

Semisimple Lie Algebras

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I. Foundations

I.1. Lie Algebras

Let K be a field.

Definition I.1.1. A *Lie algebra over K* is a K -vector space $\mathfrak{g}$ with a map

$$[-, -]: \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

(called the Lie bracket or commutator) satisfying:

- (L1) $[-, -]$ is bilinear.
- (L2) For all $x \in \mathfrak{g}$, we have $[x, x] = 0$.
- (L3) The Jacobi identity holds for all $x, y, z \in \mathfrak{g}$:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Remarks. • From (L1) and (L2), we obtain

$$0 = [x + y, x + y] = [x, y] + [y, x],$$

that is, (L2)': $[x, y] = -[y, x]$.

- Conversely, if $2 \neq 0$ in K , (L2)' implies

$$[x, x] = -[x, x] \iff 2[x, x] = 0 \iff [x, x] = 0,$$

that is, (L2).

- The notion of a Lie subalgebra (or simply subalgebra) is clear.
- We will (almost) always assume that $\dim_K \mathfrak{g} < \infty$.
- Homomorphisms of Lie algebras are K -linear maps $\varphi: \mathfrak{g} \rightarrow \mathfrak{g}'$ satisfying

$$\varphi([x, y]) = [\varphi(x), \varphi(y)].$$

- φ is an isomorphism $\iff \ker \varphi = \{0\}$ and $\text{im } \varphi = \mathfrak{g}'$. For $x', y' \in \mathfrak{g}'$, choose $x, y \in \mathfrak{g}$ with $\varphi(x) = x'$ and $\varphi(y) = y'$. Then

$$\begin{aligned} \varphi^{-1}([x', y']) &= \varphi^{-1}([\varphi(x), \varphi(y)]) = \varphi^{-1}(\varphi([x, y])) \\ &= [x, y] = [\varphi^{-1}(x'), \varphi^{-1}(y')]. \end{aligned}$$

I.2. Examples

- (i) Let V be a K -vector space with $\dim_K V = n < \infty$. Then

$$\mathfrak{g} = \text{End}(V)$$

becomes a Lie algebra under

$$[x, y] = xy - yx$$

where xy denotes composition: (L1) and (L2) are clear. For (L3), we have

$$\begin{aligned} [x, [y, z]] &= x[y, z] - [y, z]x \\ &= x(yz - zy) - (yz - zy)x \\ &= xyz - xzy - yzx + zyx. \end{aligned}$$

Cyclically permuting and summing gives zero. From now on, we denote this Lie algebra by $\mathfrak{gl}(V)$ to distinguish it from the associative algebra $\text{End}(V)$.

Choosing a basis of V gives $\mathfrak{gl}(V) \cong \mathfrak{gl}_n(K)$ with

$$\mathfrak{gl}_n(K) = \text{Mat}_n(K), \quad [X, Y] = XY - YX.$$

Subalgebras of $\mathfrak{gl}(V)$ or $\mathfrak{gl}_n(K)$ are called *linear Lie algebras*.

(ii) The following linear Lie algebras are called *classical Lie algebras*.

Type A_n . We call

$$\mathfrak{sl}_{n+1}(K) = \{x \in \mathfrak{gl}_{n+1}(K) : \text{tr } x = 0\}$$

the *special linear Lie algebra*. It is a subalgebra of $\mathfrak{gl}_{n+1}(K)$, since

$$\text{tr}([x, y]) = \text{tr}(xy - yx) = \text{tr}(xy) - \text{tr}(yx) = 0.$$

We have

$$\dim_K \mathfrak{sl}_{n+1}(K) = \dim_K \ker(\text{tr}: \mathfrak{gl}_{n+1}(K) \rightarrow K) = (n+1)^2 - 1 = n^2 + 2n.$$

Type D_n . We call

$$\mathfrak{so}_{2n}(K) = \{x \in \mathfrak{gl}_{2n}(K) : x^T m + mx = 0\}$$

with

$$m = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \quad I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

the (even) *orthogonal Lie algebra*. Writing

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we have

$$x^T m = \begin{pmatrix} c^T & a^T \\ d^T & b^T \end{pmatrix}, \quad mx = \begin{pmatrix} c & d \\ a & b \end{pmatrix}.$$

Thus $x \in \mathfrak{so}_{2n}(K)$ if and only if

$$b^T = -b, \quad c^T = -c, \quad a^T = -d$$

(so $d^T = -a$). Verification that this is a subalgebra: Exercise 1.1. We have

$$\dim_K \mathfrak{so}_{2n}(K) = 2 \cdot \frac{n(n-1)}{2} + n^2 = 2n^2 - n,$$

if $2 \neq 0$ in K .

Type C_n . We call

$$\mathfrak{sp}_{2n}(K) = \{x \in \mathfrak{gl}_{2n}(K) : x^T s + s x = 0\}$$

with

$$s = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

the *symplectic Lie algebra*. For

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have $x \in \mathfrak{sp}_{2n}(K)$ if and only if

$$b^T = b, \quad c^T = c, \quad a^T = -d$$

(so $d^T = -a$). We have

$$\dim_K \mathfrak{sp}_{2n}(K) = 2 \cdot \frac{n(n+1)}{2} + n^2 = 2n^2 + n.$$

Type B_n . Here we assume that $2 \neq 0$ in K . We call

$$\mathfrak{so}_{2n+1}(K) = \{x \in \mathfrak{gl}_{2n+1}(K) : x^T \ell + \ell x = 0\}$$

with

$$\ell = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$$

the (odd) *orthogonal Lie algebra*. We have

$$x = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \mathfrak{so}_{2n+1}(K)$$

if and only if

$$a = 0, \quad d = -c^T, \quad g = -b^T, \quad h^T = -h, \quad f^T = -f, \quad e^T = -i.$$

It follows that

$$\dim_K \mathfrak{so}_{2n+1}(K) = 2n + 2 \frac{n(n-1)}{2} + n^2 = 2n^2 + n,$$

if $2 \neq 0$ in K , that is,

$$\dim_K \mathfrak{so}_{2n+1}(K) = \dim_K \mathfrak{sp}_{2n}(K).$$

Observation. The relations $a^T = -d$ and $e^T = -i$, respectively, imply

$$\mathfrak{so}_{2n+1}(K) \subseteq \mathfrak{sl}_{2n+1}(K), \quad \mathfrak{sp}_{2n}(K) \subseteq \mathfrak{sl}_{2n}(K), \quad \mathfrak{so}_{2n}(K) \subseteq \mathfrak{sl}_{2n}(K).$$

(iii) We write

$$\begin{aligned} \mathfrak{t}_n(K) &= \{x \in \mathfrak{gl}_n(K) : x_{ij} = 0 \text{ for } i > j\}, \\ \mathfrak{n}_n(K) &= \{x \in \mathfrak{gl}_n(K) : x_{ij} = 0 \text{ for } i \geq j\}, \\ \mathfrak{d}_n(K) &= \{x \in \mathfrak{gl}_n(K) : x_{ij} = 0 \text{ for } i \neq j\}. \end{aligned}$$

These are the algebras of (upper) triangular matrices, strictly (upper) triangular matrices, and diagonal matrices. Clearly, $\mathfrak{d}_n(K)$ is abelian, that is, $[x, y] = 0$ for all $x, y \in \mathfrak{d}_n(K)$.

- (iv) For those familiar with Lie groups: Let G be a real or complex Lie group. Then the left-invariant vector fields on G , with the commutator bracket, form a Lie algebra

$$\mathfrak{g} = \text{Lie}(G)$$

over $\mathbb{R}$ or $\mathbb{C}$, respectively.

In particular,

$$\text{Lie}(\text{SL}_n \mathbb{C}) \cong \mathfrak{sl}_n \mathbb{C}, \quad \text{Lie}(\text{SO}_n \mathbb{C}) \cong \mathfrak{so}_n \mathbb{C}, \quad \text{Lie}(\text{Sp}_{2n} \mathbb{C}) \cong \mathfrak{sp}_{2n} \mathbb{C}.$$

I.3. Derivations

Definition I.3.1. A K -algebra is a K -vector space A with a bilinear map

$$A \times A \longrightarrow A.$$

Examples. • $\text{Mat}_{n \times n}(K)$ is a K -algebra under matrix multiplication.

- The Hamilton quaternions $\mathbb{H}$ form an $\mathbb{R}$ -algebra.
- Any Lie algebra $\mathfrak{g}$. In this case, we replace $\cdot$ by the bracket $[-, -]$.

Definition I.3.2. Let A be a K -algebra. A *derivation* of A is a K -linear map $\delta: A \rightarrow A$ satisfying the Leibniz rule

$$\delta(ab) = \delta(a)b + a\delta(b)$$

for all $a, b \in A$.

Proposition I.3.3. The set $\text{Der } A$ of derivations of A is a Lie subalgebra of $\mathfrak{gl}(A)$.

Proof. Clearly, $\text{Der } A$ is a vector subspace. Let $\delta, \delta' \in \text{Der } A$ and $a, b \in A$. Then

$$\begin{aligned} [\delta, \delta'](ab) &= \delta\delta'(ab) - \delta'\delta(ab) \\ &= \delta(\delta'(a)b + a\delta'(b)) - \delta'(\delta(a)b + a\delta(b)) \\ &= \delta\delta'(a)b + \delta'(a)\delta(b) + \delta(a)\delta'(b) + a\delta\delta'(b) \\ &\quad - \delta'\delta(a)b - \delta(a)\delta'(b) - \delta'(a)\delta(b) - a\delta'\delta(b) \\ &= [\delta, \delta'](a)b + a[\delta, \delta'](b). \end{aligned}$$

Thus $[\delta, \delta'] \in \text{Der } A$. □

Note: If $\mathfrak{g}$ is a Lie algebra, then $\text{Der } \mathfrak{g}$ is also a Lie algebra.

Proposition I.3.4. For $x \in \mathfrak{g}$, define $\text{ad}_x \in \text{End}_K(\mathfrak{g})$ by

$$\text{ad}_x(y) = [x, y]$$

for $y \in \mathfrak{g}$. Then $\text{ad}_x \in \text{Der } \mathfrak{g}$.

Proof. We have

$$\begin{aligned} \text{ad}_x([y, z]) &= [x, [y, z]] \\ &= -[y, [z, x]] - [z, [x, y]] \\ &= [[x, y], z] + [y, [x, z]] \\ &= [\text{ad}_x(y), z] + [y, \text{ad}_x(z)]. \end{aligned}$$

□

Proposition I.3.5. $\text{ad}: \mathfrak{g} \rightarrow \text{Der } \mathfrak{g}$ is a homomorphism.

Proof. For $x, y, z \in \mathfrak{g}$, we have

$$\begin{aligned} [\text{ad}_x, \text{ad}_y](z) &= \text{ad}_x \text{ad}_y(z) - \text{ad}_y \text{ad}_x(z) \\ &= [x, [y, z]] - [y, [x, z]] \\ &= -[z, [x, y]] - [y, [z, x]] - [y, [x, z]] \\ &= [[x, y], z] \\ &= \text{ad}_{[x, y]}(z). \end{aligned}$$

□

Definition I.3.6. $\text{ad}: \mathfrak{g} \rightarrow \text{Der } \mathfrak{g}$ is called the *adjoint representation* of $\mathfrak{g}$. Derivations in the image of ad are called *inner derivations*.

It is sometimes useful to write $\text{ad}_{\mathfrak{g}}$ for ad . For a diagonal matrix x , we have $\text{ad}_{\mathfrak{d}_n(K)}(x) = 0$, but in general $\text{ad}_{\mathfrak{gl}_n(K)}(x) \neq 0$.

I.4. Ideals

Definition I.4.1. Let $\mathfrak{g}$ be a Lie algebra over K . An *ideal* in $\mathfrak{g}$ is a K -vector subspace $\mathfrak{a} \subseteq \mathfrak{g}$ such that

$$[x, y] \in \mathfrak{a}$$

for all $x \in \mathfrak{g}$ and $y \in \mathfrak{a}$. Notation: $\mathfrak{a} \triangleleft \mathfrak{g}$.

Examples. • $\{0\} \triangleleft \mathfrak{g}$ and $\mathfrak{g} \triangleleft \mathfrak{g}$ (trivial).

- The center

$$Z(\mathfrak{g}) = \{z \in \mathfrak{g} : [z, x] = 0 \text{ for all } x \in \mathfrak{g}\}.$$

- The derived algebra

$$[\mathfrak{g}, \mathfrak{g}] = \text{span}_K\{[x, y] : x, y \in \mathfrak{g}\}.$$

Note: $[\mathfrak{g}, \mathfrak{g}] = 0 \iff \mathfrak{g}$ is abelian.

- $[\mathfrak{sl}_n(K), \mathfrak{sl}_n(K)] = \mathfrak{sl}_n(K)$, where we assume $n \neq 2$ if $2 = 0$ in K .

Ideals can be added and multiplied:

$$\mathfrak{a} + \mathfrak{b} = \{x + y : x \in \mathfrak{a}, y \in \mathfrak{b}\},$$

$$[\mathfrak{a}, \mathfrak{b}] = \text{span}_K\{[x, y] : x \in \mathfrak{a}, y \in \mathfrak{b}\},$$

and the results are again ideals.

Definition I.4.2. $\mathfrak{g}$ is called *simple* if $\{0\}$ and $\mathfrak{g}$ are the only ideals of $\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}] \neq 0$.

Note:

$$\mathfrak{g} \text{ simple} \implies Z(\mathfrak{g}) = 0, \quad [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}.$$

Theorem I.4.3. $\mathfrak{sl}_2(K)$ is simple if $\text{char } K \neq 2$.

Proof. Let

$$\mathfrak{sl}_2(K) = \langle x, y, h \rangle$$

with

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h.$$

That is, x, y, h is a basis of $\mathfrak{g} = \mathfrak{sl}_2(K)$ consisting of eigenvectors with eigenvalues $2, -2, 0$ for the endomorphism $\text{ad } h: \mathfrak{g} \rightarrow \mathfrak{g}$.

Now let $0 \neq \mathfrak{a} \triangleleft \mathfrak{g}$ and

$$0 \neq z = ax + by + ch \in \mathfrak{a}.$$

Then

$$\text{ad } x(\text{ad } x(z)) = \text{ad } x(bh - 2cx) = -2bx \in \mathfrak{a},$$

and

$$(\text{ad } y)^2(z) = -2ay \in \mathfrak{a}.$$

If $a \neq 0$ or $b \neq 0$, then $y \in \mathfrak{a}$ or $x \in \mathfrak{a}$, since $\text{char } K \neq 2$, and hence $\mathfrak{a} = \mathfrak{g}$. If $a = b = 0$, then $h \in \mathfrak{a}$, and again $\mathfrak{a} = \mathfrak{g}$. $\square$

Definition I.4.4. Let $\mathfrak{a} \triangleleft \mathfrak{g}$. We define the *factor algebra* or *quotient algebra* of $\mathfrak{g}$ by $\mathfrak{a}$ to be the quotient vector space $\mathfrak{g}/\mathfrak{a}$ with bracket

$$[x + \mathfrak{a}, y + \mathfrak{a}] = [x, y] + \mathfrak{a}.$$

This is well-defined, since for $x - x', y - y' \in \mathfrak{a}$ we have

$$\begin{aligned} [x' + \mathfrak{a}, y' + \mathfrak{a}] &= [x', y'] + \mathfrak{a} \\ &= [x', y'] + [x - x', y] - [y - y', x'] + \mathfrak{a} \\ &= [x', y'] + [x, y] - [x', y] - [y, x'] + [y', x'] + \mathfrak{a} \\ &= [x, y] + \mathfrak{a}. \end{aligned}$$

Note: For a homomorphism $\varphi: \mathfrak{g} \rightarrow \mathfrak{g}'$, we have $\ker \varphi \triangleleft \mathfrak{g}$. Conversely, if $\mathfrak{a} \triangleleft \mathfrak{g}$, then $\mathfrak{a} = \ker(\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{a})$.

Universal property: Let $\mathfrak{a} \triangleleft \mathfrak{g}$. For every $\varphi: \mathfrak{g} \rightarrow \mathfrak{g}'$ with $\mathfrak{a} \subseteq \ker \varphi$, there is a unique homomorphism $\bar{\varphi}: \mathfrak{g}/\mathfrak{a} \rightarrow \mathfrak{g}'$ satisfying $\bar{\varphi} \circ q = \varphi$:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{q} & \mathfrak{g}/\mathfrak{a} \\ & \searrow \varphi & \downarrow \bar{\varphi} \\ & & \mathfrak{g}'. \end{array}$$

Proposition I.4.5. (a) For a homomorphism $\varphi: \mathfrak{g} \rightarrow \mathfrak{g}'$, we have

$$\mathfrak{g}/\ker \varphi \xrightarrow[\sim]{\bar{\varphi}} \text{im } \varphi.$$

(b) Let $\mathfrak{a}, \mathfrak{b} \triangleleft \mathfrak{g}$ and $\mathfrak{a} \subseteq \mathfrak{b}$. Then $\mathfrak{b}/\mathfrak{a} \triangleleft \mathfrak{g}/\mathfrak{a}$ and

$$(\mathfrak{g}/\mathfrak{a})/(\mathfrak{b}/\mathfrak{a}) \cong \mathfrak{g}/\mathfrak{b}.$$

(c) Let $\mathfrak{a}, \mathfrak{b} \triangleleft \mathfrak{g}$. Then

$$(\mathfrak{a} + \mathfrak{b})/\mathfrak{a} \cong \mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b}).$$

Definition I.4.6. A *representation* of $\mathfrak{g}$ is a homomorphism

$$\mathfrak{g} \longrightarrow \mathfrak{gl}(V).$$

The main example is $\text{ad}: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$. Note that $\ker \text{ad} = Z(\mathfrak{g})$. Hence

$$\mathfrak{g} \text{ simple} \implies \mathfrak{g} \text{ linear.}$$

In fact, every finite-dimensional Lie algebra is linear (Ado–Iwasawa).

Let $A \leq \mathfrak{g}$ be a vector subspace. We also define the *normalizer*

$$N_{\mathfrak{g}}(A) = \{x \in \mathfrak{g} : [x, A] \subseteq A\}$$

and the *centralizer*

$$Z_{\mathfrak{g}}(A) = \{x \in \mathfrak{g} : [x, A] = 0\}.$$

Both are subalgebras by (L3).

I.5. Automorphisms

Let $\text{char } K = 0$. If $x \in \mathfrak{g}$ has $\text{ad } x$ nilpotent, that is, $(\text{ad } x)^N = 0$ for some $N > 0$, we define

$$\exp(\text{ad } x) = \sum_{n \geq 0} \frac{(\text{ad } x)^n}{n!}.$$

Proposition I.5.1. We have $\exp(\text{ad } x) \in \text{Aut } \mathfrak{g}$.

Proof. The identity

$$\exp(\text{ad } x)([y, z]) = [\exp(\text{ad } x)y, \exp(\text{ad } x)z]$$

follows from Proposition I.3.4 and the Leibniz rule. If we write

$$\exp(\text{ad } x) = 1 + y,$$

then

$$1 - y + y^2 - y^3 + \cdots$$

is an inverse of $1 + y$; the sum is finite because y is nilpotent. □

Definition I.5.2.

$$\text{Int } \mathfrak{g} = \langle \exp(\text{ad } x) : \text{ad } x \text{ nilpotent} \rangle_{\text{Aut } \mathfrak{g}}$$

is called the group of *inner automorphisms*.

We have

$$\text{Int } \mathfrak{g} \triangleleft \text{Aut } \mathfrak{g},$$

since for $\varphi \in \text{Aut } \mathfrak{g}$ we have

$$\varphi \text{ad } x \varphi^{-1} = \text{ad}(\varphi x),$$

hence

$$\varphi \exp(\text{ad } x) \varphi^{-1} = \exp(\text{ad}(\varphi x)).$$

Example. If $K = \bar{K}$ and $\text{char } K = 0$, then

$$\text{Int}(\mathfrak{sl}_2(K)) = \text{Aut}(\mathfrak{sl}_2(K)) = \text{PSL}_2(K),$$

$$\text{Aut}(\mathfrak{sl}_3(K)) = \text{PSL}_3(K) \rtimes \mathbb{Z}/2$$

(for now, without proof).

I.6. Solvable Lie Algebras

Let $\mathfrak{g}$ be a Lie algebra over K .

Definition I.6.1. The *derived series* of $\mathfrak{g}$ is defined by

$$D^0 \mathfrak{g} = \mathfrak{g}, \quad D^{n+1} \mathfrak{g} = [D^n \mathfrak{g}, D^n \mathfrak{g}].$$

$\mathfrak{g}$ is called *solvable* if $D^N \mathfrak{g} = 0$ for some $N > 0$.

