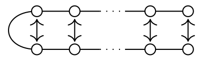
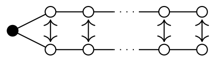
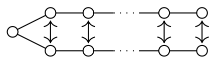
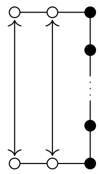
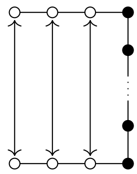
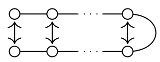
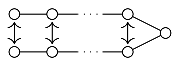
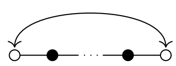
List of Satake and Vogan Diagrams of Real Simple Lie Algebras

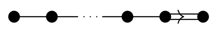
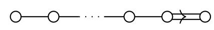
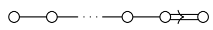
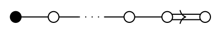
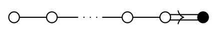
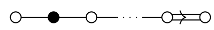
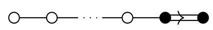
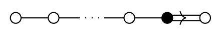
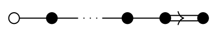
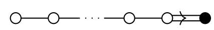
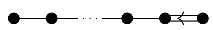
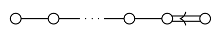
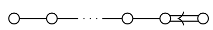
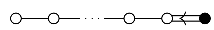
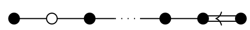
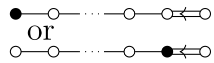
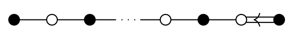
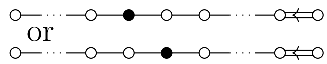
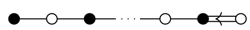
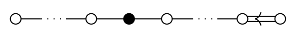
Define $\mathfrak{gl}(n, F)$ for a (skew) field F and $n \in \mathbb{N}$ as the Lie algebra of n-by-n matrices over F and define the *classical matrix algebras* as follows:

$$
\begin{aligned}
\mathfrak{sl}(n,\mathbb{R}) &= \{x \in \mathfrak{gl}(n,\mathbb{R}) \mid \operatorname{Tr} x = 0\} && (n \geq 2)\\
\mathfrak{sl}(n,\mathbb{C}) &= \{x \in \mathfrak{gl}(n,\mathbb{C}) \mid \operatorname{Tr} x = 0\} && (n \geq 2)\\
\mathfrak{sl}(n,\mathbb{H}) &= \{x \in \mathfrak{gl}(n,\mathbb{H}) \mid \operatorname{Re}\operatorname{Tr} x = 0\} && (n \geq 1)\\
\mathfrak{so}(n) &= \{x \in \mathfrak{gl}(n,\mathbb{R}) \mid x^T + x = 0\} && (n \geq 2)\\
\mathfrak{so}(n,\mathbb{C}) &= \{x \in \mathfrak{gl}(n,\mathbb{C}) \mid x^T + x = 0\} && (n \geq 2)\\
\mathfrak{so}(m,n) &= \{x \in \mathfrak{gl}(m+n,\mathbb{R}) \mid x^T \cdot I_{m,n} + I_{m,n}x = 0\} && (m+n \geq 3)\\
\mathfrak{so}^*(2n) &= \left\{x \in \mathfrak{sl}(2n,\mathbb{C}) \middle| {}^{x^T I_{n,n} J_{n,n} + I_{n,n} J_{n,n} x = 0}_{\text{and } \overline{x}^T I_{n,n} + I_{n,n} x = 0} \right\} && (n \geq 2)\\
\mathfrak{su}(n) &= \{x \in \mathfrak{sl}(n,\mathbb{C}) \mid \overline{x}^T + x = 0\} && (n \geq 2)\\
\mathfrak{su}(m,n) &= \{x \in \mathfrak{sl}(n,\mathbb{C}) \mid \overline{x}^T I_{m,n} + I_{m,n}x = 0\} && (m+n \geq 2)\\
\mathfrak{sp}(2n,\mathbb{R}) &= \{x \in \mathfrak{gl}(2n,\mathbb{R}) \mid x^T J_{n,n} + J_{n,n}x = 0\} && (n \geq 1)\\
\mathfrak{sp}(2n,\mathbb{C}) &= \{x \in \mathfrak{gl}(2n,\mathbb{C}) \mid x^T J_{n,n} + J_{n,n}x = 0\} && (n \geq 1)\\
\mathfrak{sp}(2n) &= \{x \in \mathfrak{gl}(n,\mathbb{H}) \mid \overline{x}^T + x = 0\} && (n \geq 1)\\
\mathfrak{sp}(2m,2n) &= \{x \in \mathfrak{gl}(m+n,\mathbb{H}) \mid \overline{x}^T I_{m,n} + I_{m,n}x = 0\} && (m+n \geq 1)
\end{aligned}
$$

The algebras $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{so}(n,\mathbb{C})$, $\mathfrak{sp}(2n,\mathbb{C})$ are the classical complex simple Lie algebras and are hence also simple when regarded as Lie algebras over the field of the real numbers [Kna96, Thm. 6.95]. Their Satake and Vogan diagrams however are not connected (let $\mathfrak{g}$ be one of those algebras regarded as real Lie algebra, then its complexification $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to two copies of $\mathfrak{g}$, especially not simple) and they are therefore not included in the list below.

