

Topology I

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I. Homology: Ideas and Axioms

I.1. Higher Homotopy Groups

Recall. Let (X, x_0) be a pointed space. Then

$$\pi_1(X, x_0) = \{\gamma : (I, \{0, 1\}) \longrightarrow (X, x_0)\} / \simeq_{\{0,1\}}$$

with $[\gamma_1] \cdot [\gamma_2] = [\gamma_1 \gamma_2]$, where

$$(\gamma_1 \gamma_2)(t) = \begin{cases} \gamma_1(2t), & 0 \leq t \leq \frac{1}{2}, \\ \gamma_2(2t - 1), & \frac{1}{2} \leq t \leq 1, \end{cases}$$

is the *fundamental group* of (X, x_0) .

Examples.

$$\begin{aligned} \pi_1(S^1, \cdot) &\cong \mathbb{Z}, & \pi_1(T^n, \cdot) &\cong \mathbb{Z}^n, \\ \pi_1(\Sigma_g, \cdot) &\cong \left\langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1} \right\rangle, \\ \pi_1(S^n, \cdot) &\cong 1 \quad \text{for } n \geq 2. \end{aligned}$$

Is $S^2 \cong S^3$? We have

$$\pi_1(S^2 \setminus \{x_0, x_1\}, \cdot) \cong \pi_1(S^1, \cdot) \cong \mathbb{Z}, \quad \pi_1(S^3 \setminus \{x_0, x_1\}, \cdot) \cong \pi_1(S^2, \cdot) \cong 1,$$

so the answer is no. We seek a generalization “ π_n .” Idea: Let $\partial I^n = I^n \setminus (0, 1)^n$.

Definition I.1.1. The *n*th homotopy group is the set

$$\pi_n(X, x_0) = \{f : (I^n, \partial I^n) \longrightarrow (X, x_0)\} / \simeq_{\partial I^n}$$

with operation $[f] \cdot [g] = [fg]$, where

$$(fg)(x_1, \dots, x_n) = \begin{cases} f(2x_1, x_2, \dots, x_n), & 0 \leq x_1 \leq \frac{1}{2}, \\ g(2x_1 - 1, x_2, \dots, x_n), & \frac{1}{2} \leq x_1 \leq 1. \end{cases}$$

Geometric alternative. Since $I^n / \partial I^n \cong S^n$, we have

$$\begin{array}{ccc} (I^n, \partial I^n) & \xrightarrow{f} & (X, x_0) \\ \downarrow & \nearrow \bar{f} & \\ (S^n, \cdot) & & \end{array}$$

and $f \mapsto \bar{f}$ is a bijection. Thus

$$\pi_n(X, x_0) = \text{Hom}_{\text{HoTop}_\bullet}((S^n, \cdot), (X, x_0))$$

and

$$[f] \cdot [g] = \left[S^n \xrightarrow{\text{collapse the equator}} S^n \vee S^n \xrightarrow{\bar{f} \vee \bar{g}} X \right].$$

As before, for $n \geq 1$:

- $\pi_n(X, x_0)$ is a group.
- π_n is a functor that factors through $\mathbf{HoTop}_\bullet$:

$$\begin{array}{ccc} & \mathbf{HoTop}_\bullet & \\ \nearrow & & \searrow \overline{\pi_n} \\ \mathbf{Top}_\bullet & \xrightarrow{\pi_n} & \mathbf{Groups}. \end{array}$$

The following, however, is new:

Theorem I.1.2. For $n \geq 2$, $\pi_n(X, x_0)$ is abelian.

Proof. For $n = 2$, move the two smaller squares within the interior while mapping the boundary to x_0 :

$$\begin{array}{ccccccc} x_0 & \begin{array}{|c|c|} \hline f & g \\ \hline \end{array} & \simeq & \begin{array}{|c|c|} \hline \boxed{f} & \boxed{g} \\ \hline \end{array} & \simeq & \begin{array}{|c|c|} \hline \boxed{g} & \boxed{f} \\ \hline \end{array} & \simeq & \begin{array}{|c|c|} \hline g & f \\ \hline \end{array} \\ & x_0 & & & & & & \end{array}$$

The same argument works for $n > 2$. □

We have

$$\pi_\ell(S^n, \cdot) \cong \begin{cases} \mathbb{Z}, & \ell = n, \\ 0, & \ell < n \end{cases}$$

(proved later in Theorem III.4.14), and, for example, $\pi_3(S^2, \cdot) \cong \langle \eta \rangle \cong \mathbb{Z}$, with the *Hopf bundle*

$$\mathbb{C}^2 \supseteq S^3 \xrightarrow{\eta} S^2 \cong \mathbb{C} \cup \{\infty\}, \quad (z_1, z_2) \mapsto z_1/z_2.$$

The computation is difficult. For $\ell > n \geq 2$, the groups $\pi_\ell(S^n, \cdot)$ are only partially known. The problem: $\pi_1 : \mathbf{Top}_\bullet \rightarrow \mathbf{Groups}$ preserves suitable pushouts:

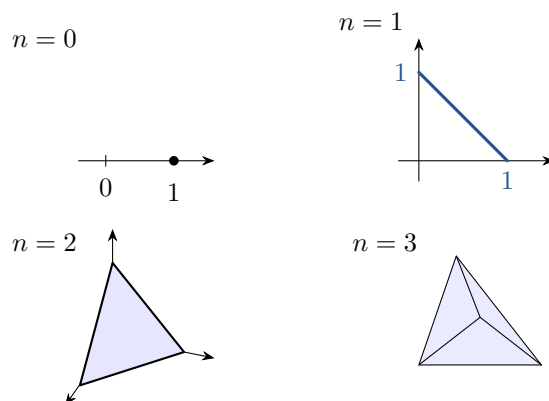
$$\begin{array}{ccc} A \longrightarrow Y & & \pi_1(A, \cdot) \longrightarrow \pi_1(Y, \cdot) \\ \text{cofibration} \downarrow & \lrcorner \downarrow & \downarrow \lrcorner \downarrow \\ X \longrightarrow Z & & \pi_1(X, \cdot) \longrightarrow \pi_1(Z, \cdot) \end{array} \quad \mapsto$$

but $\pi_n : \mathbf{Top}_\bullet \rightarrow \mathbf{Ab}$ does not for $n \geq 2$; for example, $|\pi_4(S^2 \vee S^2, \cdot)| = \infty$, but $|\pi_4(S^2, \cdot)| < \infty$. We therefore seek an invariant that distinguishes “higher-dimensional” objects and is easier to compute: homology!