Examples. An abelian $\mathfrak{g}$ is solvable. In contrast, a simple Lie algebra, such as $\mathfrak{sl}_2(K)$ for $\text{char } K \neq 2$, is not solvable.

Proposition I.6.2. The Lie algebra $\mathfrak{t}_n(K)$ is solvable.

Proof. We have

$$D^1 \mathfrak{t}_n(K) = [\mathfrak{t}_n(K), \mathfrak{t}_n(K)] = \mathfrak{n}_n(K)$$

(exercise). Let E_{ij} be the matrix with a one in position (i, j) and zeros everywhere else. Then

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}.$$

Thus let $i < j$, $k < \ell$, and, by (L2)', assume without loss of generality that $i \neq \ell$. Then

$$[E_{ij}, E_{kl}] = \begin{cases} E_{il}, & \text{if } j = k, \\ 0, & \text{otherwise,} \end{cases}$$

and if $j = k$, we have

$$\ell - i = (\ell - k) + (k - i) = (\ell - k) + (j - i).$$

For $r \geq 1$, it follows that

$$D^r \mathfrak{t}_n(K) = \langle E_{ij} : j - i \geq 2^{r-1} \rangle_K.$$

Thus $D^N \mathfrak{t}_n(K) = 0$ for $N \gg 0$. □

Proposition I.6.3. If $\mathfrak{g}$ is solvable, then so is every subalgebra $\mathfrak{h} \leq \mathfrak{g}$ and every quotient $\mathfrak{g}/\mathfrak{a}$ for $\mathfrak{a} \triangleleft \mathfrak{g}$. If $\mathfrak{a} \triangleleft \mathfrak{g}$ and both $\mathfrak{a}$ and $\mathfrak{g}/\mathfrak{a}$ are solvable, then $\mathfrak{g}$ is solvable as well.

Proof. We have $D^n \mathfrak{h} \subseteq D^n \mathfrak{g}$ (clear), and for the projection $\varphi: \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{a}$,

$$\varphi(D^n \mathfrak{g}) = (D^n \mathfrak{g} + \mathfrak{a})/\mathfrak{a} = D^n(\mathfrak{g}/\mathfrak{a})$$

(induction on n , exercise). The last assertion follows because $D^r(\mathfrak{g}/\mathfrak{a}) = 0$ first gives $D^r \mathfrak{g} \subseteq \mathfrak{a}$, and then $D^{r+s} \mathfrak{g} = 0$ as soon as $D^s \mathfrak{a} = 0$. □

Proposition I.6.4. If $\mathfrak{a}, \mathfrak{b} \triangleleft \mathfrak{g}$ are solvable, then $\mathfrak{a} + \mathfrak{b}$ is also solvable.

Proof. We have

$$(\mathfrak{a} + \mathfrak{b})/\mathfrak{b} \cong \mathfrak{a}/(\mathfrak{a} \cap \mathfrak{b})$$

by Proposition I.4.5(c), and this is solvable by Proposition I.6.3. Thus $\mathfrak{a} + \mathfrak{b}$ is solvable, again by Proposition I.6.3. □

Definition I.6.5. The *solvable radical* of $\mathfrak{g}$ is the unique maximal solvable ideal

$$\text{Rad } \mathfrak{g} \triangleleft \mathfrak{g}.$$

Uniqueness. Let $\mathfrak{a}, \mathfrak{b} \triangleleft \mathfrak{g}$ be maximal solvable ideals. Then $\mathfrak{a} + \mathfrak{b}$ is solvable by Proposition I.6.4, so $\mathfrak{a} + \mathfrak{b} = \mathfrak{a}$ and, symmetrically, $\mathfrak{a} + \mathfrak{b} = \mathfrak{b}$. Hence $\mathfrak{a} = \mathfrak{b}$. $\square$

Definition I.6.6. The Lie algebra $\mathfrak{g}$ is called *semisimple* if

$$\text{Rad } \mathfrak{g} = 0.$$

Example. If $\mathfrak{g}$ is simple, then $\mathfrak{g}$ is also semisimple: Since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, the algebra $\mathfrak{g}$ is not solvable, so

$$\text{Rad } \mathfrak{g} \neq \mathfrak{g}.$$

Since $\text{Rad } \mathfrak{g} \triangleleft \mathfrak{g}$, it follows that $\text{Rad } \mathfrak{g} = 0$. The zero algebra is semisimple but not simple.

For arbitrary $\mathfrak{g}$, the quotient $\mathfrak{g}/\text{Rad } \mathfrak{g}$ is semisimple (exercise).

With $\mathfrak{a} = \text{Rad } \mathfrak{g}$ and $\mathfrak{s} = \mathfrak{g}/\mathfrak{a}$, we have a short exact sequence

$$0 \longrightarrow \mathfrak{a} \xrightarrow{i} \mathfrak{g} \xrightarrow{p} \mathfrak{s} \longrightarrow 0$$

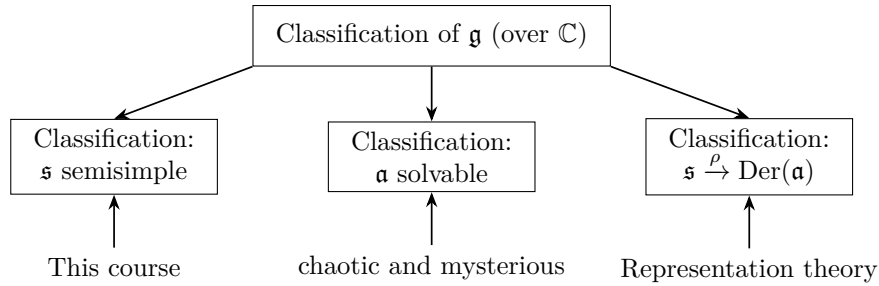
with $\mathfrak{a}$ solvable and $\mathfrak{s}$ semisimple. That is, at each term, the kernel of the following arrow equals the image of the preceding arrow. In other words, every Lie algebra is an extension of a semisimple Lie algebra by a solvable Lie algebra.

If $K = \mathbb{R}$ or $\mathbb{C}$, the extension splits, that is, there is a homomorphism $s: \mathfrak{s} \rightarrow \mathfrak{g}$ with $p \circ s = \text{id}_{\mathfrak{s}}$. In this case,

$$\mathfrak{g} \cong \mathfrak{a} \rtimes \mathfrak{s}$$

with $\rho: \mathfrak{s} \rightarrow \text{Der } \mathfrak{a}$, $\rho(x) = \text{ad}_{\mathfrak{g}}(s(x))|_{\mathfrak{a}}$, where the semidirect product $\mathfrak{a} \rtimes_{\rho} \mathfrak{s}$ of Lie algebras $\mathfrak{a}, \mathfrak{s}$ with $\rho: \mathfrak{s} \rightarrow \text{Der } \mathfrak{a}$ is defined to be the vector space $\mathfrak{a} \oplus \mathfrak{s}$ with

$$[(a_1, s_1), (a_2, s_2)] = ([a_1, a_2] + \rho(s_1)a_2 - \rho(s_2)a_1, [s_1, s_2]).$$



I.7. Nilpotent Lie Algebras

Let $\mathfrak{g}$ be a Lie algebra over K .

Definition I.7.1. The *lower central series* of $\mathfrak{g}$ is defined by

$$C^0 \mathfrak{g} = \mathfrak{g}, \quad C^n \mathfrak{g} = [\mathfrak{g}, C^{n-1} \mathfrak{g}].$$

$\mathfrak{g}$ is called *nilpotent* if $C^N \mathfrak{g} = 0$ for some $N > 0$.

Clearly,

$$\mathfrak{g} \text{ nilpotent} \implies \mathfrak{g} \text{ solvable.}$$

Example. $\mathfrak{n}_n(K)$ is nilpotent, since

$$C^r \mathfrak{n}_n(K) = \langle E_{ij} : j - i \geq r + 1 \rangle_K.$$

In contrast, $\mathfrak{t}_n(K)$ is solvable but not nilpotent for $n \geq 2$, since

$$C^r \mathfrak{t}_n(K) = \mathfrak{n}_n(K)$$

for $r \geq 1$.

Proposition I.7.2. (i) If $\mathfrak{g}$ is nilpotent, then so are $\mathfrak{h} \leq \mathfrak{g}$ and $\mathfrak{g}/\mathfrak{a}$ for $\mathfrak{a} \triangleleft \mathfrak{g}$.

(ii) If $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, then $\mathfrak{g}$ is also nilpotent.

(iii) If $\mathfrak{g}$ is nilpotent and $\mathfrak{g} \neq 0$, then $Z(\mathfrak{g}) \neq 0$.

Proof. (i) As in Proposition I.6.3. For (ii): If

$$C^N(\mathfrak{g}/Z(\mathfrak{g})) = 0,$$

then $C^N \mathfrak{g} \subseteq Z(\mathfrak{g})$, and hence

$$C^{N+1} \mathfrak{g} \subseteq [\mathfrak{g}, Z(\mathfrak{g})] = \{0\}.$$

For (iii): If N is minimal with $C^N \mathfrak{g} = 0$, then $0 \neq C^{N-1} \mathfrak{g} \subseteq Z(\mathfrak{g})$. □

Note:

$$\mathfrak{g} \text{ nilpotent} \iff \exists N > 0 : \text{ad } x_1 \text{ad } x_2 \cdots \text{ad } x_N(y) = 0$$

for all $x_1, \dots, x_N, y \in \mathfrak{g}$. In particular, $(\text{ad } x)^N = 0$, that is, all $x \in \mathfrak{g}$ are *ad-nilpotent*.

Theorem I.7.3, Engel. Let $\mathfrak{g}$ be a Lie algebra over K such that every $\text{ad } x$ is nilpotent. Then $\mathfrak{g}$ is nilpotent.

The proof requires some preparation.

Lemma I.7.4. If $x \in \mathfrak{gl}(V)$ is nilpotent, then $\text{ad } x \in \mathfrak{gl}(\mathfrak{gl}(V))$ is also nilpotent.

Proof. Consider

$$\text{ad } x = \lambda_x - \rho_x,$$

where $\lambda_x, \rho_x : \mathfrak{gl}(V) \rightarrow \mathfrak{gl}(V)$ are given by $\lambda_x(y) = xy$ and $\rho_x(y) = yx$. Then λ_x and ρ_x commute and are nilpotent, so $\lambda_x - \rho_x$ is also nilpotent. □

Warning. $\text{id}_V \in \mathfrak{gl}(V)$ is *ad-nilpotent* but not nilpotent.

Theorem I.7.5. Let $\mathfrak{g} \leq \mathfrak{gl}(V)$ with $0 < \dim_K V < \infty$. If every $x \in \mathfrak{g}$ is nilpotent, then there is a $0 \neq v \in V$ with

$$\mathfrak{g}v = 0.$$

Proof. Induction on $\dim \mathfrak{g}$. Clear for $\mathfrak{g} = 0$. Now let $\mathfrak{h} < \mathfrak{g}$ be maximal. By Lemma I.7.4, $\mathfrak{h}$ acts via ad on $\mathfrak{g}$ by nilpotent operators, and hence also on $\mathfrak{g}/\mathfrak{h}$. By the induction hypothesis, there is an $x + \mathfrak{h} \neq 0$ with

$$\text{ad}(\mathfrak{h})(x + \mathfrak{h}) = 0,$$

that is, $[\mathfrak{h}, x] \subseteq \mathfrak{h}$. Thus $\mathfrak{h} \subsetneq N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}$ by the maximality of $\mathfrak{h}$, so $\mathfrak{h} \triangleleft \mathfrak{g}$. But then $\dim_K \mathfrak{g}/\mathfrak{h} = 1$; otherwise, the preimage $p^{-1}(\mathfrak{k})$, for a one-dimensional subspace $\mathfrak{k} \leq \mathfrak{g}/\mathfrak{h}$ and the projection $p : \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$, would contradict the maximality of $\mathfrak{h}$.

Thus we have $\mathfrak{g} = \mathfrak{h} \oplus Kx$ as vector spaces. By the induction hypothesis,

$$W = \{w \in V : \mathfrak{h}w = 0\} \neq \{0\}.$$

We have $\mathfrak{g}W \subseteq W$: For $y \in \mathfrak{h}$, $z \in \mathfrak{g}$, and $w \in W$,

$$y(zw) = zyw + [y, z]w = 0.$$

Since x is nilpotent, there is a $0 \neq v \in W$ with $xv = 0$. Then $\mathfrak{h}v = 0$ as well, so $\mathfrak{g}v = 0$. $\square$

Proof of Theorem I.7.3. Induction on $\dim \mathfrak{g}$. Trivial for $\mathfrak{g} = 0$. For $\mathfrak{g} \neq 0$, the algebra $\text{ad } \mathfrak{g} \leq \mathfrak{gl}(\mathfrak{g})$ satisfies the hypotheses of Theorem I.7.5, so there is a $0 \neq x \in \mathfrak{g}$ with

$$\text{ad } \mathfrak{g}(x) = 0,$$

that is, $x \in Z(\mathfrak{g})$. Every element of $\mathfrak{g}/Z(\mathfrak{g})$ is ad-nilpotent, and

$$\dim \mathfrak{g}/Z(\mathfrak{g}) < \dim \mathfrak{g}.$$

Thus $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent by the induction hypothesis. Hence $\mathfrak{g}$ is nilpotent by Proposition I.7.2(ii). $\square$

Definition I.7.6. Let V be a K -vector space. A *complete flag* in V is a sequence of subspaces

$$0 = V_0 < V_1 < \cdots < V_n = V$$

with $\dim_K V_i = i$.

Theorem I.7.7. Let $\mathfrak{g} \leq \mathfrak{gl}(V)$ with $0 < \dim_K V < \infty$. If every $x \in \mathfrak{g}$ is nilpotent, then there is a complete flag V_i in V with

$$\mathfrak{g}V_i \subseteq V_{i-1}$$

for all $i = 1, \dots, n$. Equivalently, there is a basis of V with respect to which $\mathfrak{g} \subseteq \mathfrak{n}_n(K)$.

Proof. By Theorem I.7.5, there is a $V_1 \leq V$, $\dim V_1 = 1$, with $\mathfrak{g}V_1 = 0 = V_0$. Since $\mathfrak{g}$ acts on V/V_1 by nilpotent operators, the induction hypothesis gives a complete flag (W_i) in V/V_1 with $\mathfrak{g}W_i \subseteq W_{i-1}$. Set $V_{i+1} = p^{-1}(W_i)$ for the projection $p: V \rightarrow V/V_1$ and $i = 1, \dots, n-1$. Then, for $i = 2, \dots, n$,

$$\mathfrak{g}V_i = \mathfrak{g}p^{-1}(W_{i-1}) \subseteq p^{-1}(\mathfrak{g}W_{i-1}) \subseteq p^{-1}(W_{i-2}) = V_{i-1}.$$

$\square$

Lemma I.7.8. Let $\mathfrak{g}$ be nilpotent and $0 \neq \mathfrak{a} \triangleleft \mathfrak{g}$. Then

$$\mathfrak{a} \cap Z(\mathfrak{g}) \neq 0.$$

Proof. Apply Theorem I.7.5 to $\text{ad } \mathfrak{g}|_{\mathfrak{a}} \leq \mathfrak{gl}(\mathfrak{a})$. There is a $0 \neq x \in \mathfrak{a}$ with $[\mathfrak{g}, x] = 0$, so $x \in \mathfrak{a} \cap Z(\mathfrak{g})$. $\square$

II. Structure Theory of Semisimple Lie Algebras

From now on, let $\text{char } K = 0$ (that is, $\mathbb{Q} \subseteq K$) and $K = \bar{K}$ (for example, $K = \overline{\mathbb{Q}}$ or $K = \mathbb{C}$).

II.1. Lie's Theorem

Theorem II.1.1. Let $\mathfrak{g} \leq \mathfrak{gl}(V)$ be solvable with $0 < \dim_K V < \infty$. Then there is a common eigenvector $v \in V$ for all $x \in \mathfrak{g}$.

Proof. Induction on $\dim \mathfrak{g}$. The base case $\mathfrak{g} = \{0\}$ is clear. Since $\mathfrak{g}$ is solvable, $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \neq \{0\}$. Choose a subspace $\mathfrak{h} \leq \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ of codimension one. Then the preimage $\mathfrak{a}$ of $\mathfrak{h}$ in $\mathfrak{g}$ is an ideal $\mathfrak{a} \triangleleft \mathfrak{g}$ of codimension one. By the induction hypothesis, there is a $0 \neq v \in V$ that is an eigenvector for all $x \in \mathfrak{a}$. The eigenvalues define a linear form $\lambda \in \mathfrak{a}^* = \text{Hom}_K(\mathfrak{a}, K)$, that is,

$$xv = \lambda(x)v$$

for all $x \in \mathfrak{a}$. Let

$$W = \{w \in V : xw = \lambda(x)w \text{ for all } x \in \mathfrak{a}\} \neq 0.$$

Claim: $\mathfrak{g}W \subseteq W$. Once this is proved, write

$$\mathfrak{g} = \mathfrak{a} \oplus Kz$$

with $z \in \mathfrak{g} \setminus \mathfrak{a}$ and choose an eigenvector $v_0 \in W$ of z (which exists because $K = \bar{K}$). Then v_0 is an eigenvector for all $x \in \mathfrak{g}$.

We now prove the claim. Let $x \in \mathfrak{g}$ and $w \in W$. We must show that $xw \in W$, that is, for all $y \in \mathfrak{a}$ we must have

$$yxw = \lambda(y)xw.$$

Since

$$yxw = (xy - [x, y])w = \lambda(y)xw - \lambda([x, y])w,$$

it remains to show that $\lambda([x, y]) = 0$.

There is nothing to prove for $w = 0$. Thus let $0 \neq w \in W$ and let n be maximal such that

$$w, xw, \dots, x^{n-1}w$$

are linearly independent. Set

$$W_i = \langle w, xw, \dots, x^{i-1}w \rangle_K.$$

Then $xW_n \subseteq W_n$ and $yW_i \subseteq W_i$ for all i . With respect to the basis $w, xw, \dots, x^{n-1}w$ of W_n , we have

$$y = \begin{pmatrix} \lambda(y) & * & \cdots & * \\ 0 & \lambda(y) & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \lambda(y) \end{pmatrix},$$

because

$$yx^i w \equiv \lambda(y)x^i w \pmod{W_i}.$$

This follows by induction: it is clear for $i = 0$, and

$$\begin{aligned} yx^i w &= yx x^{i-1} w \\ &= xyx^{i-1} w - [x, y]x^{i-1} w. \end{aligned}$$

By the induction hypothesis,

$$yx^{i-1} w = \lambda(y)x^{i-1} w + w'$$

with $w' \in W_{i-1}$ and $xw' \in W_i$. Also, $[x, y] \in \mathfrak{a}$, so $[x, y]$ acts on $x^{i-1} w$ modulo W_i by the eigenvalue $\lambda([x, y])$. In particular,

$$\mathrm{tr}_{W_n}(y) = n\lambda(y).$$

Replacing y by $[x, y] \in \mathfrak{a}$ gives

$$0 = \mathrm{tr}_{W_n}([x, y]) = n\lambda([x, y]).$$

Since $\mathrm{char} K = 0$, we have $\lambda([x, y]) = 0$. □

Theorem II.1.2, Lie. Let $\mathfrak{g} \leq \mathfrak{gl}(V)$ be solvable with $\dim_K V = n < \infty$. Then there is a complete flag (V_i) in V such that

$$\mathfrak{g}V_i \subseteq V_i.$$

Equivalently, there is a basis of V with respect to which $\mathfrak{g} \subseteq \mathfrak{t}_n(K)$.

