E(G_1 \times G_2) = EG_1 \times EG_2$. Apply the Künneth formula III.3.3(iii).

(ii) $B(G_1 * G_2) = BG_1 \vee BG_2$ and $E(G_1 * G_2) = \widetilde{BG_1 \vee BG_2}$. Apply Mayer–Vietoris (technical).

(iii) $EG_0 = \text{res}_{G_0}^G EG$. Apply III.3.3(iv). $\square$

Why is $b_n^{(2)}(G) := b_n^{(2)}(EG)$ or $H_n^{(2)}(G) := H_n^{(2)}(EG)$ the right definition for ℓ^2 -group homology?

Recall (Topology II).

$$H_n(G) := \text{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}).$$

Fundamental theorem of homological algebra: There is a projective resolution $P_* \rightarrow \mathbb{Z}$ of the trivial right $\mathbb{Z}G$ -module $\mathbb{Z}$, and

$$\text{Tor}_n^{\mathbb{Z}G}(\underbrace{\mathbb{Z}}_{\text{trivial right module}}, \underbrace{\mathbb{Z}}_{\text{trivial left module}}) = H_n(P_* \otimes_{\mathbb{Z}G} \mathbb{Z}).$$

If BG is a classifying space for G (that is, $\widetilde{BG} = EG \simeq \bullet$) with $BG^0 = \bullet$, then

$$\cdots \longrightarrow C_2^{\text{CW}}(EG) \longrightarrow C_1^{\text{CW}}(EG) \longrightarrow C_0^{\text{CW}}(EG) \cong \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z}$$

is a free (in particular, projective) resolution of $\mathbb{Z}$. Hence

$$\begin{aligned} H_n(G) &= H_n(C_*^{\text{CW}}(EG) \otimes_{\mathbb{Z}G} \mathbb{Z}) \\ &= H_n(C_*^{\text{CW}}(BG)) = H_n(BG), \end{aligned}$$

so

$$b_n(G) = b_n(BG) \rightsquigarrow b_n^{(2)}(G) = b_n^{(2)}(EG).$$

IV.3. Group-theoretic applications

Definition IV.3.1. A group G is called *finitely co-Hopfian* if G is not isomorphic to any of its proper subgroups of finite index.

Example. $|G| < \infty$. **Counterexample.** $G = \mathbb{Z}$.

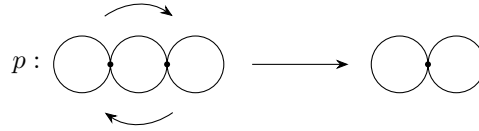
Theorem IV.3.2. Let G be of type F_∞ , and suppose $b_n^{(2)}(G) > 0$ for some $n \geq 0$. Then G is finitely co-Hopfian.

Proof. Let $H \leq G$ with $2 \leq [G : H] < \infty$ and $H \cong G$. Then

$$b_n^{(2)}(G) = b_n^{(2)}(H) = [G : H] b_n^{(2)}(G),$$

so $b_n^{(2)}(G) = 0$. □

Thus F_k is finitely co-Hopfian for $k \geq 2$. However:



This is a regular covering with $\text{Deck}(p) = \mathbb{Z}/2$, that is, $F_3 \leq F_2$ with $[F_2 : F_3] = 2$.

Remark. The same proof establishes Theorem IV.3.2 for any multiplicative group invariant, for example, $\chi(G)$. But $b_n^{(2)}(G)$ for $n \geq 0$ is finer!

Let $P = \langle S \mid R \rangle$ be a finite presentation of a finitely presented group G . Set $g(P) := |S|$ and $r(P) := |R|$.

Definition IV.3.3. The *deficiency* of a group G is

$$\text{def } G := \max_{P \text{ finite}} \{g(P) - r(P)\}.$$

For a finite presentation P of G , let X_P be the presentation complex. By attaching higher-dimensional cells, we extend X_P to a model for BG with $(BG)^2 = X_P$. The inclusion $j : X_P \rightarrow BG$ induces

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_2(X_P) & \longrightarrow & C_1(X_P) & \longrightarrow & C_0(X_P) \longrightarrow 0 \\ & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ C_3(BG) & \longrightarrow & C_2(BG) & \longrightarrow & C_1(BG) & \longrightarrow & C_0(BG) \longrightarrow 0. \end{array}$$

Thus we obtain isomorphisms

$$H_i(X_P) \xrightarrow{\cong} H_i(BG) \quad \text{for } i = 0, 1$$

and an epimorphism

$$H_2(X_P) \twoheadrightarrow H_2(BG).$$

Hence

$$\begin{aligned} g(P) - r(P) &= 1 - \chi(X_P) \\ &= 1 - b_0(X_P) + b_1(X_P) - b_2(X_P) \\ &= b_1(X_P) - b_2(X_P) \leq b_1(G) - b_2(G), \end{aligned}$$

so also $\text{def } G \leq b_1(G) - b_2(G)$.