I.2. The Idea of Homology

A “basic building block” of many topological spaces is the *standard n -simplex*

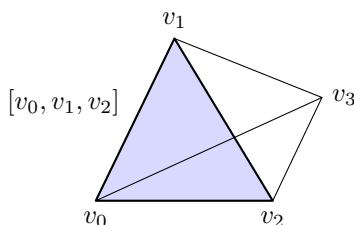
$$\Delta^n = \left\{ (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n x_i = 1, x_i \geq 0 \right\}.$$



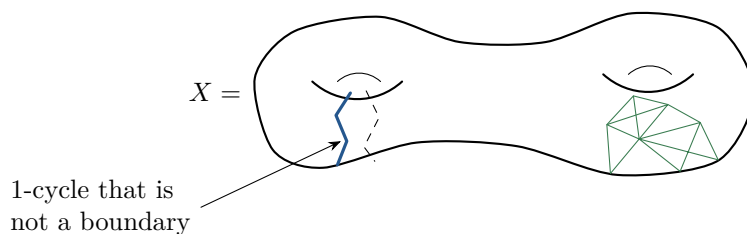
Equivalently, Δ^n is the *convex hull* of the standard basis $\{v_0, v_1, \dots, v_n\}$ of $\mathbb{R}^{n+1}$. Notation:

$$\Delta^n = [v_0, v_1, \dots, v_n].$$

We call v_i the i th *vertex* (plural: vertices) of Δ^n and $[v_0, \dots, \hat{v}_i, \dots, v_n]$ the i th *face*; the hat means “omitted.” Geometrically, the i th face lies “opposite” the i th vertex. Convention: $\emptyset$ is the only face of $[v_0]$.



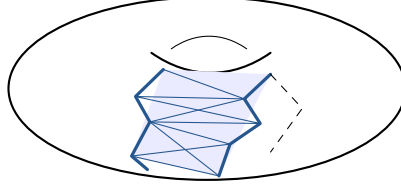
Many relevant topological spaces (for example, smooth manifolds) can be obtained by “gluing” simplices together:



- An n -chain is a choice of finitely many n -simplices (later this corresponds to an n -chain with coefficients in $\mathbb{Z}/2$).
- An n -boundary is the “boundary” of an $(n+1)$ -chain: the n -chain consisting of those faces of the simplices in the $(n+1)$ -chain that are not incident to an even number (usually two) of simplices.
- An n -cycle is an n -chain with empty boundary.
- Every n -boundary is an n -cycle (because the boundary of the boundary of an $(n+1)$ -simplex is empty).
- The n th *homology* measures the failure of the converse:

$$H_n(X; \mathbb{Z}/2) = \{n\text{-cycles}\} / \{n\text{-boundaries}\},$$

that is, we identify two n -cycles if together they form the boundary of an $(n+1)$ -chain (if they are *homologous*):



We can already anticipate connections between π_n and H_n , but also major differences; for example, $H_\ell(S^n; \mathbb{Z}/2) = 0$ for $\ell > n$ by definition.

I.3. Simplicial Homology

Goal. Make the concepts above precise ($\mathbb{Z}$ instead of $\mathbb{Z}/2$). Let

$$\partial\Delta^n := \bigcup_{i=0}^n [v_0, \dots, \widehat{v_i}, \dots, v_n], \quad \mathring{\Delta}^n := \Delta^n \setminus \partial\Delta^n$$

be the *boundary* and *interior* of the n -simplex.

Definition I.3.1. A Δ -complex is a topological space X with a family of maps $\{\sigma_\alpha^n : \Delta^n \rightarrow X\}$ for each $n \geq 0$, such that:

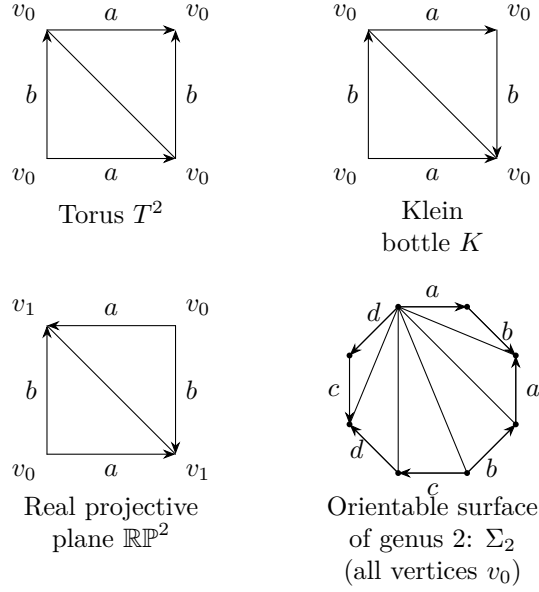
- (i) Each map $\sigma_\alpha^n|_{\mathring{\Delta}^n}$ is injective.
- (ii) Every point $x \in X$ lies in the image of $\sigma_\alpha^n|_{\mathring{\Delta}^n}$ for exactly one σ_α^n .
- (iii) Each restriction $\sigma_\alpha^n|_{[v_0, \dots, \widehat{v_i}, \dots, v_n]}$ equals some σ_β^{n-1} when we identify $[v_0, \dots, \widehat{v_i}, \dots, v_n]$ with $[v_0, \dots, v_{n-1}]$ by the unique linear homeomorphism preserving the order of the vertices (this convention is used from now on).
- (iv) A subset $A \subseteq X$ is open if and only if $(\sigma_\alpha^n)^{-1}(A) \subseteq \Delta^n$ is open for all σ_α^n .
- We call $\sigma_\alpha^n : \Delta^n \rightarrow X$ an n -simplex of the Δ -complex X .
- We have a filtration $X^0 \subseteq X^1 \subseteq \dots \subseteq X^\ell \subseteq \dots \subseteq X$ by n -skeleta,

$$X^n = \bigcup_{\sigma_\alpha^\ell, \ell \leq n} \text{im}(\sigma_\alpha^\ell).$$

- $\dim X = \ell$ if $X^\ell = X$ and $X^{\ell-1} \neq X$; $\dim X = \infty$ if no such ℓ exists.
- X is of *finite type* if, for each n , there are only finitely many n -simplices.
- X is *finite* if there are only finitely many simplices in total.

Definition I.3.2. A *simplicial complex* is a Δ -complex in which each simplex is determined by its vertices. This allows a combinatorial description (*abstract simplicial complex*).

Examples of Δ -complexes.



Note: As a simplicial complex, T^2 has at least 14 2-simplices, 21 1-simplices, and 7 vertices!

Definition I.3.3. Let X be a Δ -complex. The n th simplicial chain module $C_n^\Delta(X)$ is the free $\mathbb{Z}$ -module with basis $\{\sigma_\alpha^n : \Delta^n \rightarrow X\}$.

Formally, the forgetful functor $G : \mathbb{Z}\text{-Mod} \rightarrow \text{Set}$ has a left adjoint $F : \text{Set} \rightarrow \mathbb{Z}\text{-Mod}$, and

$$C_n^\Delta(X) := F(\{\sigma_\alpha^n : \Delta^n \rightarrow X\}).$$

Thus, elements of $C_n^\Delta(X)$ are formal $\mathbb{Z}$ -linear combinations

$$\sum_{\alpha} k_{\alpha} \sigma_{\alpha}^n \in C_n^\Delta(X),$$

where $k_{\alpha} = 0$ for all but finitely many α , and are called n -chains.

Definition I.3.4. The n th boundary map (or differential) is the homomorphism

$$\begin{aligned} \partial_n : C_n^\Delta(X) &\longrightarrow C_{n-1}^\Delta(X), \\ \sigma_{\alpha}^n &\longmapsto \sum_{i=0}^n (-1)^i \sigma_{\alpha}^n|_{[v_0, \dots, \widehat{v}_i, \dots, v_n]}. \end{aligned}$$

We also formally set $C_n^\Delta(X) = 0$ for $n < 0$, so $\partial_n = 0$ for $n \leq 0$.