Proof. Induction on $\dim V$ using Theorem II.1.1, as in the proof of Theorem I.7.7. □

Corollary II.1.3. Let $\mathfrak{g}$ be solvable. Then there are ideals $\mathfrak{g}_i \triangleleft \mathfrak{g}$ with

$$\{0\} = \mathfrak{g}_0 \subseteq \mathfrak{g}_1 \subseteq \cdots \subseteq \mathfrak{g}_n = \mathfrak{g}, \quad \dim \mathfrak{g}_i = i.$$

Proof. Apply Theorem II.1.2 to $\mathrm{ad}(\mathfrak{g}) \leq \mathfrak{gl}(\mathfrak{g})$; $\mathrm{ad}(\mathfrak{g})$ is solvable by Proposition I.6.3. □

Corollary II.1.4. Let $\mathfrak{g}$ be solvable. If $x \in [\mathfrak{g}, \mathfrak{g}]$, then $\mathrm{ad}_{\mathfrak{g}}(x)$ is nilpotent. In particular, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Proof. By Theorem II.1.2, we have $\mathrm{ad}(\mathfrak{g}) \subseteq \mathfrak{t}_n(K)$, so

$$\mathrm{ad}([\mathfrak{g}, \mathfrak{g}]) = [\mathrm{ad} \mathfrak{g}, \mathrm{ad} \mathfrak{g}] \subseteq \mathfrak{n}_n(K).$$

Thus $\mathrm{ad}_{\mathfrak{g}}(x)$ is nilpotent. Hence

$$\mathrm{ad}_{[\mathfrak{g}, \mathfrak{g}]}(x) = \mathrm{ad}_{\mathfrak{g}}(x)|_{[\mathfrak{g}, \mathfrak{g}]}$$

is also nilpotent. Theorem I.7.3 (Engel) implies that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. □

Thus every solvable Lie algebra is an extension of an abelian Lie algebra by a nilpotent Lie algebra:

$$0 \longrightarrow [\mathfrak{g}, \mathfrak{g}] \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \longrightarrow 0.$$

II.2. Jordan–Chevalley Decomposition

In this section only, let $\text{char } K$ be arbitrary.

Definition II.2.1. Let V be a finite-dimensional K -vector space. An element $x \in \text{End } V$ is called *semisimple* if x is diagonalizable (over $K = \bar{K}$).

Proposition II.2.2. Let $x \in \text{End } V$.

- (i) There are unique $x_s, x_n \in \text{End } V$ such that x_s is semisimple, x_n is nilpotent,

$$x = x_s + x_n$$

$$\text{and } [x_s, x_n] = 0.$$

- (ii) We have $x_s = p(x)$ and $x_n = q(x)$ for some $p, q \in K[X]$ with $p(0) = q(0) = 0$.

We call $x = x_s + x_n$ the *Jordan(–Chevalley) decomposition*.

Proof. Let

$$\prod_i (X - a_i)^{m_i}$$

be the characteristic polynomial of x . By the Chinese remainder theorem, there is a $p \in K[X]$ with

$$p \equiv a_i \pmod{(X - a_i)^{m_i}}$$

for all i and $p \equiv 0 \pmod{X}$. Set

$$q(X) = X - p(X), \quad x_s = p(x), \quad x_n = q(x).$$

Then $x_s v = a_i v$ for $v \in \ker(x - a_i)^{m_i}$. Hence x_s is semisimple, and $x_n = x - x_s$ is nilpotent, as can be checked in a Jordan basis.

If $x = x'_s + x'_n$ is another Jordan decomposition, then

$$x_s - x'_s = x'_n - x_n.$$

Since x_s and x_n are polynomials in x , they commute with x'_s and x'_n . In particular, $[x_s, x'_s] = 0 = [x_n, x'_n]$. The left-hand side is therefore semisimple and the right-hand side nilpotent. Since they are equal, both are zero. $\square$

Lemma II.2.3. Let $x \in \text{End } V$. Then

$$\text{ad } x = \text{ad}(x_s) + \text{ad}(x_n)$$

is the Jordan decomposition of $\text{ad } x \in \text{End}(\text{End } V)$.

Proof. As in Exercise 1.3, we see that if $y \in \text{End } V$ is semisimple with eigenvalues $a_1, \dots, a_n$, then $\text{ad } y$ has eigenvalues $a_i - a_j$ on the matrix units E_{ij} . Thus $\text{ad } y$ is semisimple. Hence $\text{ad } x_s$ is semisimple, $\text{ad } x_n$ is nilpotent by Lemma I.7.4, and

$$[\text{ad } x_s, \text{ad } x_n] = \text{ad}([x_s, x_n]) = 0.$$

The claim follows from Proposition II.2.2(i). $\square$

Lemma II.2.4. Let A be a finite-dimensional K -algebra. Then, for all $\delta \in \text{Der } A$,

$$\delta_s, \delta_n \in \text{Der } A.$$

Proof. It suffices to show that $\sigma = \delta_s \in \text{Der } A$. Let

$$A = \bigoplus_a A_a$$

be the generalized eigenspace decomposition of δ (which is the eigenspace decomposition of σ). Then

$$A_a A_b \subseteq A_{a+b},$$

because

$$(\delta - (a+b)\text{id})^n(xy) = \sum_{i=0}^n \binom{n}{i} ((\delta - a\text{id})^{n-i}x)((\delta - b\text{id})^i y).$$

For $n \gg 0$, the right-hand side vanishes. It follows that

$$\sigma(xy) = (a+b)xy = (\sigma x)y + x(\sigma y)$$

for all $x \in A_a, y \in A_b$. Hence $\sigma \in \text{Der } A$. □

II.3. Killing Form and Cartan Criteria

We seek criteria for the solvability of $\mathfrak{g}$ (equivalently, the nilpotence of $[\mathfrak{g}, \mathfrak{g}]$).

Proposition II.3.1. Let $A \subseteq B \subseteq \mathfrak{gl}(V)$ be subspaces and let

$$M = \{x \in \mathfrak{gl}(V) : [x, B] \subseteq A\}.$$

Suppose that $x \in M$ satisfies

$$\text{tr}(xy) = 0$$

for all $y \in M$. Then x is nilpotent.

Proof. Let $x = s + n$ be the Jordan decomposition (we must show that $s = 0$). Fix a Jordan basis with respect to which

$$s = \text{diag}(a_1, \dots, a_m).$$

Let

$$E = \langle a_1, \dots, a_m \rangle_{\mathbb{Q}} \subseteq K$$

(we must show that $E = 0 \iff E^* = 0$). For $f \in E^*$, let $y \in \mathfrak{gl}(V)$ have matrix representation

$$\text{diag}(f(a_1), \dots, f(a_m))$$

with respect to the Jordan basis. As observed above,

$$\text{ad } s(E_{ij}) = (a_i - a_j)E_{ij}, \quad \text{ad } y(E_{ij}) = (f(a_i) - f(a_j))E_{ij}.$$

Choose $r(X) \in K[X]$ such that

$$r(a_i - a_j) = f(a_i) - f(a_j), \quad r(0) = 0.$$

This is well-defined, since $a_i - a_j = a_k - a_\ell$ implies

$$f(a_i) - f(a_j) = f(a_k) - f(a_\ell).$$

By construction, we have

$$\text{ad } y = r(\text{ad } s).$$

By Lemma II.2.3, $\text{ad } s = (\text{ad } x)_s$. Thus Proposition II.2.2(ii) gives a $p \in K[X]$ with $p(0) = 0$ and

$$\text{ad } s = p(\text{ad } x).$$

Thus $\text{ad } y = (r \circ p)(\text{ad } x)$. Since $\text{ad } x(B) \subseteq A$ by assumption, we also have $\text{ad } y(B) \subseteq A$, that is, $y \in M$. The assumption now gives

$$0 = \text{tr}(xy) = \sum_i a_i f(a_i).$$

Since $f \in \text{Hom}_{\mathbb{Q}}(E, \mathbb{Q})$, applying f to this equation yields

$$0 = f\left(\sum_i a_i f(a_i)\right) = \sum_i f(a_i)^2.$$

Since $f(a_i) \in \mathbb{Q}$, it follows that $f(a_i) = 0$ for all i . Thus $f = 0$. Since $f \in E^*$ was arbitrary, we have $E^* = 0$, hence $E = 0$, and therefore $s = 0$. $\square$

Lemma II.3.2. For $x, y, z \in \text{End } V$, we have

$$\text{tr}([x, y]z) = \text{tr}(x[y, z]).$$

Proof.

$$\text{tr}((xy - yx)z) = \text{tr}(xyz) - \text{tr}(yxz) = \text{tr}(xyz) - \text{tr}(xzy) = \text{tr}(x(yz - zy)).$$

$\square$

Theorem II.3.3, Cartan Criterion for Solvability. Let $\mathfrak{g} \leq \mathfrak{gl}(V)$ be such that

$$\text{tr}(xy) = 0$$

for all $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$. Then $\mathfrak{g}$ is solvable.

Proof. Apply Proposition II.3.1 with $A = [\mathfrak{g}, \mathfrak{g}]$ and $B = \mathfrak{g}$ (so that $\mathfrak{g} \subseteq M$). We have $\text{tr}(xy) = 0$ for $x \in [\mathfrak{g}, \mathfrak{g}]$, $y \in \mathfrak{g}$. We need $\text{tr}(xy) = 0$ for $x \in [\mathfrak{g}, \mathfrak{g}]$, $y \in M$. To this end, let $x = [u, v]$ be a generator. Then Lemma II.3.2 gives

$$\text{tr}([u, v]y) = \text{tr}(u[v, y]) = \text{tr}([v, y]u) = 0,$$

because $y \in M$. We may now conclude that every $x \in [\mathfrak{g}, \mathfrak{g}]$ is nilpotent. Thus $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent by Theorem I.7.7, and hence $\mathfrak{g}$ is solvable. $\square$

Corollary II.3.4. Let $\mathfrak{g}$ be a Lie algebra such that

$$\text{tr}(\text{ad } x \text{ ad } y) = 0$$

for all $x \in [\mathfrak{g}, \mathfrak{g}]$, $y \in \mathfrak{g}$. Then $\mathfrak{g}$ is solvable.

Proof. $\text{ad } \mathfrak{g}$ is solvable by Theorem II.3.3, so $\mathfrak{g}$ is solvable by Proposition I.6.3. $\square$

Definition II.3.5. Let $\mathfrak{g}$ be a Lie algebra. The *Killing form* of $\mathfrak{g}$ is the symmetric bilinear form

$$B_{\mathfrak{g}}: \mathfrak{g} \times \mathfrak{g} \longrightarrow K, \quad B_{\mathfrak{g}}(x, y) = \text{tr}(\text{ad } x \text{ ad } y).$$

By Lemma II.3.2, B is associative:

$$B([x, y], z) = B(x, [y, z]).$$

Lemma II.3.6. Let $\mathfrak{a} \triangleleft \mathfrak{g}$. Then

$$B_{\mathfrak{a}} = B_{\mathfrak{g}}|_{\mathfrak{a} \times \mathfrak{a}}.$$

Proof. Observation: For $\varphi: V \rightarrow V$ with $\varphi(V) \subseteq W \subseteq V$, we have

$$\mathrm{tr} \varphi = \mathrm{tr}(\varphi|_W).$$

For $x, y \in \mathfrak{a}$, we have

$$(\mathrm{ad} x)(\mathrm{ad} y)(\mathfrak{g}) \subseteq \mathfrak{a}.$$

Thus

$$\begin{aligned} \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}} x \mathrm{ad}_{\mathfrak{g}} y) &= \mathrm{tr}((\mathrm{ad}_{\mathfrak{g}} x \mathrm{ad}_{\mathfrak{g}} y)|_{\mathfrak{a}}) \\ &= \mathrm{tr}(\mathrm{ad}_{\mathfrak{a}} x \mathrm{ad}_{\mathfrak{a}} y). \end{aligned}$$

□

Example. For $\mathfrak{g} = \mathfrak{sl}_2(K) = \langle x, h, y \rangle$, we have

$$\mathrm{ad} x = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad} h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \mathrm{ad} y = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

Thus B has matrix

$$\begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

with respect to this basis, with determinant -128 . Hence B is nondegenerate (for $\mathrm{char} K \neq 2$), that is,

$$\mathrm{Rad} B = \{x \in \mathfrak{g} : B(x, y) = 0 \text{ for all } y \in \mathfrak{g}\} = \{0\}.$$

Observation. $\mathrm{Rad} B \triangleleft \mathfrak{g}$ by the associativity of B .

Theorem II.3.7, Cartan Criterion for Semisimplicity. A Lie algebra is semisimple if and only if its Killing form is nondegenerate.

Proof. First suppose that $\mathrm{Rad} \mathfrak{g} = 0$. By definition,

$$B(x, y) = 0$$

for all $x \in \mathrm{Rad} B$, $y \in \mathfrak{g}$. In particular, Lemma II.3.6 and Corollary II.3.4 imply that $\mathrm{Rad} B \triangleleft \mathfrak{g}$ is solvable. Thus

$$\mathrm{Rad} B \subseteq \mathrm{Rad} \mathfrak{g} = 0,$$

and B is nondegenerate.

Now suppose that $\mathrm{Rad} B = 0$. It suffices to show that $\mathfrak{a} \subseteq \mathrm{Rad} B$ for every abelian ideal $\mathfrak{a} \triangleleft \mathfrak{g}$. To this end, let $x \in \mathfrak{a}$ and $y \in \mathfrak{g}$. Then we have

$$\mathfrak{g} \xrightarrow{\mathrm{ad} y} \mathfrak{g} \xrightarrow{\mathrm{ad} x} \mathfrak{a} \xrightarrow{\mathrm{ad} y} \mathfrak{a} \xrightarrow{\mathrm{ad} x} \{0\}.$$

Thus $(\mathrm{ad} x \mathrm{ad} y)^2 = 0$, that is, $\mathrm{ad} x \mathrm{ad} y$ is nilpotent, and therefore

$$\mathrm{tr}(\mathrm{ad} x \mathrm{ad} y) = 0.$$

Hence $x \in \mathrm{Rad} B$.

□

II.4. Decomposition into Simple Ideals

For $A \subseteq \mathfrak{g}$, let

$$A^\perp = \{x \in \mathfrak{g} : B(x, a) = 0 \text{ for all } a \in A\}.$$

Proposition II.4.1. Let $\mathfrak{g}$ be semisimple and $\mathfrak{a} \triangleleft \mathfrak{g}$. Then

$$\mathfrak{g} \cong \mathfrak{a} \oplus \mathfrak{a}^\perp$$

with $\mathfrak{a}^\perp \triangleleft \mathfrak{g}$, and $\mathfrak{a}, \mathfrak{a}^\perp$ are semisimple.

Proof. We have $\mathfrak{a}^\perp \triangleleft \mathfrak{g}$, since for $x \in \mathfrak{a}^\perp$, $y \in \mathfrak{g}$, $z \in \mathfrak{a}$,

$$B([x, y], z) = B(x, [y, z]) = 0.$$

Together with Corollary II.3.4, this also shows that $\mathfrak{a} \cap \mathfrak{a}^\perp$ is solvable and hence trivial. Since

$$\{0\} = \mathfrak{a} \cap \mathfrak{a}^\perp = \text{Rad}(B|_{\mathfrak{a} \times \mathfrak{a}}),$$

$\mathfrak{a}$ is semisimple by Theorem II.3.7. Hence $\mathfrak{a}^\perp$ is also semisimple. Since $B_{\mathfrak{a}}$ is nondegenerate, for each $x \in \mathfrak{g}$ there is an $a \in \mathfrak{a}$ such that

$$B(x, a') = B(a, a')$$

for all $a' \in \mathfrak{a}$. Thus $x - a \in \mathfrak{a}^\perp$. Hence $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$. □

Note: Whenever $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ with $\mathfrak{a}, \mathfrak{b} \triangleleft \mathfrak{g}$, we have, for $a \in \mathfrak{a}$, $b \in \mathfrak{b}$,

$$[a, b] \in \mathfrak{a} \cap \mathfrak{b},$$

so $[a, b] = 0$. Moreover, $\mathfrak{a} \perp \mathfrak{b}$ with respect to B , that is,

$$B_{\mathfrak{g}} = B_{\mathfrak{a}} \perp B_{\mathfrak{b}}$$

by Lemma II.3.6.

Theorem II.4.2. A Lie algebra $\mathfrak{g}$ is semisimple if and only if it decomposes uniquely into simple ideals:

$$\mathfrak{g} \cong \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_t.$$

Proof. Given such a decomposition, $B_{\mathfrak{g}}$ is an orthogonal sum of nondegenerate forms and is therefore itself nondegenerate. Thus $\mathfrak{g}$ is semisimple by Theorem II.3.7.

If $\mathfrak{g}$ is semisimple, choose a minimal $0 \neq \mathfrak{a} \triangleleft \mathfrak{g}$ and obtain

$$\mathfrak{g} \cong \mathfrak{a} \oplus \mathfrak{a}^\perp$$

as in Proposition II.4.1. For $\mathfrak{b} \triangleleft \mathfrak{a}$, we then also have $\mathfrak{b} \triangleleft \mathfrak{g}$, so $\mathfrak{a}$ is simple. Since $\mathfrak{a}^\perp$ is semisimple, it decomposes inductively into a direct sum of simple ideals, which are also ideals in $\mathfrak{g}$.

Uniqueness: If $\mathfrak{a} \triangleleft \mathfrak{g}$ is simple, then $[\mathfrak{a}, \mathfrak{g}] \triangleleft \mathfrak{a}$ implies that

$$[\mathfrak{a}, \mathfrak{g}] = \mathfrak{a},$$

because $Z(\mathfrak{a}) = 0$. Hence

$$\mathfrak{a} = [\mathfrak{a}, \mathfrak{g}] = \bigoplus_i [\mathfrak{a}, \mathfrak{g}_i],$$

that is, $\mathfrak{a} = \mathfrak{g}_i$ for some i . □

Corollary II.4.3. For semisimple $\mathfrak{g}$, the following hold:

- (i) $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.
- (ii) Every ideal $\mathfrak{a} \triangleleft \mathfrak{g}$ is a direct sum of simple ideals of $\mathfrak{g}$.

Proof. (i) follows by substituting the decomposition. For (ii): By Proposition II.4.1, $\mathfrak{a}$ is semisimple. By Theorem II.4.2, $\mathfrak{a}$ decomposes into simple ideals, which are also ideals in $\mathfrak{g}$. $\square$

II.5. Abstract Jordan Decomposition

Lemma II.5.1. Let $\mathfrak{g}$ be a Lie algebra. Then

$$\mathrm{ad} \mathfrak{g} \triangleleft \mathrm{Der} \mathfrak{g}.$$

Proof. For $x, y \in \mathfrak{g}$ and $\delta \in \mathrm{Der} \mathfrak{g}$, we have

$$[\delta, \mathrm{ad} x](y) = \delta([x, y]) - [x, \delta(y)] = [\delta x, y] = \mathrm{ad}(\delta x)(y). \quad (*)$$

$\square$

Theorem II.5.2. For semisimple $\mathfrak{g}$, we have

$$\mathrm{ad} \mathfrak{g} = \mathrm{Der} \mathfrak{g}.$$

Proof. By Theorem II.3.7, $\mathrm{ad} \mathfrak{g} \cong \mathfrak{g}$ has nondegenerate Killing form. By Lemma II.3.6 and Lemma II.5.1,

$$B_{\mathrm{ad} \mathfrak{g}} = B_{\mathrm{Der} \mathfrak{g}}|_{\mathrm{ad} \mathfrak{g} \times \mathrm{ad} \mathfrak{g}}.$$

As before, we obtain

$$\mathrm{Der} \mathfrak{g} = \mathrm{ad} \mathfrak{g} \oplus (\mathrm{ad} \mathfrak{g})^\perp, \quad \mathrm{ad} \mathfrak{g} \cap (\mathrm{ad} \mathfrak{g})^\perp = \{0\}$$

and

$$[\mathrm{ad} \mathfrak{g}, (\mathrm{ad} \mathfrak{g})^\perp] = \{0\}.$$

For $\delta \in (\mathrm{ad} \mathfrak{g})^\perp$, $(*)$ implies

$$\mathrm{ad}(\delta x) = 0$$

for all $x \in \mathfrak{g}$, so $\delta = 0$. $\square$

Thus Lemma II.2.4 implies, for semisimple $\mathfrak{g}$ and $x \in \mathfrak{g}$, that

$$\mathrm{ad} x = (\mathrm{ad} x)_s + (\mathrm{ad} x)_n$$

with $(\mathrm{ad} x)_s, (\mathrm{ad} x)_n \in \mathrm{ad} \mathfrak{g}$.

Definition II.5.3. Let $\mathfrak{g}$ be semisimple and $x \in \mathfrak{g}$. Then

$$x = x_s + x_n$$

is the unique *Jordan decomposition* of x , where $x_s, x_n \in \mathfrak{g}$ are uniquely determined by

$$\mathrm{ad}(x_s) = (\mathrm{ad} x)_s, \quad \mathrm{ad}(x_n) = (\mathrm{ad} x)_n.$$

Note:

- x_s is ad-semisimple.
- x_n is ad-nilpotent.
- $[x_s, x_n] = 0$.
- If $\mathfrak{g} \leq \mathfrak{gl}(V)$, this agrees with the Jordan decomposition as endomorphisms (proof in Section II.7).