Example.

$$\text{def}(\mathbb{Z}^n) = \frac{n(3-n)}{2} = n - \binom{n}{2}.$$

Indeed, “ $\geq$ ” holds because

$$\mathbb{Z}^n = \langle x_1, \dots, x_n \mid [x_i, x_j] \ (i < j) \rangle$$

and “ $\leq$ ” because

$$b_1(\mathbb{Z}^n) - b_2(\mathbb{Z}^n) = b_1(T^n) - b_2(T^n) = n - \binom{n}{2}.$$

Example. Let G be a *one-relator group*, that is, there is a presentation $P = \langle x_1, \dots, x_g \mid r \rangle$ with $r \neq 1 \in F_g$. If G is torsion-free, it is known that $X_P \simeq BG$. Therefore

$$\begin{aligned} g - 1 \leq \text{def } G &\leq b_1(G) - b_2(G) = -\chi(G) + 1 \\ &= -(1 - g + 1) + 1 = g - 1, \end{aligned}$$

so $\text{def } G = g - 1$.

Theorem IV.3.4.

$$\text{def } G \leq 1 - b_0^{(2)}(G) + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Proof. The same argument! □

Remark. This bound may be better or worse than $b_1(G) - b_2(G)$ (or equal to it).

Theorem IV.3.5. Let G be a nontrivial torsion-free one-relator group with g generators. Then $b_1^{(2)}(G) = g - 2$ and $b_2^{(2)}(G) = 0$.

(Already seen, specifically for surface groups.)

Proof. $BG = X_P$ has no 3-cells, so

$$H_2^{(2)}(G) = \ker(d_2^{(2)} : C_2^{(2)}(\tilde{X}_P) \longrightarrow C_1^{(2)}(\tilde{X}_P))$$

and $C_2^{(2)}(\tilde{X}_P) \cong \ell^2 G$. Since $r \neq 1 \in F_g$ by definition, we have $d_2^{(2)} \neq 0$. Therefore

$$0 \leq b_2^{(2)}(G) < 1.$$

By [Jaikin-Zapirain, López-Álvarez; 2018], G satisfies the Atiyah conjecture III.4.3 (with $R = \mathbb{C}$), and hence $b_2^{(2)}(G) = 0$. Thus

$$b_1^{(2)}(G) = -\chi(G) = -(1 - g + 1) = g - 2.$$

□

V. Lück's approximation theorem

V.1. Functional calculi

Let H be a separable Hilbert space and $T \in B(H)$. For a “function” f , we want to define “ $f(T)$.” If

$$f(z) = \sum_{k=0}^n a_k z^k \in \mathbb{C}[z],$$

we can set $f(T) = \sum_{k=0}^n a_k T^k$. If

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{C}[[z]]$$

converges in $U = \{z \in \mathbb{C} : |z| < \|T\| + \varepsilon\}$ for some $\varepsilon > 0$, then

$$\left\| \sum_{k=n+1}^m a_k T^k \right\| \leq \sum_{k=n+1}^m |a_k| \|T\|^k,$$

so $(\sum_{k=0}^n a_k T^k)_n$ is a Cauchy sequence in the Banach space $B(H)$, so the following exists:

$$f(T) = \sum_{k=0}^{\infty} a_k T^k.$$

- Recall: A power series converging in U defines a holomorphic function $f: U \rightarrow \mathbb{C}$.
- For holomorphic f , Cauchy's formula gives

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where the closed curve $\gamma: I \rightarrow \mathbb{C}$ winds once around the point z .

- Idea: Develop “operator-valued” integration and set

$$f(T) = \frac{1}{2\pi i} \sum_{\gamma \in \Gamma} \int_{\gamma} f(\zeta) (\zeta \operatorname{id}_H - T)^{-1} d\zeta,$$

where Γ is a finite set of curves such that every point in the spectrum of T is encircled once.