Lemma I.3.5. We have $\partial_{n-1} \circ \partial_n = 0$.

Proof.

$$\begin{aligned} &\partial_{n-1} \left(\sum_{i=0}^n (-1)^i \sigma_{\alpha}^n|_{[v_0, \dots, \widehat{v}_i, \dots, v_n]} \right) \\ &= \sum_{i=0}^n \left(\sum_{j < i} (-1)^{i+j} \sigma_{\alpha}^n|_{[v_0, \dots, \widehat{v}_j, \dots, \widehat{v}_i, \dots, v_n]} + \sum_{j > i} (-1)^{i+j-1} \sigma_{\alpha}^n|_{[v_0, \dots, \widehat{v}_i, \dots, \widehat{v}_j, \dots, v_n]} \right) = 0. \end{aligned}$$

□

By the lemma, $(C_*^\Delta(X), \partial_*)$ is a *chain complex*, that is, a sequence of R -modules C_* over a commutative ring with 1 (in our case $R = \mathbb{Z}$), with homomorphisms

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots,$$

such that $\partial_n \circ \partial_{n+1} = 0$, or equivalently, $\text{im } \partial_{n+1} \subseteq \ker \partial_n$, or in words: “ n -boundaries are n -cycles.”

Definition I.3.6. The n th homology of a chain complex (C_*, ∂_*) is the R -module

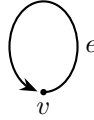
$$H_n(C_*) = \ker \partial_n / \text{im } \partial_{n+1}.$$

Definition I.3.7. The n th simplicial homology of a Δ -complex X is the $\mathbb{Z}$ -module

$$H_n^\Delta(X) = H_n(C_*^\Delta(X)).$$

Thus, two n -cycles $z_1, z_2 \in Z_n(X) := \ker \partial_n$ define the same homology class $[z_1] = [z_2] \in H_n^\Delta(X)$ if $z_2 - z_1 \in B_n(X) := \text{im } \partial_{n+1}$ is an n -boundary.

Example 1. $X = S^1$, with one vertex v and one edge e :

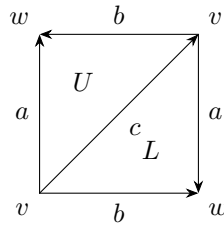


$$C_*^\Delta(X) = \left(\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}e \xrightarrow{\partial_1} \mathbb{Z}v \longrightarrow 0 \right).$$

$\partial_1(e) = v - v = 0$; that is, all differentials in the chain complex are trivial, so $H_*^\Delta(X) = C_*^\Delta(X)$, and hence

$$H_\ell(S^1) = \begin{cases} \mathbb{Z}, & \ell = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

Example 2. $X = \mathbb{RP}^2$.



The directions of the arrows determine the maps $\sigma_\alpha^n : \Delta^n \rightarrow X$.

$$\begin{aligned} \partial_2(U) &= -a + b + c, & \partial_1(a) &= -v + w, \\ \partial_2(L) &= a - b + c, & \partial_1(b) &= -v + w, \\ \partial_1(c) &= v - v = 0. \end{aligned}$$

$$\cdots \longrightarrow 0 \xrightarrow{\partial_3} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}} \mathbb{Z}^2 \longrightarrow 0.$$

The equality $\partial_2\left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right) = 0$ implies $n = m$ and $2n = 0$, so $\left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. Thus $Z_2(X) = 0$ and $H_2^\Delta(X) = 0$.

We have $\partial_1 \begin{pmatrix} k \\ \ell \\ m \end{pmatrix} = 0$ if and only if $k = -\ell$, so

$$\ker \partial_1 = \left\{ \begin{pmatrix} k \\ -k \\ m \end{pmatrix} : k, m \in \mathbb{Z} \right\}.$$

We have

$$\partial_2 \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} -r + s \\ r - s \\ r + s \end{pmatrix}.$$

Setting $p = -r + s$, we therefore obtain

$$\operatorname{im}(\partial_2) = \left\{ \begin{pmatrix} p \\ -p \\ p + 2q \end{pmatrix} : p, q \in \mathbb{Z} \right\}.$$

Writing $m = k + m'$, we see that $H_1^\Delta(X) \cong \mathbb{Z}/2\mathbb{Z}$. Finally, it follows from

$$\operatorname{im}(\partial_1) = \left\{ \begin{pmatrix} k \\ -k \end{pmatrix} : k \in \mathbb{Z} \right\},$$

that $H_0^\Delta(X) \cong \mathbb{Z}$, generated by $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \operatorname{im}(\partial_1)$. In summary:

$$H_\ell^\Delta(\mathbb{RP}^2) \cong \begin{cases} \mathbb{Z}, & \ell = 0, \\ \mathbb{Z}/2, & \ell = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Smith normal form provides an algorithm for computing the homology of a chain complex of finite-rank free modules over a principal ideal domain R : For $A \in \operatorname{Mat}(k, \ell; R)$, there exist $S \in \operatorname{GL}(k, R)$ and $T \in \operatorname{GL}(\ell, R)$ such that

$$SAT = \begin{pmatrix} a_1 & & & & \\ & \ddots & & & 0 \\ & & a_n & & \\ & & & 0 & \\ 0 & & & & \ddots \\ & & & & & 0 \end{pmatrix}, \quad n \leq \min\{k, \ell\}, \quad a_1 \mid a_2 \mid \cdots \mid a_n, \quad a_i \neq 0.$$

The a_i are called *invariant factors* and are unique up to multiplication by units. Thus homomorphisms between abstract finite-rank free R -modules also have well-defined invariant factors.

Lemma I.3.8. Let $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ be a chain complex of finite-rank free R -modules over a principal ideal domain R . Then

$$H_1(C_*) \cong R^b \oplus R/(a_1) \oplus \cdots \oplus R/(a_n),$$

where the a_i denote the invariant factors of ∂_2 and $b = \operatorname{rank}(\ker \partial_1) - n$.

Proof. $C_1^0 := \ker \partial_1$ is free and has a free complement C_1' , so that $C_1 = C_1^0 \oplus C_1'$. Find bases of C_2 and C_1^0 such that $\partial_2 : C_2 \rightarrow C_1^0$ has Smith normal form. Choose arbitrary bases of C_1' and C_0 . With the corresponding matrix representations of ∂_2 and ∂_1 , the lemma is clear. $\square$

I.4. Relative Simplicial Homology with Coefficients

Goal. For a commutative ring R with 1 and a subcomplex $A \subseteq X$, we define $H_n^\Delta(X, A; R)$.

Definition I.4.1. Let X be a Δ -complex. A *subcomplex* is a subspace $A \subseteq X$ that is a union of simplices. (X, A) is called a Δ -pair.

Note. Since $(\sigma_\alpha^n)^{-1}(A)$ is a union of faces of Δ^n , a subcomplex $A \subseteq X$ is always closed.