II.6. Complete Reducibility of Representations

Let $\mathfrak{g}$ be semisimple. Recall that a representation of $\mathfrak{g}$ is a homomorphism

$$\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$$

(see Definition I.6.6). Equivalently, V is a $\mathfrak{g}$ -module, that is,

$$(ax + by)v = a(xv) + b(yv), \tag{M1}$$

$$x(av + bw) = a(xv) + b(xw), \tag{M2}$$

$$[x, y]v = x(yv) - y(xv) \tag{M3}$$

for all $x, y \in \mathfrak{g}$, $v, w \in V$, $a, b \in K$.

A homomorphism of $\mathfrak{g}$ -modules $\varphi: V \rightarrow W$ satisfies

$$\varphi(xv) = x\varphi(v).$$

The kernel and image are submodules, and $V/\ker \varphi \cong \operatorname{im} \varphi$.

V is called *irreducible* if V has exactly two invariant subspaces: $\{0\}$ and V .

V is called *completely reducible* if

$$V \cong V_1 \oplus \cdots \oplus V_t$$

with V_i irreducible. Equivalently, every submodule $W \subseteq V$ has a complement, that is,

$$V \cong W \oplus W'$$

for some submodule $W' \subseteq V$.

Theorem II.6.1, Weyl. Let $\mathfrak{g}$ be semisimple. Then every representation

$$\varphi: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$$

is completely reducible. Here $\dim_K V < \infty$ is implicit (as always).

Proof. For $\varphi = \operatorname{ad}$, this is the statement of Theorem II.4.2. For $K = \mathbb{C}$, we use Weyl's unitary trick: There is a compact, simply connected Lie group U with

$$\operatorname{Lie}(U) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g}$$

(see Example I.2(iv)). The representation $\varphi|_{\operatorname{Lie}(U)}$ induces a representation

$$\Phi: U \rightarrow \operatorname{GL}(V).$$

Choose a positive definite Hermitian form $\langle -, - \rangle_0$ on V and set

$$\langle v, w \rangle = \int_U \langle gv, gw \rangle_0 d\mu_U(g)$$

for the Haar measure μ_U on U with $\mu_U(U) = 1$. If $W \subseteq V$ is φ -invariant, it is also Φ -invariant. Thus $W^\perp$ is a Φ -invariant complement, since for $v \in W^\perp$, $g \in U$, $w \in W$ we have

$$\langle gv, w \rangle = \langle v, g^{-1}w \rangle = 0.$$

Thus $W^\perp$ is also φ -invariant. □

II.7. Preservation of the Jordan Decomposition

Theorem II.7.1. Let $\mathfrak{g} \leq \mathfrak{gl}(V)$ be semisimple and $x \in \mathfrak{g}$. Then $\mathfrak{g}$ contains the semisimple and nilpotent parts of x as an endomorphism.

Corollary II.7.2. The abstract and ordinary Jordan decompositions of $x \in \mathfrak{g} \leq \mathfrak{gl}(V)$ in a semisimple Lie algebra coincide.

Proof. Let $x = x_s + x_n$ be the ordinary Jordan decomposition. By Lemma II.2.3,

$$\mathrm{ad}_{\mathfrak{gl}(V)} x = \mathrm{ad}_{\mathfrak{gl}(V)} x_s + \mathrm{ad}_{\mathfrak{gl}(V)} x_n$$

is the Jordan decomposition of $\mathrm{ad}_{\mathfrak{gl}(V)} x$. By Theorem II.7.1, we have $x_s, x_n \in \mathfrak{g}$, and since semisimplicity and nilpotence are preserved under restriction to invariant subspaces,

$$\mathrm{ad}_{\mathfrak{g}} x = \mathrm{ad}_{\mathfrak{g}} x_s + \mathrm{ad}_{\mathfrak{g}} x_n$$

is the ordinary Jordan decomposition in $\mathfrak{gl}(\mathfrak{g})$. □

Proof of Theorem II.7.1. Let $x = x_s + x_n$ be the ordinary Jordan decomposition in $\mathfrak{gl}(V)$. We must show that $x_s, x_n \in \mathfrak{g}$.

Since $\mathrm{ad}_{\mathfrak{gl}(V)} x(\mathfrak{g}) \subseteq \mathfrak{g}$, Proposition II.2.2(ii) implies that

$$(\mathrm{ad}_{\mathfrak{gl}(V)} x)_s(\mathfrak{g}) \subseteq \mathfrak{g}, \quad (\mathrm{ad}_{\mathfrak{gl}(V)} x)_n(\mathfrak{g}) \subseteq \mathfrak{g},$$

that is, $x_s, x_n \in N_{\mathfrak{gl}(V)}(\mathfrak{g})$. For a submodule $W \subseteq V$, let

$$\mathfrak{g}_W = \{y \in \mathfrak{gl}(V) : yW \subseteq W \text{ and } \mathrm{tr}(y|_W) = 0\}.$$

For example, $\mathfrak{g}_V = \mathfrak{sl}(V)$. Since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, we have

$$\mathfrak{g} \subseteq N_{\mathfrak{gl}(V)}(\mathfrak{g}) \cap \bigcap_{W \subseteq V \text{ } \mathfrak{g}\text{-invariant}} \mathfrak{g}_W =: \mathfrak{g}'.$$

Observation: $x_s, x_n \in \mathfrak{g}'$. Indeed, x_s and x_n preserve every $\mathfrak{g}$ -invariant subspace W , and $\mathrm{tr}(x_n|_W) = 0$ and $\mathrm{tr}(x_s|_W) = \mathrm{tr}(x|_W) = 0$. Thus it suffices to show that $\mathfrak{g}' = \mathfrak{g}$.

By Theorem II.6.1,

$$\mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{g}^\perp,$$

as a $\mathfrak{g}$ -module via ad (since $\mathfrak{g} \subseteq N_{\mathfrak{gl}(V)}(\mathfrak{g}_W)$ for all W). Here $\mathfrak{g}^\perp$ denotes an invariant complement. Now let $y \in \mathfrak{g}^\perp$. Since $\mathfrak{g}' \subseteq N_{\mathfrak{gl}(V)}(\mathfrak{g})$, we have

$$[x, y] \in \mathfrak{g} \cap \mathfrak{g}^\perp = \{0\}$$

for all $x \in \mathfrak{g}$. For every irreducible submodule $W \subseteq V$, we therefore have

$$y|_W = \lambda \text{id}_W$$

by Schur's lemma. On the other hand, $\text{tr}(y|_W) = 0$ since $y \in \mathfrak{g}_W$, so $\lambda = 0$. Thus $y|_W = 0$ for every irreducible W . Theorem II.6.1 implies that $y = 0$. $\square$

Lemma II.7.3, Schur. Let $\varphi: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be irreducible and $y \in \mathfrak{gl}(V)$ with

$$[\varphi(x), y] = 0$$

for all $x \in \mathfrak{g}$. Then there is $\lambda \in K$ with

$$y = \lambda \text{id}_V.$$

Proof. Since $K = \bar{K}$, y has an eigenvector $0 \neq v_0 \in V$ with eigenvalue $\lambda \in K$. Set

$$y' = y - \lambda \text{id}_V.$$

Then $W = \ker y' \subseteq V$ is a submodule, since for $v \in W$ and $x \in \mathfrak{g}$ we also have

$$y'(xv) = y'\varphi(x)v = \varphi(x)y'v = x(y'v) = 0.$$

Since $v_0 \in W$ and V is irreducible, we obtain $W = V$. $\square$

Theorem II.7.4. Let $\varphi: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a representation with $\mathfrak{g}$ semisimple. For $x \in \mathfrak{g}$, let

$$x = x_s + x_n$$

be the (abstract) Jordan decomposition in $\mathfrak{g}$. Then

$$\varphi(x) = \varphi(x_s) + \varphi(x_n)$$

is the ordinary Jordan decomposition in $\mathfrak{gl}(V)$.

Proof. Let $\{x_i\} \subseteq \mathfrak{g}$ be a basis of eigenvectors for $\text{ad } x_s$, that is,

$$[x_s, x_i] = \lambda_i x_i.$$

Then

$$[\varphi(x_s), \varphi(x_i)] = \lambda_i \varphi(x_i).$$

That is, $\varphi(\mathfrak{g})$ is spanned by eigenvectors of $\text{ad}_{\varphi(\mathfrak{g})} \varphi(x_s)$, so $\text{ad}_{\varphi(\mathfrak{g})} \varphi(x_s)$ is semisimple. Moreover,

$$(\text{ad}_{\varphi(\mathfrak{g})} \varphi(x_n))^k(\varphi(x)) = \varphi((\text{ad}_{\mathfrak{g}} x_n)^k(x)) = 0$$

for $k \gg 0$ and all $x \in \mathfrak{g}$. Thus $\text{ad}_{\varphi(\mathfrak{g})} \varphi(x_n)$ is nilpotent. Hence

$$\varphi(x) = \varphi(x_s) + \varphi(x_n)$$

is the abstract Jordan decomposition in $\varphi(\mathfrak{g})$. By Corollary II.7.2, it is also the ordinary Jordan decomposition. $\square$

II.8. Representation Theory of $\mathfrak{sl}_2(K)$

In this section, we consider

$$\mathfrak{sl}_2(K) = \langle x, h, y \rangle_K$$

with

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h.$$

Let V be an $\mathfrak{sl}_2(K)$ -module. By Theorem II.7.4, V decomposes into eigenspaces of h :

$$V = \bigoplus_{\lambda \in K} V_\lambda, \quad V_\lambda = \{v \in V : hv = \lambda v\},$$

where all eigenvalues $\lambda \in K$ since $K = \bar{K}$. If $V_\lambda \neq \{0\}$, we call λ a *weight* of h in V , and V_λ is called a *weight space*.

Lemma II.8.1. For $v \in V_\lambda$, we have

$$xv \in V_{\lambda+2}, \quad yv \in V_{\lambda-2}.$$

Proof.

$$\begin{aligned} h(xv) &= ([h, x] + xh)v = (2x + xh)v \\ &= (2 + \lambda)xv, \end{aligned}$$

and similarly for y . □

Remark. In particular, $x, y \in \text{End } V$ are nilpotent, as we also already know from the preceding section.

For $\lambda \in K$ with $V_\lambda \neq \{0\}$ and $V_{\lambda+2} = \{0\}$, we call every $0 \neq v \in V_\lambda$ a *vector of highest weight* λ .

Now let V be irreducible and let $0 \neq v_0 \in V_\lambda$ be a highest weight vector. Set

$$v_{-1} = 0, \quad v_i = \frac{1}{i!} y^i v_0$$

for $i \geq 0$ (since $\text{char } K = 0$).

Lemma II.8.2. (a) $hv_i = (\lambda - 2i)v_i$.

(b) $yv_i = (i + 1)v_{i+1}$.

(c) $xv_i = (\lambda - i + 1)v_{i-1}$.

Proof. (a) and (b) are clear. For (c), the case $i = 0$ holds because $xv_0 = 0$. The case $i = 1$ follows directly from $xyv_0 = hv_0$. For $i \geq 2$, we compute inductively:

$$\begin{aligned} ixv_i &= xyv_{i-1} = ([x, y] + yx)v_{i-1} \\ &= hv_{i-1} + yxv_{i-1} \\ &= (\lambda - 2(i-1))v_{i-1} + (\lambda - (i-1) + 1)yv_{i-2} \\ &= (\lambda - 2i + 2)v_{i-1} + (\lambda - i + 2)(i-1)v_{i-1} \\ &= i(\lambda - i + 1)v_{i-1}. \end{aligned}$$

□

Observations. • By (a), the vectors $v_i \neq 0$ are linearly independent.

- Since $\dim_K V < \infty$, there is a minimal m with $v_m \neq 0$ and $v_{m+1} = 0$; hence also $v_{m+i} = 0$ for $i \geq 1$.
- Thus

$$0 \neq \langle v_0, v_1, \dots, v_m \rangle_K \subseteq V$$

is a $\mathfrak{g}$ -submodule, and therefore equals V since V is irreducible.

- With respect to the basis $v_0, \dots, v_m$, we have

$$h \in \mathfrak{d}_{m+1}(K), \quad x \in \mathfrak{n}_{m+1}(K), \quad y \in \mathfrak{n}_{m+1}(K)^T.$$

- For $i = m + 1$, (c) says

$$0 = (\lambda - m)v_m,$$

so

$$\lambda = m = \dim V - 1 \in \mathbb{Z}_{\geq 0}.$$

- By (a), $\dim V_\lambda = 1$ if $V_\lambda \neq \{0\}$.
- In particular, the vector of highest weight ($= \dim V - 1$) is unique up to scalars.

Theorem II.8.3. Let V be an irreducible $\mathfrak{sl}_2(K)$ -module and $m = \dim V - 1$. Then:

(a)

$$V \cong V_m \oplus V_{m-2} \oplus \dots \oplus V_{-m}$$

(a direct sum of K -vector spaces), where each V_μ is a one-dimensional weight space of h .

(b) V has a unique vector of highest weight m (up to scalars).

(c) The action of x, h, y on V is uniquely and explicitly determined by Lemma II.8.2 after a suitable choice of basis. In particular, there is exactly one irreducible $\mathfrak{sl}_2(K)$ -module in each dimension ≥ 1 (up to isomorphism).

Proof. Only existence in (c) remains. But this follows by defining the $\mathfrak{sl}_2(K)$ -module

$$V = \langle v_0, \dots, v_m \rangle$$

using Lemma II.8.2(a),(b),(c) (Exercise 5.1). □

By Theorem II.6.1 (Weyl), we also understand general $\mathfrak{sl}_2(K)$ -modules.

Corollary II.8.4. If V is an $\mathfrak{sl}_2(K)$ -module, all eigenvalues of h are integers and occur with the same multiplicity as their negatives. The number of irreducible summands in a direct sum decomposition of V is

$$\dim V_0 + \dim V_1.$$

II.9. Root Space Decomposition

In this section, $\{0\} \neq \mathfrak{g}$ is semisimple.

Definition II.9.1. A subalgebra $\mathfrak{t} \leq \mathfrak{g}$ is called *toral* if every $x \in \mathfrak{t}$ is semisimple (that is, $x = x_s$).

Observation. If every $x \in \mathfrak{g}$ were nilpotent (that is, $x = x_n$), then $\mathfrak{g}$ would be nilpotent by Theorem I.7.3 (Engel). Hence there is $x \in \mathfrak{g}$ with $x_s \neq 0$, and thus $\langle x_s \rangle_K \leq \mathfrak{g}$ is toral. Therefore there is always a nontrivial toral $\mathfrak{t} \leq \mathfrak{g}$.

Lemma II.9.2. Every toral subalgebra $\mathfrak{t} \leq \mathfrak{g}$ is abelian.

Proof. We must show that, for every $x \in \mathfrak{t}$, zero is the only eigenvalue of $\text{ad}_{\mathfrak{t}}(x)$. Suppose that

$$[x, y] = ay$$

with $0 \neq y \in \mathfrak{t}$ and $0 \neq a \in K$. Write

$$x = \sum_i \lambda_i x_i$$

in a basis of eigenvectors x_i of $\text{ad}_{\mathfrak{t}}(y)$ with eigenvalues μ_i . Then

$$-ay = \text{ad}_{\mathfrak{t}}(y)(x) = \sum_i \lambda_i \mu_i x_i,$$

that is, $\lambda_i \mu_i \neq 0$ for some i . But

$$\text{ad}_{\mathfrak{t}}(y)(-ay) = 0,$$

so $-ay$ is an eigenvector of $\text{ad}_{\mathfrak{t}}(y)$ with eigenvalue zero, a contradiction. $\square$

Definition II.9.3. A *Cartan subalgebra* is a maximal toral subalgebra $\mathfrak{h} \leq \mathfrak{g}$.

Warning. This definition is equivalent to the general definition only under our assumptions ($\mathfrak{g}$ semisimple, $\text{char } K = 0$, $K = \bar{K}$): $\mathfrak{h} \leq \mathfrak{g}$ is nilpotent and $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

Example. The diagonal matrices in $\mathfrak{sl}_n(K)$ form a Cartan subalgebra $\mathfrak{h} \leq \mathfrak{sl}_n(K)$. More generally, this holds in types A_n, B_n, C_n, D_n (Exercise 5.2).

Now let $\mathfrak{h} \leq \mathfrak{g}$ be a Cartan subalgebra. Then the family

$$\{\text{ad}_{\mathfrak{g}}(h) : h \in \mathfrak{h}\}$$

consists of commuting endomorphisms, since

$$\begin{aligned} [h_1, [h_2, x]] &= -[x, [h_1, h_2]] - [h_2, [x, h_1]] \\ &= [h_2, [h_1, x]], \end{aligned}$$

because $[h_1, h_2] = 0$ by Lemma II.9.2. Thus all $\text{ad}_{\mathfrak{g}}(h)$ for $h \in \mathfrak{h}$ are simultaneously diagonalizable:

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_{\alpha},$$

with

$$\mathfrak{g}_{\alpha} = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

Here

$$\mathfrak{g}_0 = Z_{\mathfrak{g}}(\mathfrak{h}) \supseteq \mathfrak{h}.$$

Definition II.9.4. We call $0 \neq \alpha \in \mathfrak{h}^*$ a *root* of $\mathfrak{g}$ with respect to $\mathfrak{h}$ if $\mathfrak{g}_{\alpha} \neq \{0\}$. The set

$$\Phi \subseteq \mathfrak{h}^*$$

of all roots is called the *root system* of $(\mathfrak{g}, \mathfrak{h})$. For $\alpha \in \Phi$, $\mathfrak{g}_{\alpha}$ is called the *root space* of the root α .

We thus have the *root space decomposition* or *Cartan decomposition*

$$\mathfrak{g} = Z_{\mathfrak{g}}(\mathfrak{h}) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}.$$

Clearly, $|\Phi| < \infty$.

Next goals:

- Show that $Z_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$ and that $B|_{\mathfrak{h} \times \mathfrak{h}}$ is nondegenerate.
- Find as many properties of $\Phi \subseteq \mathfrak{h}^* \cong \mathfrak{h}$ as possible.
- Define abstract root systems accordingly.
- Classify abstract root systems by Dynkin diagrams.
- Show that

$$\{\mathfrak{g} \text{ semisimple}\} / \cong \xleftrightarrow{1:1} \{\text{abstract root systems}\} / \cong.$$

Proposition II.9.5. (i) For $\alpha, \beta \in \mathfrak{h}^*$, we have

$$[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subseteq \mathfrak{g}_{\alpha+\beta}.$$

(ii) For $0 \neq \alpha \in \mathfrak{h}^*$, every $x \in \mathfrak{g}_{\alpha}$ is ad-nilpotent.

(iii) Let $\alpha, \beta \in \mathfrak{h}^*$ with $\alpha + \beta \neq 0$. Then $\mathfrak{g}_{\alpha} \perp \mathfrak{g}_{\beta}$ with respect to B .

Proof. For (i), let $x \in \mathfrak{g}_{\alpha}$, $y \in \mathfrak{g}_{\beta}$, $h \in \mathfrak{h}$. Then

$$\begin{aligned} [h, [x, y]] &= -[y, [h, x]] - [x, [y, h]] \\ &= -[y, \alpha(h)x] - [x, -\beta(h)y] \\ &= \alpha(h)[x, y] + \beta(h)[x, y] \\ &= (\alpha + \beta)(h)[x, y]. \end{aligned}$$

(ii) follows from (i). For (iii), let $h \in \mathfrak{h}$ with $(\alpha + \beta)(h) \neq 0$, $x \in \mathfrak{g}_{\alpha}$, $y \in \mathfrak{g}_{\beta}$. Then

$$\begin{aligned} B(x, y) &= \frac{1}{(\alpha + \beta)(h)} (B(\alpha(h)x, y) + B(x, \beta(h)y)) \\ &= \frac{1}{(\alpha + \beta)(h)} (B([h, x], y) + B(x, [h, y])) = 0. \end{aligned}$$

□

Proposition II.9.6. $B|_{Z_{\mathfrak{g}}(\mathfrak{h}) \times Z_{\mathfrak{g}}(\mathfrak{h})}$ is nondegenerate.

Proof. By Theorem II.3.7, B is nondegenerate. By Proposition II.9.5,

$$Z_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}_0 \perp \mathfrak{g}_{\alpha}$$

for all $0 \neq \alpha \in \mathfrak{h}^*$. Thus if $x \in Z_{\mathfrak{g}}(\mathfrak{h})$ satisfies $B(x, Z_{\mathfrak{g}}(\mathfrak{h})) = 0$, then $x = 0$. □

Lemma II.9.7. For $x, y \in \mathfrak{gl}(V)$ with y nilpotent and $[x, y] = 0$, xy is nilpotent and in particular

$$\text{tr}(xy) = 0.$$

Theorem II.9.8. Let $\mathfrak{h} \leq \mathfrak{g}$ be a Cartan subalgebra. Then

$$Z_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}.$$

Proof. Set $\mathfrak{z} := Z_{\mathfrak{g}}(\mathfrak{h})$. We proceed step by step.