Bump [Bum13, Table 28.1], Helgason [Hel78, Chapter X] and Knapp [Kna96, §6.10] now give the following classification for the real forms $\mathfrak{g}$ of complex simple Lie-Algebras with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and their respective Satake and Vogan diagrams:

Type	$\mathfrak{g}$	$\mathfrak{k}$	Satake diagram	Vogan diagram
A	$\mathfrak{su}(n+1)$	$\mathfrak{su}(n+1)$		
A I	$\mathfrak{sl}(n+1,\mathbb{R})$, $n \equiv 1 \bmod 2$	$\mathfrak{so}(n+1)$		
A I	$\mathfrak{sl}(n+1,\mathbb{R})$, $n \equiv 0 \bmod 2$	$\mathfrak{so}(n+1)$		
A II	$\mathfrak{sl}(\frac{1}{2}(n+1),\mathbb{H})$, $n \equiv 1 \bmod 2$	$\mathfrak{sp}(n+1)$		
A III	$\mathfrak{su}(2,n-1)$, $n \geq 4$			
A III	$\mathfrak{su}(3,n-2)$, $n \geq 6$			
A III	⋮		⋮	⋮
A III	$\mathfrak{su}(\frac{n}{2},\frac{n}{2}+1)$, $n \equiv 0 \bmod 2$			
A III	$\mathfrak{su}(\frac{n-1}{2},\frac{n-1}{2})$, $n \equiv 1 \bmod 2$			
A IV	$\mathfrak{su}(1,n)$, $n \geq 2$			

Type	$\mathfrak{g}$	$\mathfrak{k}$	Satake diagram	Vogan diagram
B	$\mathfrak{so}(2n{+}1)$	$\mathfrak{so}(2n{+}1)$		
B I	$\mathfrak{so}(2,2n{-}1)$, $n{\geq}2$			
B I	$\mathfrak{so}(4,2n{-}3)$, $n{\geq}3$			
B I	$\mathfrak{so}(6,2n{-}5)$, $n{\geq}4$			
B I	⋮		⋮	⋮
B I	$\mathfrak{so}(2n{-}2,3)$, $n{\geq}2$			
B II	$\mathfrak{so}(2n,1)$			
C	$\mathfrak{sp}(2n)$	$\mathfrak{sp}(2n)$		
C I	$\mathfrak{sp}(2n,\mathbb{R})$			
C II	$\mathfrak{sp}(2n{-}2,2)$, $n{\geq}3$	$\mathfrak{sp}(2n{-}2)$ $\oplus\mathfrak{sp}(2)$		
C II	⋮	⋮	⋮	⋮
C II	$\mathfrak{sp}(n{+}1,n{-}1)$, $n{\equiv}1 \bmod 2$, $n{\geq}3$	$\mathfrak{sp}(n{+}1)$ $\oplus\mathfrak{sp}(n{-}1)$		
C II	$\mathfrak{sp}(n,n)$, $n{\equiv}0 \bmod 2$, $n{\geq}4$	$\mathfrak{sp}(n)$ $\oplus\mathfrak{sp}(n)$		