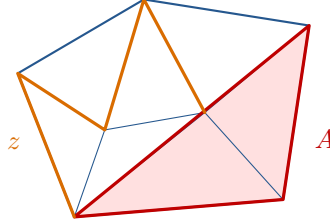
Definition I.4.2. The n th relative simplicial chain module of (X, A) is the quotient module

$$C_n^\Delta(X, A) = C_n^\Delta(X) / C_n^\Delta(A).$$

Since $\partial_n(C_n^\Delta(A)) \subseteq C_{n-1}^\Delta(A)$, we obtain

$$\partial_n : C_n^\Delta(X, A) \longrightarrow C_{n-1}^\Delta(X, A),$$

and we still have $\partial_{n-1} \circ \partial_n = 0$; that is, $(C_*^\Delta(X, A), \partial_*)$ is a chain complex. A *relative cycle* $z \in Z_n(X, A) := \ker \partial_n$ is represented by a chain whose boundary lies in A :



Definition I.4.3. The n th relative simplicial homology of a Δ -pair (X, A) is the $\mathbb{Z}$ -module

$$H_n^\Delta(X, A) = H_n(C_*^\Delta(X, A)).$$

Example. Let $X = \Delta^n$, $A = \partial\Delta^n$, with the canonical Δ -structure. Then

$$C_\ell^\Delta(X, A) \cong \begin{cases} \mathbb{Z}, & \ell = n, \\ 0, & \ell \neq n. \end{cases}$$

In particular, all differentials vanish, so we also have

$$H_\ell^\Delta(X, A) \cong \begin{cases} \mathbb{Z}, & \ell = n, \\ 0, & \ell \neq n. \end{cases}$$

Now let R be a commutative ring with 1 and let $C_n^\Delta(X; R)$ be the free R -module with basis $\{\sigma_\alpha^n : \Delta^n \rightarrow X\}$. Since $1 \in R$, we also have $-1 \in R$, and hence there is also a map $\partial_n : C_n^\Delta(X; R) \rightarrow C_{n-1}^\Delta(X; R)$ as in Definition I.3.4. Set

$$C_n^\Delta(X, A; R) = C_n^\Delta(X; R) / C_n^\Delta(A; R).$$

Since $\partial_n(C_n^\Delta(A; R)) \subseteq C_{n-1}^\Delta(A; R)$, these modules form a chain complex of free R -modules: The simplices in $\{\sigma_\alpha^n : \Delta^n \rightarrow X\}$ whose images are not contained in A form a basis. We obtain the n th relative homology with coefficients in R

$$H_n^\Delta(X, A; R) = H_n(C_*^\Delta(X, A; R)).$$

We define $C_n^\Delta(X; R) := C_n^\Delta(X, \emptyset; R)$ and $H_n^\Delta(X; R) := H_n^\Delta(X, \emptyset; R)$ and observe that $H_n^\Delta(X; \mathbb{Z}) = H_n^\Delta(X)$.

Example. $X = \mathbb{RP}^2$, $R = \mathbb{Z}/2\mathbb{Z}$ (a field, so R -modules are vector spaces; results from linear algebra apply!).

$$\partial_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \partial_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

$\text{rank}(\partial_2) = 1$, $\text{rank}(\partial_1) = 1$. By the rank-nullity theorem:

$$H_\ell^\Delta(\mathbb{RP}^2; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2, & \ell = 0, 1, 2, \\ 0, & \text{otherwise.} \end{cases}$$

Intuition. $H_n^\Delta(X, A; R)$ measures how much $H_n^\Delta(X; R)$ differs from $H_n^\Delta(A; R)$. We want to make this precise.

Definition I.4.4. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of morphisms in $R\text{-Mod}$ is *exact* at B if $\ker g = \text{im } f$.

Note (for such an exact sequence):

- g is injective if and only if $f = 0$.
- f is surjective if and only if $g = 0$.

A *long exact sequence* in $R\text{-Mod}$ is a sequence

$$\cdots \longrightarrow A_{n+1} \longrightarrow A_n \longrightarrow A_{n-1} \longrightarrow \cdots,$$

that is exact at every term.

Theorem I.4.5. (Long exact sequence) For a Δ -pair (X, A) , there is a long exact sequence

$$\cdots \xrightarrow{d_{n+1}} H_n^\Delta(A; R) \xrightarrow{i_n} H_n^\Delta(X; R) \xrightarrow{j^n} H_n^\Delta(X, A; R) \xrightarrow{d_n} H_{n-1}^\Delta(A; R) \longrightarrow \cdots.$$

Here i_n and j^n are induced by

$$Z_n^\Delta(A) \longrightarrow Z_n^\Delta(X), \quad Z_n^\Delta(X) \longrightarrow Z_n^\Delta(X, A),$$

and d_n is given by the boundary of a representative:

$$d_n([c + C_n^\Delta(A)]) = [\partial c] \in H_{n-1}^\Delta(A), \quad \partial c \in C_{n-1}^\Delta(A).$$

(We often suppress the coefficient ring R in the notation.)

Proof. Exactness at $H_n^\Delta(A)$: Let $z + B_n^\Delta(A) \in \ker i_n$. Then there is a $c \in C_{n+1}^\Delta(X)$ with $\partial c = z$, so $c + C_{n+1}^\Delta(A)$ is a relative cycle and

$$d_{n+1}([c + C_{n+1}^\Delta(A)]) = z + B_n^\Delta(A).$$

Conversely, let $z + B_n^\Delta(A) \in \text{im } d_{n+1}$. Then there exist $c \in C_{n+1}^\Delta(X)$ and $b \in C_{n+1}^\Delta(A)$ with $\partial c = z + \partial b$, so $z = \partial(c - b) \in B_n^\Delta(X)$; that is, $i_n(z + B_n^\Delta(A)) = 0$.

Exactness at $H_n^\Delta(X; R)$: Let $z + B_n^\Delta(X) \in \ker j^n$. Then the relative class of z is a relative boundary; that is, there is a $c \in C_{n+1}^\Delta(X)$ with $z - \partial c \in C_n^\Delta(A)$, and since $\partial(z - \partial c) = \partial z = 0$, we even have $z - \partial c \in Z_n^\Delta(A)$. We obtain

$$i_n(z - \partial c + B_n^\Delta(A)) = z - \partial c + B_n^\Delta(X) = z + B_n^\Delta(X).$$

Conversely, let $z + B_n^\Delta(X) \in \text{im } i_n$. Then there is a $c \in C_{n+1}^\Delta(X)$ such that $z - \partial c \in Z_n^\Delta(A)$, so the relative class of z lies in $B_n^\Delta(X, A)$; that is, $j^n(z + B_n^\Delta(X)) = 0$.

Exactness at $H_n^\Delta(X, A)$: Exercise. □

Corollary I.4.6.

$$(H_n^\Delta(X, A) = 0 \text{ for } n \geq 0) \iff (H_n^\Delta(A) \xrightarrow[\cong]{i_n} H_n^\Delta(X) \text{ for } n \geq 0). \quad \square$$

Question. Is it always true that $H_n^\Delta(A) \subseteq H_n^\Delta(X)$ and $H_n^\Delta(X, A) \cong H_n^\Delta(X)/H_n^\Delta(A)$?