Definition V.1.1. The *spectrum* of $T \in B(H)$ is

$$\sigma(T) = \{\zeta \in \mathbb{C} : \zeta \operatorname{id}_H - T \text{ is not bijective}\}.$$

(Equivalently: “not invertible,” II.2.1.)

Remark.

- $\sigma(T) \subseteq \mathbb{C}$ is always nonempty and compact.
-

$$r(T) := \sup_{\zeta \in \sigma(T)} |\zeta| = \lim_{n \rightarrow \infty} \|T^n\|^{1/n} \leq \|T\|.$$

Theorem V.1.2 (Holomorphic functional calculus). Let $T \in B(H)$ and let U be an open neighborhood of $\sigma(T)$. Then there is a unique $\mathbb{C}$ -algebra homomorphism

$$\mathcal{O}(U) = \{f: U \rightarrow \mathbb{C} \text{ holomorphic}\} \longrightarrow B(H), \quad f \longmapsto f(T),$$

such that

- $\chi_U(T) = \text{id}_H$,
- $\text{id}_U(T) = T$,
- it is continuous with respect to uniform convergence on compact subsets of U .

“Proof”. Existence: See above. Uniqueness: The homomorphism is already determined on polynomials, and holomorphic functions are locally uniform limits of polynomials on compact sets (power series representation). $\square$

Remark.

- More generally, we have a holomorphic functional calculus not only for $B(H)$ but for every Banach algebra.
- We have $\sigma(f(T)) = f(\sigma(T))$.

For a self-adjoint operator $T = T^*$, we have $\sigma(T) \subseteq \mathbb{R}$ and $r(T) = \|T\|$. We can extend the domain from $\mathcal{O}(U)$ to $C(\sigma(T), \mathbb{C})$:

Theorem V.1.3 (Continuous functional calculus). Let $T \in B(H)$ be self-adjoint. Then there is a unique isometric $*$ -embedding of C^* -algebras

$$C(\sigma(T), \mathbb{C}) \longrightarrow B(H), \quad f \longmapsto f(T),$$

such that $\chi_{\sigma(T)}(T) = \text{id}_H$ and $\text{id}_{\sigma(T)}(T) = T$.

Remark.

- More generally, for unital C^* -algebras in place of $B(H)$.
- Again, $\sigma(f(T)) = f(\sigma(T))$.
- If $\lambda \in \sigma(T)$ is an eigenvalue with eigenvector $x \in H$, then x is an eigenvector of $f(T)$ with eigenvalue $f(\lambda)$.
- $f(T) = f(T)^* \iff f \in C(\sigma(T), \mathbb{R})$.
- $f(T) \geq 0 \iff f \geq 0$.

Thus, for self-adjoint $T \in B(H)$ and $x \in H$,

$$\Phi_{x,T}: C(\sigma(T), \mathbb{C}) \longrightarrow \mathbb{C}, \quad f \longmapsto \langle x, f(T)x \rangle$$

is a positive linear functional, that is, $\Phi_{x,T}(f) \geq 0$ if $f \in C(\sigma(T), \mathbb{R}_{\geq 0})$.

Theorem V.1.4 (Riesz–Markov). Let X be a compact Hausdorff space and let $\Phi: C(X, \mathbb{C}) \rightarrow \mathbb{C}$ be a positive linear functional. Then there is a unique regular Borel measure μ on X such that

$$\Phi(f) = \int f d\mu$$

for all $f \in C(X, \mathbb{C})$. We have $\mu(X) = \|\Phi\|$.

Definition V.1.5. The *spectral measure* of $T \in B(H)$, $T = T^*$, for $x \in H$ is the unique $\mu_{x,T}$ with $\Phi_{x,T} = \int \cdot d\mu_{x,T}$.

Thus

$$\langle x, f(T)x \rangle = \int f d\mu_{x,T} \quad \text{for } f \in C(\sigma(T), \mathbb{C}).$$

Observation. The right-hand side is defined for $f \in \mathcal{B}(\sigma(T), \mathbb{C})$, the bounded Borel measurable functions $f: \sigma(T) \rightarrow \mathbb{C}$.

- Obtain $\langle x, f(T)y \rangle$ for $x, y \in H$ by polarization.
- Riesz lemma: There is a unique $z \in H$ with $\langle z, y \rangle = \langle x, f(T)y \rangle$ for all $y \in H$.
- Define $z =: f(T)^*x$; this gives $f(T)^*$ and hence $f(T)$.