- 1) $x \in \mathfrak{z} \implies x_s, x_n \in \mathfrak{z}$: We must show that $\text{ad}(x)_{s,n}(\mathfrak{h}) = \{0\}$. Since $\text{ad } x(\mathfrak{h}) = \{0\}$, Proposition II.2.2(ii) implies that

$$(\text{ad } x)_{s,n}(\mathfrak{h}) = \{0\}.$$

But $(\text{ad } x)_{s,n} = \text{ad}(x_{s,n})$.

- 2) $x \in \mathfrak{z}$ semisimple $\implies x \in \mathfrak{h}$: $\mathfrak{h} + Kx$ is toral since $[\mathfrak{h}, x] = 0$. Since $\mathfrak{h}$ is maximal toral, we obtain $x \in \mathfrak{h}$.
- 3) $B|_{\mathfrak{h} \times \mathfrak{h}}$ is nondegenerate: Let $h \in \mathfrak{h}$ with $B(h, \mathfrak{h}) = 0$. If $x \in \mathfrak{z}$ is nilpotent, then $B(x, h) = 0$ by Lemma II.9.7. But then also $B(x, h) = 0$ for all $x \in \mathfrak{z}$ by 1) and 2). Proposition II.9.6 implies that $h = 0$.
- 4) $\mathfrak{z}$ is nilpotent: If $x \in \mathfrak{z}$ is semisimple, then $x \in \mathfrak{h}$ by 2), and thus $\text{ad}_{\mathfrak{z}}(x) = 0$ is nilpotent. If $x \in \mathfrak{z}$ is nilpotent, so is $\text{ad}_{\mathfrak{z}}(x) = \text{ad}_{\mathfrak{g}}(x)|_{\mathfrak{z}}$. For arbitrary $x \in \mathfrak{z}$, we have $x = x_s + x_n$ with $x_s, x_n \in \mathfrak{z}$ by 1). Since $[x_s, x_n] = 0$, $\text{ad}_{\mathfrak{z}}(x)$ is therefore nilpotent. By Theorem I.7.3 (Engel), $\mathfrak{z}$ is nilpotent.
- 5) $\mathfrak{h} \cap [\mathfrak{z}, \mathfrak{z}] = \{0\}$: Since

$$B(\mathfrak{h}, [\mathfrak{z}, \mathfrak{z}]) = B([\mathfrak{h}, \mathfrak{z}], \mathfrak{z}) = B(\{0\}, \mathfrak{z}) = 0$$

this follows from 3).

- 6) $\mathfrak{z}$ is abelian: Since $\mathfrak{z}$ is nilpotent by 4), $[\mathfrak{z}, \mathfrak{z}] \neq \{0\}$ also implies that

$$[\mathfrak{z}, \mathfrak{z}] \cap Z(\mathfrak{z}) \neq \{0\}$$

by Lemma I.7.8. Thus let $0 \neq z \in [\mathfrak{z}, \mathfrak{z}] \cap Z(\mathfrak{z})$. By 2) and 5), z is not semisimple, that is, $0 \neq z_n \in \mathfrak{z}$ by 1). Hence $z_n \in Z(\mathfrak{z})$, since $\text{ad}(z_n) = (\text{ad } z)_n = p(\text{ad } z)$ for a polynomial $p \in K[X]$ with $p(0) = 0$ by Proposition II.2.2(ii). But then Lemma II.9.7 implies that

$$B(z_n, z') = 0$$

for all $z' \in \mathfrak{z}$, contradicting Proposition II.9.6.

- 7) $\mathfrak{z} = \mathfrak{h}$: Suppose there were $z \in \mathfrak{z} \setminus \mathfrak{h}$. By 1) and 2), we then have $0 \neq z_n \in \mathfrak{z}$. By 6) and Lemma II.9.7, we then have $B(z_n, \mathfrak{z}) = 0$, again contradicting Proposition II.9.6.

□

Corollary II.9.9. $B|_{\mathfrak{h} \times \mathfrak{h}}$ is nondegenerate.

We thus obtain an isomorphism

$$\mathfrak{h}^* \xrightarrow{\cong} \mathfrak{h}, \quad \alpha \longmapsto t_{\alpha},$$

where $t_{\alpha} \in \mathfrak{h}$ is uniquely determined by

$$B(t_{\alpha}, h) = \alpha(h)$$

for all $h \in \mathfrak{h}$. In particular, we have the finite set

$$\{t_{\alpha} : \alpha \in \Phi\} \subseteq \mathfrak{h}.$$

III. Root Systems

Let $\mathfrak{g}$ be semisimple with Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and root system $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$.

III.1. Properties of the Root Space Decomposition

Theorem III.1.1. Let $\alpha \in \Phi$. Then:

- (i) $K\Phi = \mathfrak{h}^*$.
- (ii) $-\alpha \in \Phi$.
- (iii) For $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, we have
$$[x, y] = B(x, y)t_\alpha.$$
- (iv) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = Kt_\alpha$.
- (v) $\alpha(t_\alpha) = B(t_\alpha, t_\alpha) \neq 0$.
- (vi) For $0 \neq x_\alpha \in \mathfrak{g}_\alpha$, there exists $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that

$$\langle x_\alpha, h_\alpha, y_\alpha \rangle_K \cong \mathfrak{sl}_2(K),$$

with

$$x_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y_\alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and $h_\alpha = [x_\alpha, y_\alpha]$.

(vii)

$$h_\alpha = \frac{2}{B(t_\alpha, t_\alpha)} t_\alpha, \quad h_{-\alpha} = -h_\alpha.$$

Proof. (i) Linear algebra:

$$K\Phi = \mathfrak{h}^* \iff \bigcap_{\alpha \in \Phi} \ker \alpha = \{0\}.$$

If (i) does not hold, there exists $0 \neq h \in \mathfrak{h}$ with $\alpha(h) = 0$ for all $\alpha \in \Phi$. But then

$$[h, \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha] = [h, \mathfrak{g}] = 0,$$

that is, $h \in Z(\mathfrak{g}) = \{0\}$, a contradiction.

(ii) Proposition II.9.5(iii) states that

$$B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$$

for all $-\alpha \neq \beta \in \mathfrak{h}^*$. Since $\mathfrak{g}_\alpha \neq 0$ and B is nondegenerate, Theorem II.3.7 implies that $\mathfrak{g}_{-\alpha} \neq 0$.

(iii) For all $h \in \mathfrak{h}$, we have

$$\begin{aligned} B(h, [x, y]) &= B([h, x], y) = B(\alpha(h)x, y) \\ &= B(t_\alpha, h)B(x, y) = B(B(x, y)t_\alpha, h). \end{aligned}$$

Thus

$$B(x, y)t_\alpha - [x, y] \in \text{Rad}(B|_{\mathfrak{h} \times \mathfrak{h}}) = \{0\}$$

by Corollary II.9.9.

(iv) By (iii), it remains to show that $B|_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}} \neq 0$. But we have just seen this in (ii).

(v) As above, we find $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ with $B(x, y) = 1$, so $[x, y] = t_\alpha$ by (iii). If $\alpha(t_\alpha) = 0$, then

$$[t_\alpha, x] = [t_\alpha, y] = 0.$$

Thus $\mathfrak{s} = \langle x, y, t_\alpha \rangle_K$ and $\text{ad}_{\mathfrak{g}}(\mathfrak{s}) \subseteq \mathfrak{gl}(\mathfrak{g})$ are solvable. By Theorem II.1.2 (Lie), $\text{ad}_{\mathfrak{g}}(t_\alpha)$ is nilpotent, since $t_\alpha \in [\mathfrak{s}, \mathfrak{s}]$. Since $t_\alpha \in \mathfrak{h}$, however, $\text{ad}_{\mathfrak{g}}(t_\alpha)$ is also semisimple, so $\text{ad}_{\mathfrak{g}}(t_\alpha) = 0$, that is, $t_\alpha \in Z(\mathfrak{g}) = \{0\}$, a contradiction.

(vi) For $0 \neq x_\alpha \in \mathfrak{g}_\alpha$, choose $y_\alpha \in \mathfrak{g}_{-\alpha}$ with

$$B(x_\alpha, y_\alpha) = \frac{2}{B(t_\alpha, t_\alpha)}.$$

Then (iii) gives

$$h_\alpha = [x_\alpha, y_\alpha] = B(x_\alpha, y_\alpha)t_\alpha = \frac{2}{B(t_\alpha, t_\alpha)}t_\alpha.$$

Moreover,

$$[h_\alpha, x_\alpha] = \alpha(h_\alpha)x_\alpha = 2x_\alpha, \quad [h_\alpha, y_\alpha] = -\alpha(h_\alpha)y_\alpha = -2y_\alpha.$$

(vii) $\alpha(h_\alpha) = 2$ forces the formula for h_α . The element $t_{-\alpha}$ is defined by

$$B(t_{-\alpha}, h) = -\alpha(h)$$

for all $h \in \mathfrak{h}$, so $t_{-\alpha} = -t_\alpha$. It follows that $h_{-\alpha} = -h_\alpha$. □

For $\alpha \in \Phi$, let $\mathfrak{sl}_2^\alpha \subseteq \mathfrak{g}$ now be constructed as in (vi).

We consider the representation

$$\text{ad}_{\mathfrak{g}}: \mathfrak{sl}_2^\alpha \longrightarrow \mathfrak{gl}(\mathfrak{g}).$$

First, consider the submodule

$$V = \mathfrak{h} \oplus \bigoplus_{c \in K^*} \mathfrak{g}_{c\alpha}$$

(see Proposition II.9.5(i)). Its weights are 0 and

$$(c\alpha)(h_\alpha) = 2c$$

for all $c \in K^*$ with $\mathfrak{g}_{c\alpha} \neq 0$. By Corollary II.8.4, we have $c \in \frac{1}{2}\mathbb{Z} \subseteq K$. We have $V_0 = \mathfrak{h}$ and $[\mathfrak{sl}_2^\alpha, \ker \alpha] = \{0\}$, that is,

$$\ker \alpha \subseteq \mathfrak{h} \subseteq V$$

is a trivial submodule that decomposes into $\dim \mathfrak{h} - 1$ irreducible one-dimensional submodules. Moreover, $\mathfrak{sl}_2^\alpha \subseteq V$ is an irreducible submodule with

$$\mathfrak{sl}_2^\alpha \cap \mathfrak{h} = Kh_\alpha \quad \text{and} \quad h_\alpha \notin \ker \alpha.$$

Thus $-2, 0, 2$ are the only even weights in V . In particular, $2\alpha \notin \Phi$. Since $\alpha \in \Phi$ was arbitrary, it follows that $\alpha/2 \notin \Phi$ as well, that is, 1 is not a weight of V . Corollary II.8.4 therefore gives

$$V = \ker \alpha \oplus \mathfrak{sl}_2^\alpha,$$

and hence $\dim \mathfrak{g}_\alpha = 1$, and $\pm\alpha$ are the only multiples of α in Φ .

Theorem III.1.2. (i) For $\alpha \in \Phi$, we have $\dim \mathfrak{g}_\alpha = 1$ and

$$\mathfrak{sl}_2^\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus Kh_\alpha, \quad Kh_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}].$$

For $0 \neq x_\alpha \in \mathfrak{g}_\alpha$, there exists a unique $y_\alpha \in \mathfrak{g}_{-\alpha}$ with

$$[x_\alpha, y_\alpha] = h_\alpha = \frac{2t_\alpha}{B(t_\alpha, t_\alpha)}.$$

(ii) If $\alpha \in \Phi$ and $c\alpha \in \Phi$ for $c \in K^*$, then $c = \pm 1$.

We now consider the action of $\mathfrak{sl}_2^\alpha$ on $\mathfrak{g}_\beta$ for $\beta \in \Phi$ with $\beta \neq \pm\alpha$.

To this end, let

$$V = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\beta+i\alpha}.$$

By Proposition II.9.5(i), this is again an $\mathfrak{sl}_2^\alpha$ -module with weights

$$(\beta + i\alpha)(h_\alpha) = \beta(h_\alpha) + 2i \in \mathbb{Z}$$

and one-dimensional weight spaces. Since $\dim_K(V_0 \oplus V_1) = 1$, V is irreducible (Corollary II.8.4). The highest and lowest weights are

$$\beta(h_\alpha) + 2q \quad \text{and} \quad \beta(h_\alpha) - 2r,$$

respectively, where q and r are the largest integers for which $\beta + q\alpha \in \Phi$ and $\beta - r\alpha \in \Phi$, respectively. Theorem II.8.3 then gives

$$\beta + i\alpha \in \Phi \quad (-r \leq i \leq q)$$

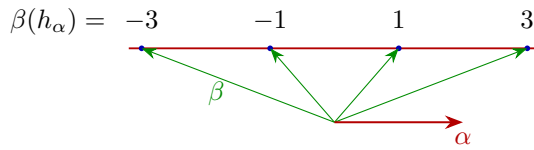
and

$$\beta(h_\alpha) + 2q = -(\beta(h_\alpha) - 2r).$$

Thus

$$\beta(h_\alpha) = r - q,$$

the “left overhang” of the α -string through β :



We call $(\beta(h_\alpha))_{\alpha, \beta \in \Phi}$ the *Cartan integers* of Φ .

Theorem III.1.3. (i) For $\alpha, \beta \in \Phi$, we have $\beta(h_\alpha) \in \mathbb{Z}$ and

$$\beta - \beta(h_\alpha)\alpha \in \Phi.$$

(ii) For $\alpha, \beta, \alpha + \beta \in \Phi$, we have

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}.$$

(iii) For $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$, let $r, q \in \mathbb{Z}_{\geq 0}$ be maximal with $\beta - r\alpha, \beta + q\alpha \in \Phi$. Then

$$\beta + i\alpha \in \Phi \quad (-r \leq i \leq q)$$

and $\beta(h_\alpha) = r - q$.

(iv) The root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \Phi$ generate $\mathfrak{g}$.

III.2. Geometry of the Root System

For $\alpha, \beta \in \mathfrak{h}^*$, set

$$(\alpha, \beta) = B(t_\alpha, t_\beta).$$

By Theorem III.1.1(i), there exists a basis $\alpha_1, \dots, \alpha_n$ of $\mathfrak{h}^*$ consisting of roots.

Proposition III.2.1. For $\beta \in \Phi$, let

$$\beta = \sum_{i=1}^n c_i \alpha_i$$

with $c_i \in K$. Then $c_i \in \mathbb{Q}$ for all i .

Proof. Applying $2(-, \alpha_j)/(\alpha_j, \alpha_j)$ gives the system of linear equations

$$\underbrace{2 \frac{(\beta, \alpha_j)}{(\alpha_j, \alpha_j)}}_{\beta(h_{\alpha_j})=:b_j \in \mathbb{Z}} = \sum_{i=1}^n \underbrace{2 \frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}}_{\alpha_i(h_{\alpha_j})=:a_{ij} \in \mathbb{Z}} c_i \iff Ac = b$$

by Theorem III.1.3(i), with

$$A = (a_{ij})^\top \in \text{Mat}_{n \times n}(\mathbb{Z}), \quad c = (c_1, \dots, c_n)^\top \in K^n, \quad b = (b_1, \dots, b_n)^\top \in \mathbb{Z}^n.$$

We have

$$\det A = \left(\prod_{j=1}^n \frac{2}{(\alpha_j, \alpha_j)} \right) \det((\alpha_i, \alpha_j)) \in \mathbb{Z} \setminus \{0\},$$

since $B|_{\mathfrak{h} \times \mathfrak{h}}$ is nondegenerate. Thus the system of linear equations has a unique solution $c \in \mathbb{Q}^n$. $\square$

For

$$E_0 = \mathbb{Q}\Phi \subseteq \mathfrak{h}^*$$

we therefore obtain $\dim_{\mathbb{Q}} E_0 = n$. Moreover, for $\lambda, \mu \in \mathfrak{h}^*$, we have

$$(\lambda, \mu) = B(t_\lambda, t_\mu) = \sum_{\alpha \in \Phi} \alpha(t_\lambda) \alpha(t_\mu) = \sum_{\alpha \in \Phi} (\alpha, \lambda)(\alpha, \mu).$$

In particular,

$$(\beta, \beta) = \sum_{\alpha \in \Phi} (\alpha, \beta)^2 \iff \frac{1}{(\beta, \beta)} = \sum_{\alpha \in \Phi} \left(\frac{(\alpha, \beta)}{(\beta, \beta)} \right)^2 \in \mathbb{Q}$$

for $\beta \in \Phi$, that is, $(\beta, \beta) \in \mathbb{Q}$, since $2(\alpha, \beta)/(\beta, \beta) \in \mathbb{Z}$. Hence (α, β) also lies in $\mathbb{Q}$. Therefore the restriction of $(-, -)$ to E_0 is a nondegenerate symmetric $\mathbb{Q}$ -bilinear form.

For $0 \neq \lambda = \sum q_i \alpha_i \in E_0$, we have

$$(\lambda, \lambda) = \text{tr}((\text{ad } t_\lambda)^2) = \sum_{\alpha \in \Phi} \alpha(t_\lambda)^2 > 0.$$

Thus $(-, -)$ is positive definite. Hence $(-, -)$ makes

$$E = \mathbb{R} \otimes_{\mathbb{Q}} E_0$$

into a Euclidean space.

Theorem III.2.2. (i) $\Phi \subseteq E \setminus \{0\}$ and $\mathbb{R}\Phi = E$.

(ii) Let $\alpha \in \Phi$ and $c \in \mathbb{R}$. If $c\alpha \in \Phi$, then $c = \pm 1$.

(iii) For $\alpha, \beta \in \Phi$, we have

$$\beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha \in \Phi.$$

(iv) For $\alpha, \beta \in \Phi$, we have $2(\beta, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$.

III.3. Abstract Root Systems

Set

$$\langle \beta, \alpha \rangle := \frac{2(\beta, \alpha)}{(\alpha, \alpha)}.$$

Observation: The map

$$\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$$

is the reflection in the hyperplane of E with normal vector α .

Definition III.3.1. Let E be a Euclidean space. A finite subset $\Phi \subseteq E$ is called an *abstract reduced root system* if:

(W1) $\Phi \subseteq E \setminus \{0\}$ and $\mathbb{R}\Phi = E$.

(W2) For $\alpha \in \Phi$, the only multiples of α in Φ are the roots $\pm\alpha$.

(W3) For $\alpha \in \Phi$, we have $\sigma_\alpha(\Phi) = \Phi$.

(W4) For $\alpha, \beta \in \Phi$, we have $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

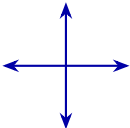
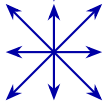
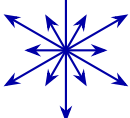
The rank of Φ is $n = \dim E$.

Definition III.3.2. The *Weyl group* of Φ is

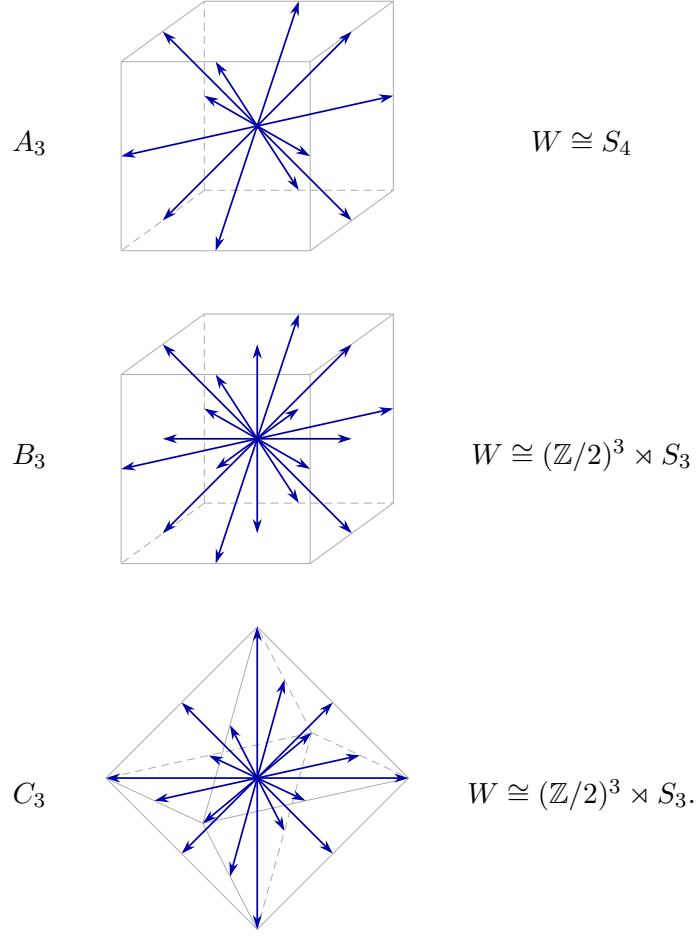
$$W(\Phi) = \langle \sigma_\alpha : \alpha \in \Phi \rangle \subseteq \text{GL}(E).$$

By (W3) and (W1), $W(\Phi)$ is finite.