Answer. No, only if $d_{n+1} = 0$ and $d_n = 0$. In that case,

$$0 \longrightarrow H_n^\Delta(A) \longrightarrow H_n^\Delta(X) \longrightarrow H_n^\Delta(X, A) \longrightarrow 0$$

is a *short exact sequence*.

Question. In this case, is it always true that $H_n^\Delta(X) \cong H_n^\Delta(A) \oplus H_n^\Delta(X, A)$?

Answer. No, only if the short exact sequence *splits*:

Lemma I.4.7. (Splitting lemma) Let

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

be a short exact sequence of R -modules. Then the following are equivalent:

- (i) There is a $p : B \rightarrow A$ with $p \circ i = \text{id}_A$.
- (ii) There is an $s : C \rightarrow B$ with $j \circ s = \text{id}_C$.
- (iii) There is an isomorphism $B \cong A \oplus C$ such that

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \begin{array}{c} \nearrow i \\ \searrow i_A \end{array} & B & \begin{array}{c} \nwarrow j \\ \downarrow \cong \\ \nearrow \text{pr}_C \end{array} & C \longrightarrow 0 \\ & & & & A \oplus C & & \end{array}$$

commutes.

Proof. (iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii) are clear.

(ii) $\Rightarrow$ (iii): We must show that $B \cong \text{im}(i) \oplus \text{im}(s)$. To this end, write $b = (b - s(j(b))) + s(j(b))$, where $b - s(j(b)) \in \ker j = \text{im}(i)$. If $b \in \text{im}(i) \cap \text{im}(s)$, then $b = s(j(b'))$ for some $b' \in B$. Since $\text{im}(i) = \ker j$, we have $0 = j(b) = j(s(j(b')))) = j(b')$, so $b = s(j(b')) = s(0) = 0$.

(i) $\Rightarrow$ (iii): We must show that $B \cong \text{im}(i) \oplus \ker(p)$. Write $b = (b - i(p(b))) + i(p(b))$. If $b \in \text{im}(i) \cap \ker(p)$, then $b = i(a)$ for some $a \in A$ and $0 = p(b) = p(i(a)) = a$, so $b = i(a) = i(0) = 0$. It remains to show that $\ker p \cong C$: For every $c \in C$, there is a $b \in B$ with $j(b) = c$ and $b = i(a) + k$ with $k \in \ker p$, so $j(b) = j(i(a) + k) = j(k)$. If $k \in \ker p \cap \ker j$, then $k = 0$ because $\ker j = \text{im } i$; that is, $j|_{\ker p} : \ker p \xrightarrow{\cong} C$. $\square$

Note.

- By (ii), a short exact sequence always splits if C is a free R -module.
- The splitting lemma does not hold in Groups. Example: For $1 \rightarrow A_3 \rightarrow S_3 \xrightarrow{\text{sgn}} \{\pm 1\} \rightarrow 1$, (ii) holds but (i) does not, because $\{\pm 1\}$ cannot be lifted to a normal subgroup of S_3 .

Theorem I.4.8. (Excision) Let X be a Δ -complex and let $A \subseteq Y \subseteq X$ be subcomplexes with $A \subseteq \mathring{Y}$, where $\mathring{Y}$ denotes the interior. Then the inclusion of Δ -pairs

$$(X \setminus \mathring{A}, Y \setminus \mathring{A}) \longrightarrow (X, Y)$$

induces isomorphisms for all $n \geq 0$:

$$H_n^\Delta(X \setminus \mathring{A}, Y \setminus \mathring{A}; R) \xrightarrow{\cong} H_n^\Delta(X, Y; R).$$

Proof. We prove the stronger statement that j induces an isomorphism

$$C_*^\Delta(X \setminus \mathring{A}, Y \setminus \mathring{A}; R) \xrightarrow[\cong]{j_*} C_*^\Delta(X, Y; R)$$

of chain complexes. The commutativity relation $j^{n-1} \circ \partial_n = \partial_n \circ j^n$ is clear. We show that j_* is surjective: Let $c + C_n^\Delta(Y) \in C_n^\Delta(X, Y)$, and assume without loss of generality that c contains no simplices in Y . Since $X \setminus Y \subseteq X \setminus A$, $c + C_n^\Delta(Y \setminus \mathring{A})$ is a preimage under j_* . We show that j_* is injective: Let $c + C_n^\Delta(Y \setminus \mathring{A}) \in \ker j_*$. Then

$$c \in C_n^\Delta(X \setminus \mathring{A}) \cap C_n^\Delta(Y) = C_n^\Delta(Y \setminus \mathring{A}).$$

□

I.5. The Eilenberg–Steenrod Axioms

Goal. An axiomatic description of homology. Drawbacks of simplicial homology:

- It is defined only for Δ -pairs.
- The category of Δ -pairs is “unwieldy” if $H_n^\Delta : \Delta\text{-pairs} \rightarrow R\text{-Mod}$ is to be a functor.
- It is unclear whether $f \simeq g$ implies $H_n^\Delta(f) = H_n^\Delta(g)$.

Hence the following wish list: Homology should be a family of functors $H_n : \mathbf{Top}^{(2)} \rightarrow R\text{-Mod}$ with

1) a factorization

$$\begin{array}{ccc} & \mathbf{HoTop}^{(2)} & \\ \nearrow & & \searrow \\ \mathbf{Top}^{(2)} & \xrightarrow{H_n} & R\text{-Mod}, \end{array}$$

2) a long exact sequence for every pair of spaces (X, A) ,

3) an excision isomorphism for $A \subseteq Y \subseteq X$ with $\overline{A} \subseteq \mathring{Y}$.

Definition I.5.1. A *homology theory* with values in $R\text{-Mod}$ is a family of functors $(H_n)_{n \in \mathbb{Z}}$,

$$H_n : \mathbf{Top}^{(2)} \longrightarrow R\text{-Mod},$$

together with a family of natural transformations

$$\partial_n : H_n \longrightarrow H_{n-1} \circ J,$$

where $J : \mathbf{Top}^{(2)} \rightarrow \mathbf{Top}^{(2)}$ is defined by $J(X, A) = (A, \emptyset)$ and

$$J(f : (X, A) \rightarrow (Y, B)) = f|_A : (A, \emptyset) \rightarrow (B, \emptyset)$$

such that:

- 1) **Homotopy invariance.** If $f, g : (X, A) \rightarrow (Y, B)$ satisfy $f \simeq_h g$ with $h_t(A) \subseteq B$ for all $t \in I$, then $H_n(f) = H_n(g)$.
- 2) **Long exact sequence.** Associated with (X, A) are the inclusions $i : (A, \emptyset) \rightarrow (X, \emptyset)$ and $j : (X, \emptyset) \rightarrow (X, A)$, which induce the long exact sequence

$$\cdots \rightarrow H_n(A, \emptyset) \xrightarrow{H_n(i)} H_n(X, \emptyset) \xrightarrow{H_n(j)} H_n(X, A) \xrightarrow{\partial_n(X, A)} H_{n-1}(A, \emptyset) \rightarrow \cdots$$

- 3) **Excision.** Let $A, Y \subseteq X$ be subspaces such that $\bar{A} \subseteq \mathring{Y}$. Then for every $n \in \mathbb{Z}$, the inclusion $j : (X \setminus A, Y \setminus A) \rightarrow (X, Y)$ induces an isomorphism

$$H_n(X \setminus A, Y \setminus A) \xrightarrow[\cong]{H_n(j)} H_n(X, Y).$$

The homology theory is also called *ordinary* if the *dimension axiom* holds:

- 4) $H_n(\cdot, \emptyset) = 0$ for $n \neq 0$.