Theorem V.1.6 (Measurable functional calculus). Let $T \in B(H)$, $T = T^*$. Then the continuous functional calculus extends uniquely to a continuous $*$ -homomorphism

$$\mathcal{B}(\sigma(T), \mathbb{C}) \longrightarrow B(H), \quad f \longmapsto f(T).$$

Remark.

- More generally, for von Neumann algebras.
- We have $\|f(T)\| \leq \|f\|_\infty$.
- If $\lambda \in \sigma(T)$ is an eigenvalue of T with eigenvector $x \in H$ satisfying $\|x\| = 1$, then $\mu_{x,T} = \delta_\lambda$, and conversely.
- In this case, $f(T)x = f(\lambda)x$.
- There is a measure μ_T on $\sigma(T)$, unique up to equivalence, with

$$\mu_T(A) = 0 \iff \mu_{x,T}(A) = 0 \quad \text{for all } x \in H.$$

- Thus

$$L^\infty(\sigma(T), \mu_T) = \mathcal{B}(\sigma(T), \mathbb{C}) / \sim_{\mu_T\text{-almost everywhere}},$$

and $L^\infty(\sigma(T), \mu_T) \hookrightarrow B(H)$ is an ultraweakly continuous $*$ -embedding.

Observation. For measurable $A \subseteq \sigma(T)$, $\chi_A = \overline{\chi_A} = \chi_A^2$ is a projection, and hence

$$P_T(A) := \chi_A(T) \in B(H)$$

is also a projection, and P_T is a *projection-valued measure* (that is, $P_T(\sigma(T)) = \text{id}_H$, and for $x \in H$, $A \mapsto \langle x, P_T(A)x \rangle$ is a Borel measure, namely $\mu_{x,T}$). “Integration” of bounded Borel functions gives

$$\int f dP_T = f(T),$$

and for all $x \in H$ we have

$$\left\langle x, \left(\int f dP_T \right) x \right\rangle = \int f d\langle x, P_T x \rangle = \int f d\mu_{x,T}.$$

For $f = \text{id}_{\sigma(T)}$, we obtain:

Theorem V.1.7 (Spectral theorem). For self-adjoint $T \in B(H)$, we have

$$T = \int \lambda dP_T(\lambda).$$

Proposition V.1.8. If $\lambda \in \sigma(T)$ is an eigenvalue, then $P_T(\{\lambda\})$ is the orthogonal projection onto the eigenspace of λ .

V.2. The approximation theorem

Definition V.2.1. A (countable, discrete) group G is called *residually finite* if it admits a (residually finite) *chain*

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots, \quad G_i \trianglelefteq G, \quad [G : G_i] < \infty, \quad \bigcap_{i=0}^{\infty} G_i = \{1\}$$

Examples. Finitely generated abelian (even nilpotent) groups, F_n , $\pi_1(M^3)$, finitely generated $G \leq \text{GL}(n, K)$ with arbitrary char K .

Counterexamples. Baumslag–Solitar groups

$$B(m, n) = \langle a, b \mid ba^mb^{-1}a^{-n} \rangle$$

except when $n = \pm 1$, $m = \pm 1$, $n = \pm m$; Higman’s group, Tarski monsters.

Theorem V.2.2 (Lück, 1994). Let G be residually finite and let X be a free G -CW complex of finite type. Then, for every chain $(G_i)_{i=0}^{\infty}$ and every $n \geq 0$, we have

$$b_n^{(2)}(G \curvearrowright X) = \lim_{i \rightarrow \infty} \frac{b_n(G_i \backslash X)}{[G : G_i]}.$$

Remark. For simply connected X , this gives the theorem from the introduction.

Corollary V.2.3. Let G be residually finite and of type F_{∞} . Then, for every chain $(G_i)_{i=0}^{\infty}$ and every $n \geq 0$, we have

$$b_n^{(2)}(G) = \lim_{i \rightarrow \infty} \frac{b_n(G_i)}{[G : G_i]}.$$

Proof. Set $X = EG$. Then $G_i \curvearrowright EG$ is also free, and hence $G_i \backslash EG$ is a model for BG_i . $\square$

- $b_n^{(2)}(\tilde{X}) > 0$ means “free homology growth along coverings.”
- $b_n^{(2)}(G) > 0$ means “free homology growth along subgroups.”