Examples.

Rank	Root system	Weyl group
1: A_1		$W \cong \mathbb{Z}/2$
2: $A_1 \times A_1$		$W \cong \mathbb{Z}/2 \times \mathbb{Z}/2$
2: A_2		$W \cong S_3$
2: B_2		$W \cong (\mathbb{Z}/2)^2 \rtimes \mathbb{Z}/2$
2: G_2		$W \cong D_6.$

Examples Rank 3.



Definition III.3.3. A morphism of root systems

$$(E, \Phi) \longrightarrow (E', \Phi')$$

is an $\mathbb{R}$ -linear map $f: E \rightarrow E'$ with $f(\Phi) \subseteq \Phi'$ and

$$\langle f(\beta), f(\alpha) \rangle = \langle \beta, \alpha \rangle.$$

This also defines isomorphisms and automorphisms.

Note that

$$\sigma_{f(\alpha)}(f(\beta)) = f(\beta) - \langle f(\beta), f(\alpha) \rangle f(\alpha) = f(\sigma_\alpha(\beta)).$$

If f is an isomorphism, then

$$W(\Phi) \longrightarrow W(\Phi'), \quad \sigma \longmapsto f\sigma f^{-1}$$

is therefore an isomorphism of groups. We have an inclusion

$$W(\Phi) \subseteq \text{Aut } \Phi$$

(injectivity requires proof; exercise: $W(\Phi) \triangleleft \text{Aut } \Phi$).

We have

$$\langle \beta, \alpha \rangle = 2 \frac{\|\beta\|}{\|\alpha\|} \cos \theta, \quad \theta = \angle(\beta, \alpha),$$

so

$$\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 4 \cos^2 \theta \in \{0, 1, 2, 3, 4\}.$$

If $\beta \neq \pm \alpha$ and $\|\beta\| \geq \|\alpha\|$, only the following possibilities remain:

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	θ	$\ \beta\ /\ \alpha\ $
0	0	$\pi/2$	—
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	$\sqrt{2}$
-1	-2	$3\pi/4$	$\sqrt{2}$
1	3	$\pi/6$	$\sqrt{3}$
-1	-3	$5\pi/6$	$\sqrt{3}$

Observation. If $\alpha, \beta \in \Phi$, $\alpha \neq \beta$, and $(\alpha, \beta) > 0$, then $\alpha - \beta \in \Phi$.

Definition III.3.4. A *base* of (E, Φ) is a subset $\Delta \subseteq \Phi$ such that:

- (i) Δ is a basis of E .
- (ii) For every $\alpha \in \Phi$, we have

$$\alpha = \sum_{\beta \in \Delta} k_{\beta} \beta,$$

where either $k_{\beta} \in \mathbb{Z}_{\geq 0}$ for all $\beta \in \Delta$ or $k_{\beta} \in \mathbb{Z}_{\leq 0}$ for all $\beta \in \Delta$.

Accordingly, α is called positive or negative. Once Δ has been chosen, we call each $\alpha \in \Delta$ a *simple root*.

Observation Construction. Choose

$$\gamma \in E \setminus \bigcup_{\alpha \in \Phi} (\mathbb{R}\alpha)^{\perp}$$

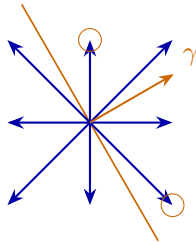
(a regular vector) and set

$$\Phi^+(\gamma) = \{\alpha \in \Phi : (\gamma, \alpha) > 0\}.$$

Then

$$\Phi = \Phi^+(\gamma) \sqcup -\Phi^+(\gamma).$$

We call $\alpha \in \Phi^+(\gamma)$ decomposable if $\alpha = \beta + \delta$ with $\beta, \delta \in \Phi^+(\gamma)$. Then the indecomposable roots in $\Phi^+(\gamma)$ form a base $\Delta = \Delta(\gamma)$, and every base arises in this way.



Observation. If $\alpha, \beta \in \Delta$ and $\alpha \neq \beta$, then $(\alpha, \beta) \leq 0$.

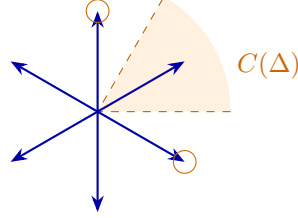
Definition III.3.5. The connected components of

$$E \setminus \bigcup_{\alpha \in \Phi} (\mathbb{R}\alpha)^{\perp}$$

are called (open) *Weyl chambers*. For a base $\Delta \subseteq \Phi$,

$$C(\Delta) = \bigcap_{\alpha \in \Delta} \{x \in E : (x, \alpha) > 0\}$$

is called the fundamental Weyl chamber of Δ . Every Weyl chamber is fundamental for exactly one base $\Delta \subseteq \Phi$.



We have

$$\Phi^+(\gamma) = \Phi^+(\gamma') \iff \Delta(\gamma) = \Delta(\gamma') \iff C(\Delta(\gamma)) = C(\Delta(\gamma')).$$

For $\sigma \in W(\Phi)$, the equality $(\sigma\alpha, \sigma\beta) = (\alpha, \beta)$ implies that

$$\sigma(\Phi^+(\gamma)) = \Phi^+(\sigma\gamma), \quad \sigma(\Delta(\gamma)) = \Delta(\sigma\gamma), \quad \sigma(C(\Delta(\gamma))) = C(\Delta(\sigma\gamma)).$$

Observations. • W acts simply transitively on the Weyl chambers, the bases, and the sets of positive roots.

- We have

$$W = \langle \sigma_\alpha : \alpha \in \Delta \rangle.$$

Definition III.3.6. We call (E, Φ) *irreducible* if

$$\Phi = \Phi_1 \sqcup \Phi_2, \quad \Phi_1 \perp \Phi_2$$

implies that $\Phi_1 = \emptyset$ or $\Phi_2 = \emptyset$.

Defining irreducibility for bases in the same way, we have

$$\Phi \text{ irreducible} \iff \Delta \text{ irreducible}.$$

Proposition III.3.7. Let Φ be irreducible. Then $W \curvearrowright E$ is irreducible.

Proof. If $0 \neq E' \subseteq E$ is W -invariant, then so is $(E')^\perp$. For $\alpha \in \Phi$, we have

$$\alpha \in E' \quad \text{or} \quad E' \subseteq (\mathbb{R}\alpha)^\perp \quad (\iff \alpha \in (E')^\perp).$$

(Exercise 7.1.) Since Φ is irreducible, we obtain $\Phi \subseteq E'$ or $\Phi \subseteq (E')^\perp$. Since $\mathbb{R}\Phi = E$ and $E' \neq \{0\}$, we obtain $\Phi \subseteq E'$ and hence $E' = E$. $\square$

Proposition III.3.8. Let Φ be irreducible. Then there are at most two distinct root lengths in Φ , and the orbits of $W \curvearrowright \Phi$ are the sets of roots of the same length.

Proof. Let $\alpha, \beta \in \Phi$ with $\|\beta\|$ maximal. By Proposition III.3.7, we have $\langle W\alpha \rangle_{\mathbb{R}} = E$, so without loss of generality, $(\alpha, \beta) \neq 0$. Then

$$\frac{\|\beta\|}{\|\alpha\|} \in \{1, \sqrt{2}, \sqrt{3}\}.$$

If all three cases occurred, there would be $\alpha_1, \alpha_2 \in \Phi$ with

$$\frac{\|\beta\|}{\|\alpha_1\|} = \sqrt{2}, \quad \frac{\|\beta\|}{\|\alpha_2\|} = \sqrt{3},$$

so

$$\frac{\|\alpha_1\|}{\|\alpha_2\|} = \sqrt{\frac{3}{2}},$$

contradicting the possible length ratios after replacing α_1 by an element of its W -orbit that is not orthogonal to α_2 .

Now let $\alpha, \beta \in \Phi$ with $\|\alpha\| = \|\beta\|$; again, without loss of generality, $(\alpha, \beta) \neq 0$. If $\beta = \pm\alpha$, both roots clearly lie in the same W -orbit. Otherwise,

$$\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = \pm 1.$$

Since $\sigma_\beta(\beta) = -\beta$, we may assume without loss of generality that $\langle \alpha, \beta \rangle = 1$. Then

$$(\sigma_\alpha \sigma_\beta \sigma_\alpha)(\beta) = \sigma_\alpha \sigma_\beta(\beta - \alpha) = \sigma_\alpha(\beta - \alpha - \beta) = \alpha.$$

□

III.4. Classification of Root Systems

Let (E, Φ) be a root system with Weyl group W and base

$$\Delta = \{\alpha_1, \dots, \alpha_n\} \subseteq \Phi.$$

Definition III.4.1. We call

$$A = (\langle \alpha_i, \alpha_j \rangle)_{i,j=1}^n \in \text{Mat}_{n \times n}(\mathbb{Z})$$

the *Cartan matrix* of Φ .

It is well defined up to simultaneous permutation of its rows and columns, since W acts transitively on the bases, and it is invertible by the argument in the proof of Proposition III.2.1.

Examples.

$$A_1 \times A_1 : \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad A_2 : \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad B_2 : \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \quad G_2 : \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

Observation. If (E', Φ') and $\Delta' = \{\alpha'_1, \dots, \alpha'_n\}$ satisfy

$$\langle \alpha'_i, \alpha'_j \rangle = \langle \alpha_i, \alpha_j \rangle,$$

then $\alpha_i \mapsto \alpha'_i$ defines an isomorphism $(E, \Phi) \cong (E', \Phi')$.

Root systems are uniquely determined by their Cartan matrices. Conversely, the root system can be recovered from a Cartan matrix (root strings).

Definition III.4.2. The *Coxeter graph* of Φ has vertex set Δ , and for $i \neq j$, the vertices α_i and α_j are joined by

$$\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle \in \{0, 1, 2, 3\}$$

edges.

$$\begin{array}{lll}
A_1 : & \circ & A_1 \times A_1 : \quad \circ \quad \circ \quad A_2 : \quad \circ \text{---} \circ \\
B_2 : & \circ \text{=}\circ & G_2 : \quad \circ \text{=}\text{=}\circ \quad A_3 : \quad \circ \text{---} \circ \text{---} \circ \\
B_3 : & \circ \text{---} \circ \text{=}\circ & C_3 : \quad \circ \text{---} \circ \text{=}\circ
\end{array}$$

To recover the Cartan matrix from the Coxeter graph, we still need to know which root is longer when $\circ \text{=}\circ$

or $\circ \text{=}\text{=}\circ$

occurs. Adding “<” or “>” accordingly gives the *Dynkin diagram* of Φ .

$$\begin{array}{ll}
B_2 : & \circ \text{>}\text{=}\circ \\
G_2 : & \circ \text{<}\text{=}\text{=}\circ \\
B_3 : & \circ \text{---} \circ \text{>}\text{=}\circ \\
C_3 : & \circ \text{---} \circ \text{<}\text{=}\circ
\end{array}$$

Observation. Φ is irreducible if and only if Δ is irreducible, which holds if and only if the Dynkin diagram is connected. For a general Φ , we have a unique decomposition

$$\Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_r$$

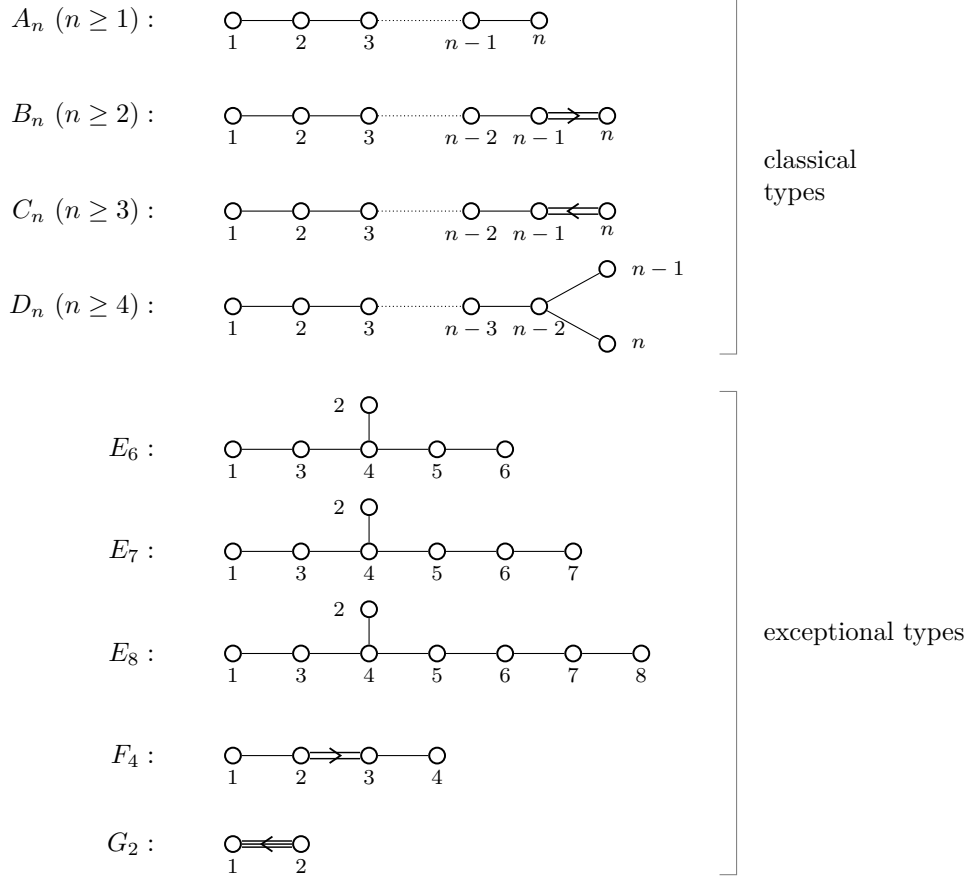
in

$$E = E_1 \oplus \cdots \oplus E_r, \quad E_i \perp E_j \ (i \neq j),$$

with (E_i, Φ_i) irreducible.

It therefore remains to classify the irreducible root systems or, equivalently, the connected Dynkin diagrams.

Theorem III.4.3. The isomorphism classes of irreducible root systems of rank n correspond to the following Dynkin diagrams:



Proof. We need to show:

- 1.) Isomorphic root systems have the same Dynkin diagram.
- 2.) Only the listed Dynkin diagrams can occur.
- 3.) Root systems with the same Dynkin diagram are isomorphic.
- 4.) Every listed diagram is the Dynkin diagram of a root system.

1.) and 3.) are clear, since the Dynkin diagram is simply a compact notation for the Cartan matrix.

For 2.), it is equivalent to show that only the underlying Coxeter graphs can occur. To this end, we call a subset

$$\{\varepsilon_1, \dots, \varepsilon_n\}$$

of linearly independent unit vectors in a Euclidean space $(E, (-, -))$ *admissible* if

$$(\varepsilon_i, \varepsilon_j) \leq 0 \quad (i \neq j), \quad 4(\varepsilon_i, \varepsilon_j)^2 \in \{0, 1, 2, 3\} \quad (i \neq j).$$

Note that $\Delta \subseteq \Phi$ becomes admissible when each $\alpha \in \Delta$ is replaced by $\alpha/\|\alpha\|$. As before, we obtain a graph Γ with n vertices by joining the i th and j th vertices by $4(\varepsilon_i, \varepsilon_j)^2$ edges for $i \neq j$. We need to determine the connected graphs arising from admissible sets.

Observations. (i) $\{\varepsilon_1, \dots, \widehat{\varepsilon_i}, \dots, \varepsilon_n\}$ is again admissible and corresponds to removing the i th vertex together with its incident edges.

(ii) The number of pairs of adjacent vertices is less than n . We have

$$0 < \left\| \sum_{i=1}^n \varepsilon_i \right\|^2 = n + 2 \sum_{i < j} (\varepsilon_i, \varepsilon_j),$$

and $(\varepsilon_i, \varepsilon_j) \neq 0$ implies $2(\varepsilon_i, \varepsilon_j) \leq -1$.

(iii) The underlying simple graph of Γ has no cycles. This follows from (i) and (ii).

(iv) At most three edges are incident to each vertex of Γ . Let ε be one of the admissible vectors, and suppose that ε is joined to $\eta_1, \dots, \eta_k$. By (iii), $(\eta_i, \eta_j) = 0$ for $i \neq j$. Let $\eta_0 \in \langle \varepsilon, \eta_1, \dots, \eta_k \rangle_{\mathbb{R}} \cap \langle \eta_1, \dots, \eta_k \rangle_{\mathbb{R}}^{\perp}$ satisfy $\|\eta_0\| = 1$. Then $(\varepsilon, \eta_0) \neq 0$, so

$$1 = \|\varepsilon\|^2 = \sum_{i=0}^k (\varepsilon, \eta_i)^2,$$

implies

$$\sum_{i=1}^k 4(\varepsilon, \eta_i)^2 < 4.$$

(v) A triple edge occurs only in $\Gamma = \circ \equiv \circ$. This follows from (iv).

(vi) Every subgraph of type A, $\underset{\varepsilon_{i+1}}{\circ} - \underset{\varepsilon_{i+2}}{\circ} - \dots - \underset{\varepsilon_{i+\ell}}{\circ}$, can be replaced by a single vertex.

We show that

$$\{\varepsilon_1, \dots, \varepsilon_i, \varepsilon_{i+\ell+1}, \dots, \varepsilon_n\} \cup \left\{ \underbrace{\sum_{j=1}^{\ell} \varepsilon_{i+j}}_{=:\varepsilon} \right\}$$

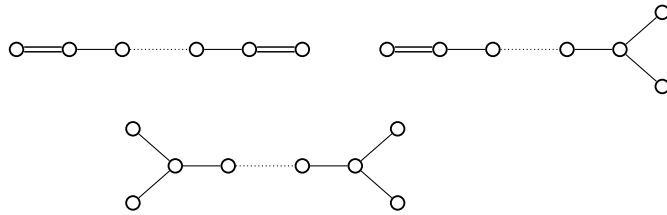
is again admissible. We have

$$\|\varepsilon\|^2 = \ell + 2 \sum_{j < k} (\varepsilon_{i+j}, \varepsilon_{i+k}) = \ell + (\ell - 1)(-1) = 1.$$

By (iii), for every other vertex $\eta \in \{\varepsilon_1, \dots, \varepsilon_i, \varepsilon_{i+\ell+1}, \dots, \varepsilon_n\}$, we have $(\eta, \varepsilon) = 0$ or $(\eta, \varepsilon) = (\eta, \varepsilon_{i+j_0})$ for some $j_0 \in \{1, \dots, \ell\}$. It follows that

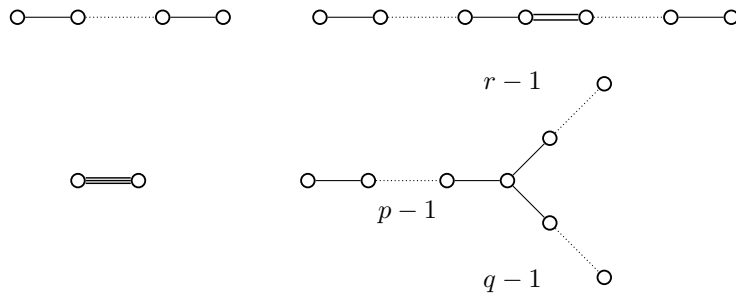
$$4(\eta, \varepsilon)^2 \in \{0, 1, 2, 3\}.$$

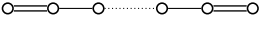
(vii) Γ has none of the following subgraphs:

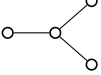


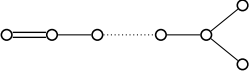
This follows from (vi) and (iv).