Otherwise, it is called *generalized*. We write $H_n(X)$ for $H_n(X, \emptyset)$. The R -modules $H_n(\cdot)$ are called the *coefficient modules* of the homology theory.

Note. If (H_*, ∂_*) is a homology theory, then $f : (X, A) \rightarrow (Y, B)$ induces a “commutative ladder with exact rows”:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H_{n+1}(X, A) & \xrightarrow{\partial_{n+1}(X, A)} & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) & \longrightarrow & \cdots \\ & & \downarrow H_{n+1}(f) & & \downarrow H_n(f|_A) & & \downarrow H_n(f) & & \downarrow H_n(f) & & \\ \cdots & \longrightarrow & H_{n+1}(Y, B) & \xrightarrow{\partial_{n+1}(Y, B)} & H_n(B) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(Y, B) & \longrightarrow & \cdots \end{array}$$

The left square commutes because ∂ is a natural transformation.

II. Singular Homology

Goal: Construction of an ordinary homology theory.

II.1. Definition of Singular Homology

Let R be a commutative ring with 1. We define H_n^{sing} as the composition of two functors

$$\text{Top}^{(2)} \xrightarrow{C_*(-;R)} R\text{-chain} \xrightarrow{H_n} R\text{-mod}.$$

Morphisms in $R\text{-chain}$ are *chain maps*, that is, families $\{f_*: C_* \rightarrow D_*\}$ of R -module homomorphisms such that $d_n \circ f_n = f_{n-1} \circ c_n$, that is,

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{c_{n+1}} & C_n & \xrightarrow{c_n} & C_{n-1} \longrightarrow \cdots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} \\ \cdots & \longrightarrow & D_{n+1} & \xrightarrow{d_{n+1}} & D_n & \xrightarrow{d_n} & D_{n-1} \longrightarrow \cdots \end{array}$$

For $z \in \ker c_n$ we have $d_n(f_n(z)) = f_{n-1}(c_n(z)) = 0$, so $f_n(\ker c_n) \subseteq \ker d_n$, and for $b \in \text{im } c_{n+1}$, $b = c_{n+1}(x)$, we have

$$f_n(b) = f_n(c_{n+1}(x)) = d_{n+1}(f_{n+1}(x)),$$

so $f_n(\text{im } c_{n+1}) \subseteq \text{im } d_{n+1}$.

- f_* induces $H_n(f_*): H_n(C_*) \rightarrow H_n(D_*)$.
- $H_n(g_* \circ f_*) = H_n(g_*) \circ H_n(f_*)$, $H_n(\text{id}_{C_*}) = \text{id}_{H_n(C_*)}$.
- $H_n: R\text{-chain} \rightarrow R\text{-mod}$ is a functor!

Now let X be a topological space.

Definition II.1.1. The n th *singular chain module* $C_n^{(\text{sing})}(X; R)$ is the free R -module with basis $\{\sigma^n: \Delta^n \rightarrow X, \sigma^n \text{ continuous}\}$. (Convention: $C_n(X; R) = 0$ for $n < 0$.)

- “Singular” because all continuous maps $\sigma^n: \Delta^n \rightarrow X$ are allowed, however singular:



- *Singular boundary map*

$$\begin{aligned} \partial_n: C_n(X; R) &\longrightarrow C_{n-1}(X; R), \\ \sigma_n &\longmapsto \sum_{i=0}^n (-1)^i \sigma_n|_{[v_0, \dots, \widehat{v_i}, \dots, v_n]}. \end{aligned}$$

- $\partial_n \circ \partial_{n+1} = 0$ as before.
- For $(X, A) \in \text{Ob}_{\text{Top}^{(2)}}$, define $C_n(X, A; R) := C_n(X; R)/C_n(A; R)$.

- $f: (X, A) \rightarrow (Y, B)$ induces

$$C_n(f; R): C_n(X, A; R) \longrightarrow C_n(Y, B; R),$$

$$[\sigma_n: \Delta^n \rightarrow X] \longmapsto [f \circ \sigma_n].$$

- By construction, $C_n(-; R)$ is functorial.
- $\partial_n(C_n(A; R)) \subseteq C_{n-1}(A; R)$, so we obtain $\partial_n: C_n(X, A; R) \rightarrow C_{n-1}(X, A; R)$.
- Clearly, $\partial_n \circ C_n(f) = C_{n-1}(f) \circ \partial_n$, so $C_*(f)$ is a chain map.
- $C_*(-; R): \text{Top}^{(2)} \rightarrow R\text{-chain}$ is a functor!

Definition II.1.2. The n th *singular homology* of a pair of spaces (X, A) with coefficients in R is the R -module

$$H_n^{\text{sing}}(X, A; R) = H_n(C_*(X, A; R)).$$

Note. $H_n^{\text{sing}} = H_n \circ C_*(-; R)$ is a functor.

Still to be constructed: a natural transformation

$$\partial_n(X, A): H_n^{\text{sing}}(X, A; R) \longrightarrow H_{n-1}^{\text{sing}}(A; R).$$

II.2. The Long Exact Sequence of a Pair of Spaces

Goal. Construct the natural transformation $\partial_n: H_n^{\text{sing}} \rightarrow H_{n-1}^{\text{sing}} \circ J$ purely algebraically and prove the long exact sequence axiom for $(H_*^{\text{sing}}, \partial_*)$.

Lemma II.2.1. (Snake Lemma) A diagram in $R\text{-mod}$

$$\begin{array}{ccccccc} 0 & \cdots \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{j} & C \longrightarrow 0 \\ & & f \downarrow & & g \downarrow & & h \downarrow \\ 0 & \longrightarrow & A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' \cdots \longrightarrow 0 \end{array}$$

with exact rows induces a natural exact sequence

$$\begin{aligned} 0 \cdots \rightarrow \ker f &\longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\delta} \text{coker } f \\ &\longrightarrow \text{coker } g \longrightarrow \text{coker } h \cdots \rightarrow 0 \end{aligned}$$

in the following diagram:

$$\begin{array}{ccccccc} 0 & \cdots \longrightarrow & \ker f & \longrightarrow & \ker g & \longrightarrow & \ker h \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \cdots \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{j} & C \longrightarrow 0 \\ & & f \downarrow & & g \downarrow & & h \downarrow \\ 0 & \longrightarrow & A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' \cdots \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \text{coker } f & \longrightarrow & \text{coker } g & \longrightarrow & \text{coker } h \cdots \longrightarrow 0 \end{array}$$

(An orange arrow labeled δ connects $\ker h$ to $\text{coker } f$.)