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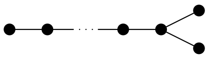
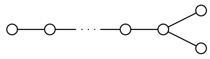
<tr><th>Type</th><th>$\mathfrak{g}$</th><th>$\mathfrak{k}$</th><th>Satake diagram</th><th>Vogan diagram</th></tr>
<tr><td>D</td><td>$\mathfrak{so}(2n)$</td><td>$\mathfrak{so}(2n)$</td><td></td><td></td></tr>
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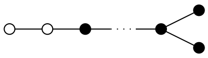
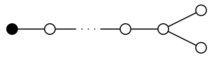
$n{\geq}4$</td><td></td><td></td><td></td></tr>
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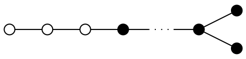
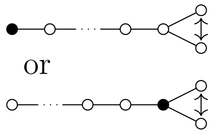
$n{\geq}5$</td><td></td><td></td><td>
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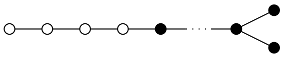
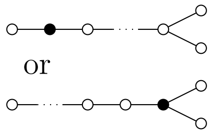
$n{\geq}6$</td><td></td><td></td><td>
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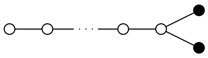
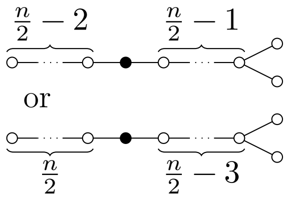
$n{\geq}6$</td><td></td><td></td><td>
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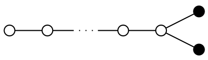
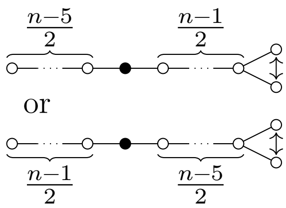
$n{\geq}5$</td><td></td><td></td><td>
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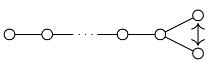
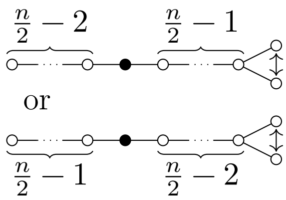
$n{\geq}4$</td><td></td><td></td><td>
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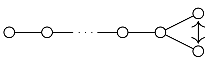
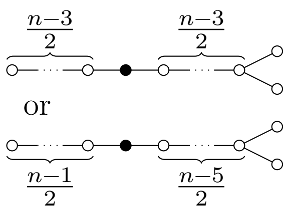
$n{\geq}5$</td><td></td><td></td><td>
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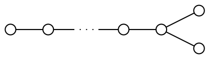
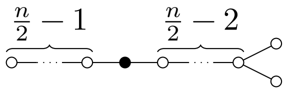
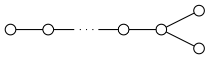
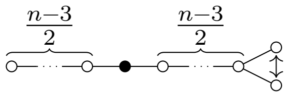
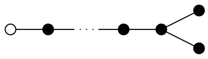
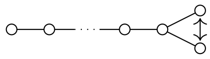
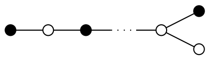
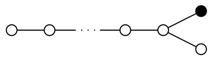
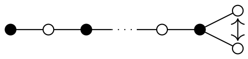
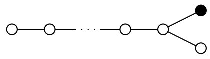
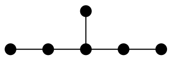
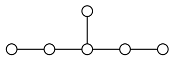
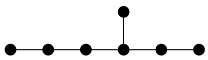
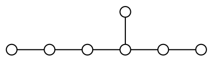
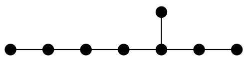
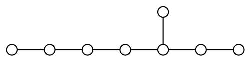
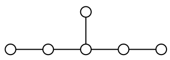
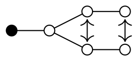
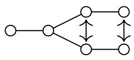
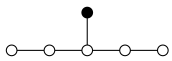
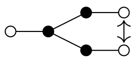
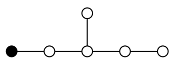
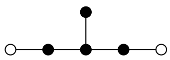
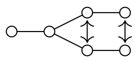
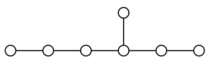
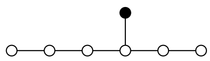
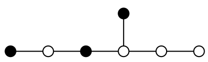
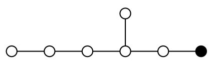
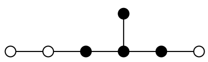
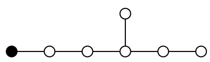
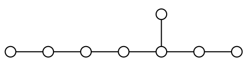
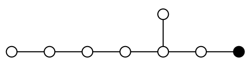
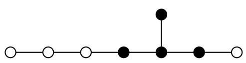
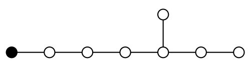
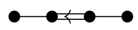
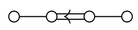
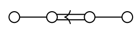
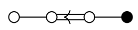
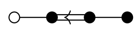
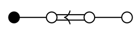
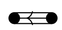
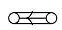
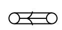
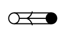
Type	$\mathfrak{g}$	$\mathfrak{k}$	Satake diagram	Vogan diagram
D I	$\mathfrak{so}(n,n)$, $n \equiv 0 \bmod 2$			
D I	$\mathfrak{so}(n,n)$, $n \equiv 1 \bmod 2$			
D II	$\mathfrak{so}(2n-1,1)$			
D III	$\mathfrak{so}^*(2n)$, $n \equiv 0 \bmod 2$			
D III	$\mathfrak{so}^*(2n)$, $n \equiv 1 \bmod 2$			
E	$\mathfrak{e}_6$	$\mathfrak{e}_6$		
E	$\mathfrak{e}_7$	$\mathfrak{e}_7$		
E	$\mathfrak{e}_8$	$\mathfrak{e}_8$		
E I		$\mathfrak{sp}(8)$		
E II		$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$		
E III		$\mathfrak{so}(10) \oplus \mathbb{R}$		
E IV		$\mathfrak{f}_4$		
E V		$\mathfrak{su}(8)$		
E VI		$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$		
E VII		$\mathfrak{e}_6 \oplus \mathbb{R}$		
E VIII		$\mathfrak{so}(16)$		
E IX		$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$		

Type	$\mathfrak{g}$	$\mathfrak{k}$	Satake diagram	Vogan diagram
F	$\mathfrak{f}_4$	$\mathfrak{f}_4$		
F I		$\mathfrak{sp}(6)\oplus\mathfrak{su}(2)$		
F II		$\mathfrak{so}(9)$		
G	$\mathfrak{g}_2$	$\mathfrak{g}_2$		
G I		$\mathfrak{su}(2)\oplus\mathfrak{su}(2)$		

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