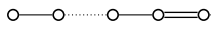
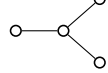
(viii) Γ must have one of the following forms: a chain with only single edges, a chain with exactly one double edge, G_2 , or a tree with exactly one branching vertex and three arms of lengths $p-1, q-1, r-1$:

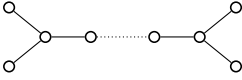


By (v), a triple edge can only occur on its own. If there were more than one double edge, Γ would have a subgraph 

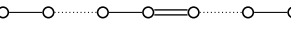
contradicting (vii). If Γ has a double edge, it has no subgraph , since otherwise there

would also be a subgraph 

contradicting (vii). By (iii), the only remaining possibility is . If Γ has only single edges, Γ can have at most one subgraph of the form .

since otherwise 

would occur, contradicting (vii). By (iii), only the first and the last graphs in (viii) remain.

(ix) If Γ has the form , then Γ has type B_n/C_n or type F_4 . This follows by a somewhat more involved argument using the Cauchy–Schwarz inequality.

(x) If Γ has the last form in (viii), then Γ has type D_n, E_6, E_7 or E_8 . This follows by a clever argument that includes deriving the inequality

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

For 4.), the classical types arise as the root systems of the classical matrix algebras in I.2(ii). For the five remaining exceptional types, one can give explicit constructions; for example, for G_2 :

$$\Phi = \pm\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_3, 2\varepsilon_1 - \varepsilon_2 - \varepsilon_3, 2\varepsilon_2 - \varepsilon_1 - \varepsilon_3, 2\varepsilon_3 - \varepsilon_1 - \varepsilon_2\}$$

in

$$E = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}.$$

□

III.5. Automorphisms of Root Systems

By Exercise 6.2, we have $W(\Phi) \triangleleft \text{Aut } \Phi$. For a fixed base $\Delta \subseteq \Phi$, let

$$\text{Sym } \Delta = \{\sigma \in \text{Aut } \Phi : \sigma(\Delta) = \Delta\}.$$

For $\sigma \in \text{Aut } \Phi$ and all $\alpha, \beta \in \Phi$, we have

$$\langle \sigma\alpha, \sigma\beta \rangle = \langle \alpha, \beta \rangle.$$

Thus each $\sigma \in \text{Sym } \Delta$ is given by a symmetry of the Dynkin diagram. Conversely, every such symmetry extends to an element $\sigma \in \text{Sym } \Delta$, so $\text{Sym } \Delta$ can be identified with the group of symmetries of the Dynkin diagram.

Lemma III.5.1. We have

$$\text{Aut } \Phi = W(\Phi) \rtimes \text{Sym } \Delta.$$

Proof. If $\sigma \in W(\Phi) \cap \text{Sym } \Delta$, then $\sigma = \text{id}_E$, since W acts simply transitively on the set of bases. For $\sigma \in \text{Aut } \Phi$, the set $\sigma(\Delta)$ is a base, so there exists $\tau \in W(\Phi)$ with $\tau\sigma(\Delta) = \Delta$, that is,

$$\sigma = \underbrace{\tau^{-1}}_{\in W(\Phi)} \circ \underbrace{(\tau \circ \sigma)}_{\in \text{Sym } \Delta}.$$

□

Clearly, $\text{Sym } \Delta \cong \mathbb{Z}/2$ for A_n ($n \geq 2$), D_n ($n \geq 5$), and E_6 , while $\text{Sym } \Delta \cong S_3$ for type D_4 (triality). For all other types, $\text{Sym } \Delta = 1$.

III.6. The Conjugacy, Isomorphism, and Existence Theorems

Classification scheme (recall that $\text{char } K = 0$, $K = \overline{K}$):

$$\begin{aligned} \{\text{simple Lie algebras}\} / \cong & \xleftrightarrow{1:1} \{\text{irreducible root systems}\} / \cong \\ & \xleftrightarrow{1:1} \{\text{connected Dynkin diagrams}\} / \cong. \end{aligned}$$

It remains to establish:

1. Well-definedness: independence of the choice of a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$.
2. Injectivity: if $(E, \Phi) \cong (E', \Phi')$ for $\mathfrak{g}, \mathfrak{g}'$, then $\mathfrak{g} \cong \mathfrak{g}'$.
3. Surjectivity: if Φ is irreducible, there is a simple $\mathfrak{g}$ with root system Φ .

Theorem III.6.1. Let $\mathfrak{g}$ be semisimple with Cartan subalgebras $\mathfrak{h}, \mathfrak{h}' \subseteq \mathfrak{g}$. Then there exists $f \in \text{Int } \mathfrak{g}$ such that

$$f(\mathfrak{h}') = \mathfrak{h}.$$

Proof. Involved; Humphreys, §§15–16. □

Theorem III.6.2. Let $\mathfrak{g}, \mathfrak{g}'$ be simple with Cartan subalgebras $\mathfrak{h} \subseteq \mathfrak{g}$, $\mathfrak{h}' \subseteq \mathfrak{g}'$, and let $\varphi: \Phi \xrightarrow{\cong} \Phi'$ be an isomorphism of root systems. This induces $f: \mathfrak{h} \rightarrow \mathfrak{h}'$ with $\varphi(\alpha) \circ f = \alpha$. Let $\Delta \subseteq \Phi$ be a base; it induces $\Delta' = \varphi(\Delta) \subseteq \Phi'$. Choose arbitrary $0 \neq x_\alpha \in \mathfrak{g}_\alpha$ and $0 \neq x'_\alpha \in \mathfrak{g}'_{\varphi(\alpha)}$ for $\alpha \in \Delta$. Then there is a unique isomorphism

$$F: \mathfrak{g} \xrightarrow{\cong} \mathfrak{g}'$$

with

$$F|_{\mathfrak{h}} = f, \quad F(x_\alpha) = x'_\alpha$$

for all $\alpha \in \Delta$.

Proof. Let $y_\alpha \in \mathfrak{g}_{-\alpha}$ and $y'_\alpha \in \mathfrak{g}'_{-\varphi(\alpha)}$ for $\alpha \in \Delta$ be as in Theorem III.1.1(vi). Let $D \subseteq \mathfrak{g} \oplus \mathfrak{g}'$ be generated by

$$\{x_\alpha \oplus x'_\alpha, y_\alpha \oplus y'_\alpha, h_\alpha \oplus h'_\alpha : \alpha \in \Delta\}.$$

For the projections

$$\begin{array}{ccc} & \mathfrak{g} \oplus \mathfrak{g}' & \\ p \swarrow & & \searrow p' \\ \mathfrak{g} & & \mathfrak{g}' \end{array}$$

both $p|_D$ and $p'|_D$ are isomorphisms (Humphreys, §14.2). Set $F = (p'|_D) \circ (p|_D)^{-1}$. □

For a set X ,

$$\mathfrak{F}(X) = \langle X \rangle_{\text{Lie}} \subseteq \bigoplus_{n \geq 0} \bigotimes_{i=0}^n KX$$

with the commutator bracket is called the free Lie algebra on X .

Theorem III.6.3, Serre. Let Φ be irreducible with base $\Delta = \{\alpha_1, \dots, \alpha_n\}$, and let $\mathfrak{g}$ be the quotient of $\mathfrak{F}(\{x_i, y_i, h_i : 1 \leq i \leq n\})$ by the ideal generated by the relations

$$\begin{aligned} [h_i, h_j] &= 0, \\ [x_i, y_i] &= h_i, & [x_i, y_j] &= 0 \quad (i \neq j), \\ [h_i, x_j] &= \langle \alpha_j, \alpha_i \rangle x_j, & [h_i, y_j] &= -\langle \alpha_j, \alpha_i \rangle y_j, \\ (\text{ad } x_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(x_j) &= 0, & (\text{ad } y_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(y_j) &= 0 \quad (i \neq j). \end{aligned}$$

Then $\mathfrak{g}$ is finite-dimensional and simple, with Cartan subalgebra $\langle h_i \rangle_K$ and root system Φ .

Proof. Humphreys, §18. □

III.7. Automorphisms of Simple Lie Algebras

Theorem III.7.1. Let $\mathfrak{g}$ be simple with Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and base $\Delta \subseteq \Phi$. Then

$$\text{Aut } \mathfrak{g} \cong \text{Int } \mathfrak{g} \rtimes \text{Sym } \Delta.$$

Proof. By Theorem III.6.2, for every $\sigma \in \text{Aut } \Phi$ we obtain an element $\bar{\sigma} \in \text{Aut } \mathfrak{g}$ inducing the action σ on Φ . After fixing a choice of simple root vectors, the map $\text{Sym } \Delta \rightarrow \text{Aut } \mathfrak{g}$, $\sigma \mapsto \bar{\sigma}$, defines an embedding as a subgroup. Let $g \in \text{Aut } \mathfrak{g}$ be arbitrary. By Theorem III.6.1, there exists $f \in \text{Int } \mathfrak{g}$ with $f(g(\mathfrak{h})) = \mathfrak{h}$. We then obtain

$$\sigma \in \text{Aut } \Phi$$

from the action of $f \circ g$ on $\Phi \subseteq \mathfrak{h}^*$. By Lemma III.5.1, there exists $\tau \in W(\Phi)$ with $\tau\sigma \in \text{Sym } \Delta$. We can now choose $\bar{\tau} \in \text{Int } \mathfrak{g}$ so that

$$g = (\bar{\tau} \circ f)^{-1} \circ \bar{\tau} \circ \bar{\sigma}$$

with $(\bar{\tau} \circ f)^{-1} \in \text{Int } \mathfrak{g}$ and $\bar{\tau} \circ \bar{\sigma} \in \text{Sym } \Delta$, where we identify $\text{Sym } \Delta$ with its image in $\text{Aut } \mathfrak{g}$. For

$$\text{Int } \mathfrak{g} \cap \text{Sym } \Delta = \{1\}$$

we refer to Bourbaki, Lie III.5. □

A. Exercises

Exercise Sheet 1

Exercise 1.1. Verify that

$$\mathfrak{so}_{2n}(K) \subseteq \mathfrak{gl}_{2n}(K)$$

is indeed a subalgebra.

Exercise 1.2. Show that, for

$$\mathfrak{t}_n(K) = \mathfrak{d}_n(K) \oplus \mathfrak{n}_n(K),$$

the following hold:

- (i) $[\mathfrak{d}_n(K), \mathfrak{n}_n(K)] = \mathfrak{n}_n(K)$,
- (ii) $[\mathfrak{t}_n(K), \mathfrak{t}_n(K)] = \mathfrak{n}_n(K)$.

Exercise 1.3. Suppose that the matrix $x \in \mathfrak{gl}_n(K)$ has n distinct eigenvalues $\alpha_1, \dots, \alpha_n \in K$. Show that

$$\alpha_i - \alpha_j, \quad 1 \leq i, j \leq n,$$

are the eigenvalues of $\text{ad } x$.

Exercise Sheet 2

Let $\mathfrak{g}$ be a Lie algebra over K .

Exercise 2.1. Let $\mathfrak{a} \triangleleft \mathfrak{g}$ be an ideal. Show by induction that

$$(D^n(\mathfrak{g}) + \mathfrak{a})/\mathfrak{a} \cong D^n(\mathfrak{g}/\mathfrak{a})$$

for all $n \geq 0$.

Exercise 2.2. Show that

$$\mathfrak{g}/\text{Rad}(\mathfrak{g})$$

is semisimple.

Exercise 2.3. Show that $\mathfrak{g}$ is solvable if and only if there is a sequence

$$\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \dots \supseteq \mathfrak{g}_k = \{0\}$$

with $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ abelian for $i = 0, \dots, k-1$.

Exercise Sheet 3

Exercise 3.1. Let $\mathfrak{g}$ be nilpotent and let $\mathfrak{h} \subsetneq \mathfrak{g}$ be a subalgebra. Show that

$$\mathfrak{h} \subsetneq N_{\mathfrak{g}}(\mathfrak{h}).$$

Exercise 3.2. Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{a} \triangleleft \mathfrak{g}$ be such that $\mathfrak{g}/\mathfrak{a}$ is nilpotent and

$$(\operatorname{ad} x)|_{\mathfrak{a}}$$

is nilpotent for all $x \in \mathfrak{g}$. Show that $\mathfrak{g}$ is nilpotent.

Exercise 3.3. Show that every nilpotent Lie algebra $\{0\} \neq \mathfrak{g}$ has an outer derivation.

Hint: Use $\mathfrak{g} \cong \mathfrak{a} \oplus Kx$ with $\mathfrak{a} \triangleleft \mathfrak{g}$. Then $Z_{\mathfrak{g}}(\mathfrak{a}) \neq \{0\}$. Choose n maximal such that

$$Z_{\mathfrak{g}}(\mathfrak{a}) \subseteq \mathcal{C}^n(\mathfrak{g})$$

and any

$$z \in Z_{\mathfrak{g}}(\mathfrak{a}) \setminus \mathcal{C}^{n+1}(\mathfrak{g}).$$

Then $\delta: \mathfrak{g} \rightarrow \mathfrak{g}$, defined by $\delta|_{\mathfrak{a}} = 0$ and $\delta(x) = z$, is an outer derivation.

Exercise Sheet 4

Exercise 4.1. Show that the Killing form B of a nilpotent Lie algebra $\mathfrak{g}$ vanishes.

Exercise 4.2. Show that a Lie algebra $\mathfrak{g}$ is solvable if and only if

$$[\mathfrak{g}, \mathfrak{g}] \subseteq \operatorname{Rad} B.$$

Exercise 4.3. Find a solvable Lie algebra $\mathfrak{g}$ with $B \neq 0$.

Note. From now on, $\operatorname{char} K = 0$ and $K = \overline{K}$ are assumed in the exercises as well.

Exercise Sheet 5

Exercise 5.1. Construct an irreducible $\mathfrak{sl}_2(K)$ -representation of each dimension ≥ 1 .

Hint: Lemma II.8.2. To verify the module axioms, it suffices to show that the endomorphisms x, h, y satisfy the same commutator relations as $x, h, y \in \mathfrak{sl}_2(K)$.

Exercise 5.2. Show that, in the classical Lie algebras of types A_n, B_n, C_n, D_n , the diagonal matrices form a Cartan subalgebra of dimension n .

Exercise 5.3. Let $\mathfrak{g}$ be semisimple and let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra. Show that

$$N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}.$$

Exercise Sheet 6

Exercise 6.1. Show that there is no semisimple Lie algebra of dimension four, five, or seven.

Exercise 6.2. Show that, for every root system $\Phi \subseteq E$, the Weyl group W is a normal subgroup of $\text{Aut}(\Phi)$.

Exercise 6.3. Let $\Phi \subseteq E$ be a root system. For $\alpha \in E \setminus \{0\}$, let

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}, \quad \Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}.$$

Prove that $\Phi^\vee$ is a root system and that there is a canonical isomorphism

$$W(\Phi) \cong W(\Phi^\vee).$$

Also show that

$$\langle \alpha^\vee, \beta^\vee \rangle = \langle \beta, \alpha \rangle$$

(for the Cartan integers), and draw $\Phi, \Phi^\vee \subseteq E$ for Φ of types A_2, B_2, G_2 . Does $\Phi \cong \Phi^\vee$ hold in general?

Exercise Sheet 7

Let (E, Φ) be a root system and let $\Delta \subseteq \Phi$ be a base.

Exercise 7.1. For a subspace $E' \subseteq E$ and $\alpha \in \Phi$, suppose that

$$\sigma_\alpha(E') = E'.$$

Show that either $\alpha \in E'$ or

$$E' \subseteq (\mathbb{R}\alpha)^\perp.$$

Exercise 7.2. Let $\alpha \in \Phi$ be simple. Show that

$$\sigma_\alpha(\Phi^+ \setminus \{\alpha\}) \subseteq \Phi^+ \setminus \{\alpha\}.$$

Exercise 7.3. Let

$$\rho = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta$$

and let $\alpha \in \Delta$. Show that

$$\sigma_\alpha(\rho) = \rho - \alpha.$$

Exercise 7.4. Let $\sigma \in W(\Phi)$ be the unique element such that

$$\sigma(\Phi^+) = \Phi^-.$$

Show that, in every expression

$$\sigma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_k}$$

of σ as a reduced word with $\alpha_i \in \Delta$, every σ_α for $\alpha \in \Delta$ must occur. For which irreducible root systems of ranks two and three do we have

$$\sigma = -\text{id}_E?$$

B. Solutions

Solution 1

1.1. Clearly, $\mathfrak{so}_{2n}(K) \subseteq \mathfrak{gl}_{2n}(K)$ is a K -linear subspace. Let $x, y \in \mathfrak{so}_{2n}(K)$, that is,

$$x^T m + m x = 0, \quad y^T m + m y = 0, \quad m = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} [x, y]^T m + m [x, y] &= (xy - yx)^T m + m(xy - yx) \\ &= y^T x^T m - x^T y^T m + mxy - myx \\ &= y^T(-mx) - x^T(-my) + (-x^T m)y - (-y^T m)x \\ &= 0. \end{aligned}$$

1.2. (i) First, for each basis vector

$$e_i = \text{diag}(0, \dots, 0, 1, 0, \dots, 0)$$

of $\mathfrak{d}_n(K)$, we show that

$$[e_i, x] \in \mathfrak{n}_n(K)$$

for all $x \in \mathfrak{n}_n(K)$. Indeed,

$$[e_i, x] = e_i x - x e_i.$$

The first matrix can have nonzero entries only in the i th row, and the second only in the i th column. Since $x_{i1} = \dots = x_{ii} = 0$ and $x_{ii} = \dots = x_{ni} = 0$, this is a strictly upper triangular matrix. Thus

$$[\mathfrak{d}_n(K), \mathfrak{n}_n(K)] \subseteq \mathfrak{n}_n(K).$$

Let E_{ij} be the $(n \times n)$ -matrix with a one in position (i, j) . For $i < j$, we have

$$[e_i, E_{ij}] = E_{ij},$$

so we also have

$$\mathfrak{n}_n(K) \subseteq [\mathfrak{d}_n(K), \mathfrak{n}_n(K)].$$

(ii) The inclusion

$$[\mathfrak{t}_n(K), \mathfrak{t}_n(K)] \subseteq \mathfrak{n}_n(K)$$

follows immediately from (i), and the reverse inclusion also follows from (i), since

$$\mathfrak{t}_n(K) = \mathfrak{d}_n(K) \oplus \mathfrak{n}_n(K).$$

1.3. By assumption, the characteristic polynomial of $x \in \mathfrak{gl}_n(K)$ splits into distinct linear factors. Hence there is a basis of eigenvectors for x , and we may assume that

$$x = \text{diag}(\alpha_1, \dots, \alpha_n).$$

Then

$$\text{ad } x(E_{ij}) = [x, E_{ij}] = xE_{ij} - E_{ij}x = \alpha_i E_{ij} - \alpha_j E_{ij} = (\alpha_i - \alpha_j)E_{ij}.$$

Solution 2

2.1. Base case: We have

$$(D^0(\mathfrak{g}) + \mathfrak{a})/\mathfrak{a} = \mathfrak{g}/\mathfrak{a} \cong D^0(\mathfrak{g}/\mathfrak{a}).$$

Induction step: Let

$$\varphi: \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{a}$$

be the canonical map. Then

$$\begin{aligned} (D^n(\mathfrak{g}) + \mathfrak{a})/\mathfrak{a} &= \varphi(D^n(\mathfrak{g})) \\ &= \varphi([D^{n-1}(\mathfrak{g}), D^{n-1}(\mathfrak{g})]) \\ &= [\varphi(D^{n-1}(\mathfrak{g})), \varphi(D^{n-1}(\mathfrak{g}))] \\ &\stackrel{\text{induction hypothesis}}{=} [D^{n-1}(\mathfrak{g}/\mathfrak{a}), D^{n-1}(\mathfrak{g}/\mathfrak{a})] \\ &= D^n(\mathfrak{g}/\mathfrak{a}). \end{aligned}$$

□

2.2. Let $\mathfrak{a} \triangleleft \mathfrak{g}/\text{Rad}(\mathfrak{g})$ be a solvable ideal and define

$$\mathfrak{b} = \varphi^{-1}(\mathfrak{a}), \quad \varphi: \mathfrak{g} \longrightarrow \mathfrak{g}/\text{Rad}(\mathfrak{g})$$

using the canonical map. Then $\mathfrak{b} \triangleleft \mathfrak{g}$, since for $x \in \mathfrak{b}$, $y \in \mathfrak{g}$ we have

$$\varphi([y, x]) = [\varphi(y), \varphi(x)] \in \mathfrak{a},$$

so $[y, x] \in \mathfrak{b}$. Moreover, $\mathfrak{b}$ is solvable by [I.6.3](#), since $\varphi(\mathfrak{b})$ and $\mathfrak{b} \cap \text{Rad}(\mathfrak{g})$ are solvable. It follows that $\mathfrak{b} \subseteq \text{Rad}(\mathfrak{g})$, so $\mathfrak{a} = \varphi(\mathfrak{b}) = \{0\}$. □

2.3. If $\mathfrak{g}$ is solvable, the derived series

$$\mathfrak{g} = D^0\mathfrak{g} \supseteq D^1\mathfrak{g} \supseteq \cdots \supseteq D^k\mathfrak{g} = 0$$

has the required properties, since

$$D^{i+1}\mathfrak{g} = [D^i\mathfrak{g}, D^i\mathfrak{g}]$$

implies that $D^i\mathfrak{g}/D^{i+1}\mathfrak{g}$ is abelian.