Proof. Diagram chase. To construct δ :

- For $c \in \ker h$, choose $b \in B$ with $j(b) = c$. Since $j'(g(b)) = h(j(b)) = h(c) = 0$, we have $g(b) \in \ker j' = \text{im}(i')$. Thus there is $a' \in A'$ with $i'(a') = g(b)$. Set $\delta(c) = a' + \text{im } f \in \text{coker } f$.
- If $b, b' \in B$ satisfy $j(b) = j(b') = c \in \ker h$, choose $a, a' \in A'$ with $i'(a) = g(b)$ and $i'(a') = g(b')$. Then $b - b' = i(a_0)$ for some $a_0 \in A$, so

$$i'(a - a') = g(b - b') = g(i(a_0)) = i'(f(a_0)),$$

that is, $a - a' = f(a_0) \in \text{im } f$, since i' is injective.

- Let $c_1, c_2 \in \ker h$ and

$$\delta(c_1) = a'_1 + \text{im } f, \quad \delta(c_2) = a'_2 + \text{im } f, \quad \delta(c_1 + c_2) = a'_3 + \text{im } f$$

and, as above, choose $b_i \in B$ with $g(b_i) = i'(a'_i)$ and $j(b_1) = c_1, j(b_2) = c_2, j(b_3) = c_1 + c_2$. Then

$$i'(a'_1 + a'_2 - a'_3) = g(b_1 + b_2 - b_3)$$

and $j(b_1 + b_2 - b_3) = c_1 + c_2 - (c_1 + c_2) = 0$, so there is $a \in A$ with $i(a) = b_1 + b_2 - b_3$, and hence

$$i'(a'_1 + a'_2 - a'_3) = g(i(a)) = i'(f(a)),$$

so $a'_1 + a'_2 - a'_3 = f(a) \in \text{im } f$, since i' is injective.

□

A *short exact sequence of chain complexes* is a sequence of chain maps

$$0 \longrightarrow A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \longrightarrow 0,$$

such that, for every $n \in \mathbb{Z}$,

$$0 \longrightarrow A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \longrightarrow 0$$

is a short exact sequence of R -modules.

Theorem II.2.2. A short exact sequence of chain complexes

$$0 \longrightarrow C_* \longrightarrow D_* \longrightarrow E_* \longrightarrow 0$$

induces a natural long exact sequence of R -modules

$$\cdots \longrightarrow H_{n+1}(E_*) \xrightarrow{\delta} H_n(C_*) \longrightarrow H_n(D_*) \longrightarrow H_n(E_*) \xrightarrow{\delta} \cdots$$

Proof. Consider:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_{n+1} & \longrightarrow & D_{n+1} & \longrightarrow & E_{n+1} & \longrightarrow & 0 \\ & & \downarrow c_{n+1} & & \downarrow d_{n+1} & & \downarrow e_{n+1} & & \\ 0 & \longrightarrow & C_n & \longrightarrow & D_n & \longrightarrow & E_n & \longrightarrow & 0. \end{array}$$

Snake Lemma:

$$C_n/B_n(C_*) \longrightarrow D_n/B_n(D_*) \longrightarrow E_n/B_n(E_*) \longrightarrow 0 \quad \text{is exact.}$$

Consider:

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_{n-1} & \longrightarrow & D_{n-1} & \longrightarrow & E_{n-1} \longrightarrow 0 \\ & & \downarrow c_{n-1} & & \downarrow d_{n-1} & & \downarrow e_{n-1} \\ 0 & \longrightarrow & C_{n-2} & \longrightarrow & D_{n-2} & \longrightarrow & E_{n-2} \longrightarrow 0. \end{array}$$

Snake Lemma:

$$0 \longrightarrow Z_{n-1}(C_*) \longrightarrow Z_{n-1}(D_*) \longrightarrow Z_{n-1}(E_*) \quad \text{is exact.}$$

Consider

$$\begin{array}{ccccccc} C_n/B_n(C_*) & \longrightarrow & D_n/B_n(D_*) & \longrightarrow & E_n/B_n(E_*) & \longrightarrow & 0 \\ \bar{c}_n \downarrow & & \bar{d}_n \downarrow & & \bar{e}_n \downarrow & & \\ 0 & \longrightarrow & Z_{n-1}(C_*) & \longrightarrow & Z_{n-1}(D_*) & \longrightarrow & Z_{n-1}(E_*). \end{array}$$

Snake Lemma:

$$H_n(C_*) \longrightarrow H_n(D_*) \longrightarrow H_n(E_*) \xrightarrow{\delta} H_{n-1}(C_*) \longrightarrow H_{n-1}(D_*) \longrightarrow H_{n-1}(E_*)$$

is exact (and natural). □

Let (X, A) be a pair of spaces. Applying this to

$$0 \longrightarrow C_*^{\text{sing}}(A; R) \longrightarrow C_*^{\text{sing}}(X; R) \longrightarrow C_*^{\text{sing}}(X, A; R) \longrightarrow 0$$

gives the natural transformation $\partial_n: H_n^{\text{sing}} \rightarrow H_{n-1}^{\text{sing}} \circ J$ as the connecting homomorphism δ , and:

Theorem II.2.3. The pair $(H_*^{\text{sing}}, \partial_*)$ satisfies the long exact sequence axiom of a homology theory with values in $R\text{-mod}$.

Remark. More generally, for a triple $B \subseteq A \subseteq X$, the short exact sequence

$$0 \longrightarrow C_*(A, B; R) \longrightarrow C_*(X, B; R) \longrightarrow C_*(X, A; R) \longrightarrow 0$$

gives the long exact sequence

$$\cdots \longrightarrow H_{n+1}(X, A; R) \longrightarrow H_n(A, B; R) \longrightarrow H_n(X, B; R) \longrightarrow H_n(X, A; R) \longrightarrow \cdots$$

II.3. Homotopy Invariance

Goal: Show that $f \simeq g \implies H_n^{\text{sing}}(f) = H_n^{\text{sing}}(g)$.

Definition II.3.1. An R -chain homotopy between R -chain maps

$$f_*, g_*: (C_*, c_*) \longrightarrow (D_*, d_*)$$

is a family of R -module homomorphisms $h_n: C_n \rightarrow D_{n+1}$ such that

$$f_n - g_n = h_{n-1} \circ c_n + d_{n+1} \circ h_n.$$

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & C_{n-1} \longrightarrow \cdots \\ & & \downarrow \downarrow & \nearrow h_n & \downarrow \downarrow & \nearrow h_{n-1} & \downarrow \downarrow \\ \cdots & \longrightarrow & D_{n+1} & \longrightarrow & D_n & \longrightarrow & D_{n-1} \longrightarrow \cdots \end{array}$$

Notation: $f_* \simeq_{h_*} g_*$.

Lemma II.3.2. If $f_* \simeq_{h_*} g_*$, then $H_n(f_*) = H_n(g_*)$ for $n \in \mathbb{Z}$.

Proof. For $z \in Z_n(C_*)$, we have

$$f_n(z) = g_n(z) + \underbrace{h_{n-1}(c_n(z))}_{=0} + \underbrace{d_{n+1}(h_n(z))}_{\in B_n(D_*)}.$$

□

Note. Chain homotopy is an equivalence relation. For $f_*^1, g_*^1: C_* \rightarrow D_*$, $f_*^2, g_*^2: D_* \rightarrow E_*$ with $f_*^1 \simeq g_*^1$ and $f_*^2 \simeq g_*^2$, we have $f_*^2 \circ f_*^1 \simeq g_*^2 \circ g_*^1$.