V.3. The proof

To show:

$$b_n^{(2)}(G \curvearrowright X) = \lim_{i \rightarrow \infty} b_n^{(2)}(G/G_i \curvearrowright G_i \backslash X)$$

(compare Example I following Definition III.3.2).

Recall: $b_n^{(2)}(G \curvearrowright X) = \dim_{\mathcal{R}(G)} \ker \Delta_n^{(2)}$, and

$$\begin{array}{ccc} C_n^{(2)}(X) & \xrightarrow{\Delta_n^{(2)}} & C_n^{(2)}(X) \\ \cong \downarrow & & \downarrow \cong \\ \ell^2(G)^k & \xrightarrow{\cdot D} & \ell^2(G)^k \end{array} \quad \begin{array}{ccc} C_n^{(2)}(G_i \backslash X) & \xrightarrow{\Delta_{n,i}^{(2)}} & C_n^{(2)}(G_i \backslash X) \\ \cong \downarrow & & \downarrow \cong \\ \ell^2(G/G_i)^k & \xrightarrow{\cdot D_i} & \ell^2(G/G_i)^k \end{array}$$

for a $*$ -invariant matrix $D \in \text{Mat}(k, k; \mathbb{Z}G)$ (here we use the fact that X^{n+1} is a finite G -CW complex!), and D_i is obtained from D via $\mathbb{Z}G \rightarrow \mathbb{Z}(G/G_i)$. That is:

We need to show

$$\dim_{\mathcal{R}(G)} \ker(\cdot D) = \lim_{i \rightarrow \infty} \dim_{\mathcal{R}(G/G_i)} \ker(\cdot D_i).$$

Now let $\varepsilon_j \in \ell^2(G)^k$ be the vector with e in the j th component and 0 in all other components. By Proposition V.1.8,

$$\begin{aligned} \dim_{\mathcal{R}(G)} \ker(\cdot D) &= \text{tr}_{\mathcal{R}(G)} \text{pr}_{\ker(\cdot D)} \\ &= \text{tr}_{\mathcal{R}(G)} P_{\cdot D}(\{0\}) \\ &= \sum_{j=1}^k \langle \varepsilon_j, P_{\cdot D}(\{0\}) \varepsilon_j \rangle = \sum_{j=1}^k \mu_{\varepsilon_j, \cdot D}(\{0\}). \end{aligned}$$

Similarly,

$$\dim_{\mathcal{R}(G/G_i)} \ker(\cdot D_i) = \sum_{j=1}^k \mu_{\varepsilon_{i,j}, \cdot D_i}(\{0\})$$

for $\varepsilon_{i,j} \in \ell^2(G/G_i)^k$ with G_i in the j th component and 0 in all other components. Setting

$$\mu := \sum_{j=1}^k \mu_{\varepsilon_j, \cdot D}, \quad \mu_i := \sum_{j=1}^k \mu_{\varepsilon_{i,j}, \cdot D_i},$$

it therefore remains to show:

$$\mu(\{0\}) = \lim_{i \rightarrow \infty} \mu_i(\{0\}).$$

We have $\|\cdot D\| \leq k^2 \|D\|_1$ and also $\|\cdot D_i\| \leq k^2 \|D\|_1$. Choosing $d := \max\{1, k^2 \|D\|_1\}$, we thus have $\text{supp } \mu, \text{supp } \mu_i \subseteq [0, d]$.

Proposition V.3.1. For all $f \in C([0, d], \mathbb{R})$, we have

$$\int f d\mu = \lim_{i \rightarrow \infty} \int f d\mu_i \quad (\text{weak convergence}).$$

Proof. First let $f \in \mathbb{R}[x]$. Then

$$\begin{aligned} \int f d\mu &= \sum_{j=1}^k \langle \varepsilon_j, f(\cdot D) \varepsilon_j \rangle \\ &= \sum_{j=1}^k \langle \varepsilon_j, \underbrace{f(D)}_{\in \text{Mat}(k, k; \mathbb{R}G)} \varepsilon_j \rangle =: (*) \end{aligned}$$

and

$$\int f d\mu_i = \sum_{j=1}^k \left\langle \varepsilon_{i,j}, \underbrace{f(D_i)}_{\in \text{Mat}(k,k;\mathbb{R}(G/G_i))} \varepsilon_{i,j} \right\rangle =: (**).$$