Conversely, if there is a sequence

$$\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \cdots \supseteq \mathfrak{g}_k = 0$$

with $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ abelian, we prove

$$D^i\mathfrak{g} \subseteq \mathfrak{g}_i$$

by induction. The base case $D^0\mathfrak{g} = \mathfrak{g} = \mathfrak{g}_0$ is clear. Since $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian, for all $x, y \in \mathfrak{g}_i$ we have

$$[x + \mathfrak{g}_{i+1}, y + \mathfrak{g}_{i+1}] = \mathfrak{g}_{i+1},$$

that is, $[x, y] + \mathfrak{g}_{i+1} = \mathfrak{g}_{i+1}$, so $[x, y] \in \mathfrak{g}_{i+1}$. Hence

$$D^{i+1}\mathfrak{g} = [D^i\mathfrak{g}, D^i\mathfrak{g}] \subseteq [\mathfrak{g}_i, \mathfrak{g}_i] \subseteq \mathfrak{g}_{i+1}.$$

Since $\mathfrak{g}_k = 0$, it follows that $D^k\mathfrak{g} = 0$.

Solution 3

3.1. Applying I.7.5 to

$$\operatorname{ad} \mathfrak{h} \subseteq \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$$

yields an element $x + \mathfrak{h} \neq 0$ such that

$$[\mathfrak{h}, x] \subseteq \mathfrak{h}.$$

That is, $x \in N_{\mathfrak{g}}(\mathfrak{h}) \setminus \mathfrak{h}$, so

$$\mathfrak{h} \subsetneq N_{\mathfrak{g}}(\mathfrak{h}).$$

3.2. Since $\mathfrak{g}/\mathfrak{a}$ is nilpotent, there is some $N \gg 0$ such that

$$\mathcal{C}^N(\mathfrak{g}/\mathfrak{a}) = \{0\},$$

that is,

$$\mathcal{C}^N(\mathfrak{g}) \subseteq \mathfrak{a}.$$

Thus, for each $x \in \mathfrak{g}$, we have

$$(\operatorname{ad} x)^N(\mathfrak{g}) \subseteq \mathcal{C}^N(\mathfrak{g}) \subseteq \mathfrak{a}.$$

By assumption, there is some $r = r(x)$ with $(\operatorname{ad} x)^r|_{\mathfrak{a}} = 0$. Thus $(\operatorname{ad} x)^{N+r} = 0$ on $\mathfrak{g}$. By Engel's theorem, $\mathfrak{g}$ is therefore nilpotent. $\square$

3.3. There is an ideal $\mathfrak{a} \triangleleft \mathfrak{g}$ of codimension one: since $\mathfrak{g} \neq 0$ is nilpotent, we have

$$0 < \dim \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}].$$

Choose any subspace $\bar{\mathfrak{a}} \subseteq \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ of codimension one and set $\mathfrak{a} = p^{-1}(\bar{\mathfrak{a}})$ for the canonical map

$$p: \mathfrak{g} \longrightarrow \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}].$$

As seen above, $\mathfrak{a} \triangleleft \mathfrak{g}$ and

$$\begin{aligned} \dim \mathfrak{a} &= \dim \bar{\mathfrak{a}} + \dim(\mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}]) \\ &= (\dim \mathfrak{g} - \dim[\mathfrak{g}, \mathfrak{g}] - 1) + \dim[\mathfrak{g}, \mathfrak{g}] \\ &= \dim \mathfrak{g} - 1. \end{aligned}$$

Choosing $x \in \mathfrak{g} \setminus \mathfrak{a}$, we obtain

$$\mathfrak{g} = \mathfrak{a} \oplus Kx.$$

If $\mathfrak{a} = 0$, then $Z_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{g} \neq 0$. Otherwise, $Z(\mathfrak{a}) \neq 0$ because $\mathfrak{a}$ is nilpotent, and because

$$Z(\mathfrak{a}) \subseteq Z_{\mathfrak{g}}(\mathfrak{a})$$

we have $Z_{\mathfrak{g}}(\mathfrak{a}) \neq 0$ in either case.

Let n be maximal such that

$$Z_{\mathfrak{g}}(\mathfrak{a}) \subseteq \mathcal{C}^n(\mathfrak{g})$$

and choose

$$z \in Z_{\mathfrak{g}}(\mathfrak{a}) \setminus \mathcal{C}^{n+1}(\mathfrak{g})$$

and $x \in \mathfrak{g} \setminus \mathfrak{a}$. Then $\mathfrak{g} = \mathfrak{a} \oplus Kx$. Define $\delta: \mathfrak{g} \rightarrow \mathfrak{g}$ by

$$\delta|_{\mathfrak{a}} = 0, \quad \delta(x) = z.$$

This is a derivation. Indeed, for $a_1, a_2 \in \mathfrak{a}$ and $\lambda_1, \lambda_2 \in K$, we have

$$[a_1 + \lambda_1 x, a_2 + \lambda_2 x] = [a_1, a_2] + \lambda_1 [x, a_2] + \lambda_2 [a_1, x] \in \mathfrak{a},$$

so $\delta([a_1 + \lambda_1 x, a_2 + \lambda_2 x]) = 0$. On the other hand,

$$\begin{aligned} & [\delta(a_1 + \lambda_1 x), a_2 + \lambda_2 x] + [a_1 + \lambda_1 x, \delta(a_2 + \lambda_2 x)] \\ &= [\lambda_1 z, a_2 + \lambda_2 x] + [a_1 + \lambda_1 x, \lambda_2 z] \\ &= \lambda_1 [z, a_2] + \lambda_1 \lambda_2 [z, x] + \lambda_2 [a_1, z] + \lambda_1 \lambda_2 [x, z] \\ &= 0. \end{aligned}$$

Suppose there were some $y \in \mathfrak{g}$ with $\delta = \text{ad } y$. Then we would have

$$[y, \mathfrak{a}] = \delta(\mathfrak{a}) = 0,$$

so $y \in Z_{\mathfrak{g}}(\mathfrak{a}) \subseteq \mathcal{C}^n(\mathfrak{g})$. Hence

$$z = \delta(x) = [y, x] \in \mathcal{C}^{n+1}(\mathfrak{g}),$$

a contradiction. Thus δ is an outer derivation.

Solution 4

4.1. Let $\mathfrak{g}$ be nilpotent and let $x, y \in \mathfrak{g}$. Then

$$\text{ad } x \text{ ad } y \in \text{End}(\mathfrak{g})$$

is nilpotent. Indeed, for all $z \in \mathfrak{g}$ we have

$$(\text{ad } x \text{ ad } y)^n z \in \mathcal{C}^{2n}(\mathfrak{g}),$$

and $\mathcal{C}^{2n}(\mathfrak{g}) = 0$ for $n \gg 0$. It follows that

$$B(x, y) = \text{Tr}(\text{ad } x \text{ ad } y) = 0.$$

4.2. Let $\mathfrak{g}$ be solvable. By Lie's theorem, there is a basis of $\mathfrak{g}$ with respect to which

$$\text{ad } \mathfrak{g} \subseteq \mathfrak{t}_n(K).$$

Consequently, for $x \in [\mathfrak{g}, \mathfrak{g}]$ we have

$$\text{ad } x \in [\mathfrak{t}_n(K), \mathfrak{t}_n(K)] \subseteq \mathfrak{n}_n(K).$$

Thus $\text{ad } x \text{ ad } y$ is a strictly upper triangular matrix for all $y \in \mathfrak{g}$, and

$$B(x, y) = \text{Tr}(\text{ad } x \text{ ad } y) = 0.$$

Hence $[\mathfrak{g}, \mathfrak{g}] \subseteq \text{Rad } B$. The converse is the statement of Corollary II.3.4.

4.3. Consider the Lie algebra $\mathfrak{g}$ with basis $\{x, y\}$ and

$$[x, y] = x.$$

To verify the Jacobi identity, it suffices to compute

$$[y, [x, y]] + [x, [y, y]] + [y, [y, x]] = -x + 0 + x = 0.$$

We have

$$D^2\mathfrak{g} = [Kx, Kx] = \{0\},$$

so $\mathfrak{g}$ is solvable. With respect to the basis $\{x, y\} \subseteq \mathfrak{g}$, we have

$$\text{ad } y = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus

$$B(y, y) = \text{Tr} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 \neq 0.$$

Solution 5

5.1. Define

$$V(m) = Kv_0 \oplus \cdots \oplus Kv_m$$

with the $\mathfrak{sl}_2(K)$ -action from II.8.2:

$$h.v_i = (m - 2i)v_i, \quad y.v_i = (i + 1)v_{i+1}, \quad x.v_i = (m - i + 1)v_{i-1},$$

where $v_{-1} = v_{m+1} = 0$. We show that this defines a homomorphism $\varphi: \mathfrak{sl}_2(K) \rightarrow \mathfrak{gl}(V(m))$.

First,

$$\begin{aligned} [h., x.]v_i &= (h.x. - x.h.)v_i \\ &= (m - i + 1)h.v_{i-1} - (m - 2i)x.v_i \\ &= (m - i + 1)(m - 2(i - 1))v_{i-1} \\ &\quad - (m - 2i)(m - i + 1)v_{i-1} \\ &= 2(m - i + 1)v_{i-1} = 2x.v_i = [h, x].v_i. \end{aligned}$$

Next,

$$\begin{aligned} [h., y.]v_i &= (h.y. - y.h.)v_i \\ &= (i + 1)h.v_{i+1} - (m - 2i)y.v_i \\ &= (i + 1)(m - 2(i + 1))v_{i+1} - (m - 2i)(i + 1)v_{i+1} \\ &= -2(i + 1)v_{i+1} = -2y.v_i = [h, y].v_i. \end{aligned}$$

Finally,

$$\begin{aligned} [x., y.]v_i &= (x.y. - y.x.)v_i \\ &= (i + 1)x.v_{i+1} - (m - i + 1)y.v_{i-1} \\ &= (i + 1)(m - i)v_i - i(m - i + 1)v_i \\ &= (m - 2i)v_i = h.v_i = [x, y].v_i. \end{aligned}$$

Let $\{0\} \neq W \subseteq V(m)$ be an invariant subspace and let

$$0 \neq w = \sum_{i=0}^m \lambda_i v_i \in W.$$

Let i_0 be maximal such that $\lambda_{i_0} \neq 0$. Then

$$0 \neq \varphi(x)^{i_0} w \in K v_0,$$

so $W = V(m)$, that is, $V(m)$ is irreducible.

5.2. Let $\mathfrak{g}$ be one of the classical Lie algebras. By [II.7.2](#), the diagonal matrices form a toral subalgebra $\mathfrak{d} \subseteq \mathfrak{g}$. If $\mathfrak{t} \supseteq \mathfrak{d}$ is toral and $x \in \mathfrak{t}$, then x is diagonalizable and commutes with all $y \in \mathfrak{d}$ by [II.9.2](#). In the respective types, the matrices

$$\begin{array}{ll} A_n : & E_{ii} - E_{n+1, n+1} \in \mathfrak{d} & \text{for } i = 1, \dots, n, \\ B_n : & E_{ii} - E_{n+i, n+i} \in \mathfrak{d} & \text{for } i = 2, \dots, n+1, \\ C_n, D_n : & E_{ii} - E_{n+i, n+i} \in \mathfrak{d} & \text{for } i = 1, \dots, n \end{array}$$

determine a simultaneous eigenspace decomposition with one-dimensional weight spaces, so x must be diagonal. Thus $\mathfrak{t} = \mathfrak{d}$, and $\mathfrak{d}$ is maximal toral. The list also shows that

$$\dim \mathfrak{d} = n.$$

5.3. Let $x \in N_{\mathfrak{g}}(\mathfrak{h})$. Write

$$x = h_0 + \sum_{\alpha \in \Phi} x_{\alpha}$$

according to the Cartan decomposition. Then, for all $h \in \mathfrak{h}$, we have

$$\mathfrak{h} \ni [h, x] = \sum_{\alpha \in \Phi} \alpha(h) x_{\alpha} \in \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha},$$

so $[h, x] \in \mathfrak{h} \cap \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha} = \{0\}$. That is, $[h, x] = 0$, and hence $x \in Z_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$ by [II.9.8](#).

Solution 6

6.1. Since roots occur in pairs $\pm\alpha$,

$$\dim \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}$$

is even, so

$$\dim \mathfrak{g} \equiv \dim \mathfrak{h} \pmod{2}.$$

Thus, if $\dim \mathfrak{g} = 4$, we would have $\dim \mathfrak{h} = 2$, since there is always a Cartan subalgebra $\mathfrak{h} \neq \{0\}$ and $\mathfrak{g}$ is not abelian. But then we would have

$$\Phi = \{\pm\alpha\}$$

for some $\alpha \in \mathfrak{h}^*$, contradicting $\langle \Phi \rangle = \mathfrak{h}^*$.

If $\dim \mathfrak{g} = 5$, then $\dim \mathfrak{h} = 1$ or 3 . If $\dim \mathfrak{h} = 1$, we must have $\Phi = \{\pm\alpha\}$, so $\dim \mathfrak{g} = 3$, a contradiction. If $\dim \mathfrak{h} = 3$, we must again have $\Phi = \{\pm\alpha\}$, again contradicting $\langle \Phi \rangle = \mathfrak{h}^*$.

If $\dim \mathfrak{g} = 7$, we initially obtain $\dim \mathfrak{h} = 1, 3$ or 5 , but we exclude the cases $\dim \mathfrak{h} = 1$ and 5 as above. If $\dim \mathfrak{h} = 3$, we must have

$$\Phi = \{\pm\alpha, \pm\beta\},$$

which again contradicts $\langle \Phi \rangle = \mathfrak{h}^*$.

6.2. Let $\sigma_\alpha \in W$ be a generator and let $\tau \in \text{Aut } \Phi$. Then, for $\beta \in \Phi$, we have

$$\begin{aligned}\tau\sigma_\alpha\tau^{-1}(\beta) &= \tau(\tau^{-1}(\beta) - \langle \tau^{-1}(\beta), \alpha \rangle \alpha) \\ &= \beta - \langle \beta, \tau(\alpha) \rangle \tau(\alpha) \\ &= \sigma_{\tau(\alpha)}(\beta).\end{aligned}$$

Since $\mathbb{R}\Phi = E$, it follows that

$$\tau\sigma_\alpha\tau^{-1} = \sigma_{\tau(\alpha)} \quad \text{in } \text{GL}(E).$$

Since every $g \in W$ has the form

$$g = \sigma_{\alpha_1} \circ \cdots \circ \sigma_{\alpha_k}$$

for $\alpha_1, \dots, \alpha_k \in \Phi$, we have

$$\tau g \tau^{-1} = \sigma_{\tau(\alpha_1)} \circ \cdots \circ \sigma_{\tau(\alpha_k)} \in W.$$

6.3. Axioms (W1) and (W2) are clear for $\Phi^\vee$. Let $\alpha^\vee, \beta^\vee \in \Phi^\vee$. Then

$$\begin{aligned}\sigma_{\alpha^\vee}(\beta^\vee) &= \sigma_\alpha(\beta^\vee) = \sigma_\alpha\left(\frac{2\beta}{(\beta, \beta)}\right) \\ &= \frac{2}{(\beta, \beta)}\sigma_\alpha(\beta) = \frac{2\sigma_\alpha(\beta)}{(\sigma_\alpha(\beta), \sigma_\alpha(\beta))} \in \Phi^\vee.\end{aligned}$$

This proves (W3). Finally,

$$\langle \beta^\vee, \alpha^\vee \rangle = \frac{2(\beta^\vee, \alpha^\vee)}{(\alpha^\vee, \alpha^\vee)} = \frac{2(\beta, \alpha)}{(\beta, \beta)} = \langle \alpha, \beta \rangle \in \mathbb{Z}.$$

This proves (W4). The isomorphism

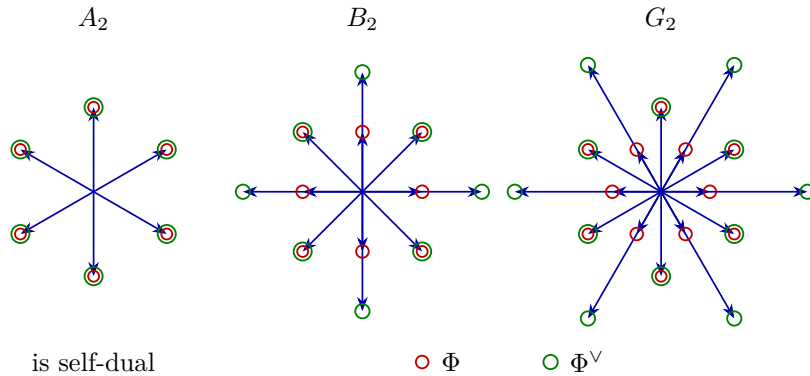
$$W(\Phi) \cong W(\Phi^\vee)$$

follows from $\sigma_\alpha = \sigma_{\alpha^\vee}$ for all $\alpha \in \Phi$. The identity

$$\langle \beta^\vee, \alpha^\vee \rangle = \langle \alpha, \beta \rangle$$

was established above.

The diagrams are:



In general, $\Phi \cong \Phi^\vee$ does not hold; for example,

$$B_3^\vee = C_3.$$

Solution 7

7.1. If there is an $x \in E'$ with $(x, \alpha) \neq 0$, then we also have

$$\sigma_\alpha(x) - x = -\frac{2(x, \alpha)}{(\alpha, \alpha)}\alpha \in E'.$$

Thus $\alpha \in E'$. Otherwise, $(x, \alpha) = 0$ for all $x \in E'$, so $E' \subseteq (\mathbb{R}\alpha)^\perp$.

7.2. Let $\beta \in \Phi^+ \setminus \{\alpha\}$ and write

$$\beta = \sum_{\gamma \in \Delta} n_\gamma \gamma, \quad n_\gamma \geq 0.$$

The coefficient n_γ for $\gamma \neq \alpha$ in

$$\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$$

is unchanged. Since at least one such coefficient is positive, $\sigma_\alpha(\beta)$ cannot be negative. Moreover, $\sigma_\alpha(\beta) \neq \alpha$, since $\sigma_\alpha(-\alpha) = \alpha$ already. Thus

$$\sigma_\alpha(\beta) \in \Phi^+ \setminus \{\alpha\}.$$

7.3. By 2), σ_α permutes the set $\Phi^+ \setminus \{\alpha\}$, and $\sigma_\alpha(\alpha) = -\alpha$. Therefore

$$\begin{aligned} \sigma_\alpha(\rho) &= \frac{1}{2} \left(\sum_{\beta \in \Phi^+ \setminus \{\alpha\}} \sigma_\alpha(\beta) - \alpha \right) \\ &= \frac{1}{2} \left(\sum_{\beta \in \Phi^+ \setminus \{\alpha\}} \beta - \alpha \right) = \rho - \alpha. \end{aligned}$$

7.4. Let

$$\sigma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$$

be a reduced word expression with $\alpha_i \in \Delta$. Suppose there were a $\beta \in \Delta$ such that

$$\beta \neq \alpha_i \quad \text{for } i = 1, \dots, t.$$

Under each reflection σ_{α_i} , the coefficient of β with respect to Δ remains equal to one. By 2), we therefore have $\sigma(\beta) \in \Phi^+$, a contradiction.

By inspection, for an irreducible Φ of rank 2 or 3, we obtain:

$$\begin{aligned} \sigma = -\text{id}_E &\iff \Phi \text{ has type } B_2, G_2, B_3 \text{ or } C_3, \\ \sigma \neq -\text{id}_E &\iff \Phi \text{ has type } A_2 \text{ or } A_3. \end{aligned}$$

For a general irreducible Φ , we have

$$-\text{id}_E \notin W(\Phi) \iff \Phi \text{ has type } A_n \ (n \geq 2), \ D_{2n+1} \ (n \geq 2) \text{ or } E_6.$$