If i is large enough that G_i contains no $g \neq e$ from the supports of the diagonal entries of $f(D)$, then $(*) = (**)$. In general, use Stone–Weierstrass. $\square$

The “Portmanteau theorem” gives

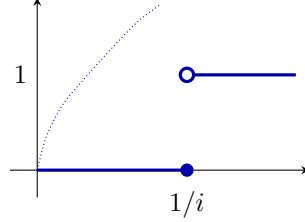
$$\mu(\{0\}) \geq \limsup_{i \rightarrow \infty} \mu_i(\{0\}).$$

It remains to show:

$$\mu(\{0\}) \leq \liminf_{i \rightarrow \infty} \mu_i(\{0\}).$$

Note: This does not follow from weak convergence $\mu_i \rightarrow \mu$! Example: $\delta_{1/i} \rightarrow \delta_0$ weakly! For example, for $k = 1$ the measure would be $\mu_i = \delta_{1/i}$ if $D_i = (\frac{1}{i} \cdot G_i)$.

But: D_i has entries in $\mathbb{Z}[G/G_i]$! This gives us a uniform bound on the *spectral distribution functions* $\lambda \mapsto \mu_i((0, \lambda))$:



Proposition V.3.2. For all $i \geq 0$ and all $\lambda \in (0, 1)$, we have

$$\mu_i((0, \lambda)) \leq \frac{k \log d}{|\log \lambda|}.$$

Proof. For a given i , let $r := k[G : G_i]$. Regard D_i as a symmetric $(r \times r)$ -matrix with coefficients in $\mathbb{Z}$. Let $\lambda_1 < \dots < \lambda_s$ be the positive eigenvalues of D_i with multiplicities $m_1, \dots, m_s$. Then

$$\mu_i(\{\lambda_j\}) = \text{tr}_{\mathcal{R}(G/G_i)} P_{D_i}(\{\lambda_j\}) = \dim_{\mathcal{R}(G/G_i)} \text{im } P_{D_i}(\{\lambda_j\}) = \frac{m_j}{[G : G_i]}.$$

Now let $\lambda \in (0, 1)$ be given and let λ_t be the largest eigenvalue $< \lambda$. Then

$$\mu_i((0, \lambda)) = \mu_i(\{\lambda_1\}) + \dots + \mu_i(\{\lambda_t\}) = \frac{m_1 + \dots + m_t}{[G : G_i]}.$$

If there is no positive eigenvalue $< \lambda$, the claim is immediate. Set

$$R := \text{rank } D_i = m_1 + \dots + m_s, \quad p(x) := \frac{\text{CharPoly}_{D_i}(x)}{x^{r-R}} \in \mathbb{Z}[x].$$

Then

$$\lambda_1^{m_1} \dots \lambda_s^{m_s} = |p(0)| \geq 1.$$

Thus

$$\begin{aligned} 1 &\leq \lambda_1^{m_1} \dots \lambda_t^{m_t} \lambda_{t+1}^{m_{t+1}} \dots \lambda_s^{m_s} \leq \lambda^{m_1 + \dots + m_t} d^r, \\ 0 &\leq (m_1 + \dots + m_t) \log \lambda + k[G : G_i] \log d, \\ \frac{m_1 + \dots + m_t}{[G : G_i]} &\leq \frac{k \log d}{-\log \lambda} = \frac{k \log d}{|\log \lambda|}. \end{aligned}$$

$\square$

From $\mu_i(\{0\}) = \mu_i([0, \lambda)) - \mu_i((0, \lambda))$ for $\lambda \in (0, 1)$, Portmanteau ($\liminf_i \mu_i(U) \geq \mu(U)$ for open U , here relative to $[0, d]$) gives

$$\liminf_{i \rightarrow \infty} \mu_i(\{0\}) \geq \mu([0, \lambda)) - \frac{k \log d}{|\log \lambda|},$$

and hence also

$$\begin{aligned} \liminf_{i \rightarrow \infty} \mu_i(\{0\}) &\geq \lim_{\lambda \rightarrow 0^+} \left(\mu([0, \lambda)) - \frac{k \log d}{|\log \lambda|} \right) \\ &= \inf_{\lambda > 0} \mu([0, \lambda)) \geq \mu(\{0\}). \end{aligned} \quad \square$$