The converse of Lemma II.3.2 is false (homework).

A *chain homotopy equivalence* is a chain map

$$f_*: (C_*, c_*) \longrightarrow (D_*, d_*)$$

with a *chain homotopy inverse* $g_*: D_* \rightarrow C_*$ such that $g_* \circ f_* \simeq \text{id}_{C_*}$ and $f_* \circ g_* \simeq \text{id}_{D_*}$. Notation: $(C_*, c_*) \simeq_{f_*} (D_*, d_*)$.

Lemma II.3.3. If $(C_*, c_*) \simeq_{f_*} (D_*, d_*)$, then $H_n(f_*)$ is an isomorphism for every $n \in \mathbb{Z}$.

Proof. By Lemma II.3.2,

$$H_n(g_* \circ f_*) = H_n(\text{id}_{C_*}) \quad \text{and} \quad H_n(f_* \circ g_*) = H_n(\text{id}_{D_*}).$$

Since we have already seen that $H_n: R\text{-chain} \rightarrow R\text{-mod}$ is a functor, it follows that

$$H_n(g_*) \circ H_n(f_*) = \text{id}_{H_n(C_*)} \quad \text{and} \quad H_n(f_*) \circ H_n(g_*) = \text{id}_{H_n(D_*)}.$$

□

Remark. Partial converse: If R is a principal ideal domain and C_* and D_* consist of free R -modules, then a chain map $f_*: C_* \rightarrow D_*$ with $H_n(f_*): H_n(C_*) \xrightarrow{\cong} H_n(D_*)$ for all $n \in \mathbb{Z}$ is a chain homotopy equivalence (without proof).

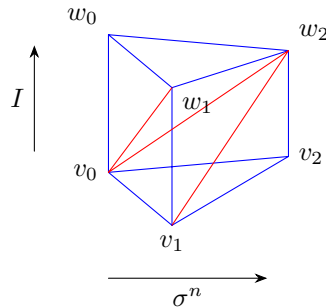
First we show: If $f, g: X \rightarrow Y$, $f \simeq_F g$, then $H_n^{\text{sing}}(f) = H_n^{\text{sing}}(g)$. Set $f_* := C_*^{\text{sing}}(f)$, $g_* := C_*^{\text{sing}}(g)$. By Lemma II.3.2, we need to construct $g_* \simeq_{P_*} f_*$, $P_n: C_n^{\text{sing}}(X; R) \rightarrow C_{n+1}^{\text{sing}}(Y; R)$. For $\sigma^n: \Delta^n \rightarrow X$, consider

$$\Delta^n \times I \xrightarrow{\sigma^n \times \text{id}_I} X \times I \xrightarrow{F} Y$$

and define

$$P_n(\sigma^n) = \sum_{i=0}^n (-1)^i (F \circ (\sigma^n \times \text{id}_I))|_{[v_0, \dots, v_i, w_i, \dots, w_n]},$$

where $v_0, \dots, v_i$ are vertices in the “bottom of the prism” $\Delta^n \times \{0\}$ and $w_i, \dots, w_n$ are vertices in the “top of the prism” $\Delta^n \times \{1\}$:



We need to show:

$$\underbrace{\partial_{n+1} \circ P_n}_{\text{boundary of the prism}} = \underbrace{g_n}_{\text{top}} - \underbrace{f_n}_{\text{bottom}} - \underbrace{P_{n-1} \circ \partial_n}_{\text{three side faces}}.$$

Formally:

$$\begin{aligned} \partial_{n+1}(P_n(\sigma^n)) &= \sum_{i=0}^n (-1)^i \partial_{n+1}(F \circ (\sigma^n \times \text{id}_I))|_{[v_0, \dots, v_i, w_i, \dots, w_n]} \\ &= \sum_{0 \leq j \leq i \leq n} (-1)^i (-1)^j F \circ (\sigma^n \times \text{id}_I)|_{[v_0, \dots, \widehat{v}_j, \dots, v_i, w_i, \dots, w_n]} \\ &\quad + \sum_{0 \leq i \leq j \leq n} (-1)^{j+1} (-1)^i F \circ (\sigma^n \times \text{id}_I)|_{[v_0, \dots, v_i, w_i, \dots, \widehat{w}_j, \dots, w_n]} \\ &= \sum_{i=0}^n F \circ (\sigma^n \times \text{id}_I)|_{[v_0, \dots, \widehat{v}_i, w_i, \dots, w_n]} - \sum_{i=0}^n F \circ (\sigma^n \times \text{id}_I)|_{[v_0, \dots, v_i, \widehat{w}_i, \dots, w_n]} \\ &\quad + \sum_{j < i} (\dots) + \sum_{i < j} (\dots) \\ &= F \circ (\sigma^n \times \text{id}_I)|_{[w_0, \dots, w_n]} - F \circ (\sigma^n \times \text{id}_I)|_{[v_0, \dots, v_n]} \\ &\quad - \left(\sum_{i < j} (-1)^i (-1)^j F \circ (\sigma^n \times \text{id}_I)|_{[v_0, \dots, v_i, w_i, \dots, \widehat{w}_j, \dots, w_n]} \right. \\ &\quad \left. + \sum_{j < i} (-1)^{i-1} (-1)^j F \circ (\sigma^n \times \text{id}_I)|_{[v_0, \dots, \widehat{v}_j, \dots, v_i, w_i, \dots, w_n]} \right) \\ &= g_n(\sigma^n) - f_n(\sigma^n) - P_{n-1}(\partial_n(\sigma^n)). \end{aligned}$$

Now, for $f, g: (X, A) \rightarrow (Y, B)$, $f \simeq_F g$, $F_t(A) \subseteq B$ for all $t \in I$, we have $P_n(C_n^{\text{sing}}(A; R)) \subseteq C_{n+1}^{\text{sing}}(B; R)$, so we obtain

$$P_n: C_n^{\text{sing}}(X, A; R) \longrightarrow C_{n+1}^{\text{sing}}(Y, B; R)$$

and still have $g_n - f_n = \partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n$. We have thus proved:

Theorem II.3.4. The pair $(H_*^{\text{sing}}, \partial_*)$ satisfies the homotopy invariance axiom of a homology theory with values in $R\text{-mod}$.

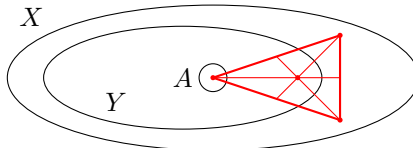
II.4. Excision

Goal: Show that, for $A^\circ \subseteq Y \subseteq X$ with $\overline{A} \subset Y^\circ$,

$$H_n^{\text{sing}}(X \setminus A, Y \setminus A; R) \cong H_n^{\text{sing}}(X, Y; R).$$

Recall: For Δ -complexes $A \subset Y \subset X$ with $A \subset Y^\circ$, we have $C_n^\Delta(X \setminus A, Y \setminus A; R) \cong C_n^\Delta(X, Y; R)$. One cannot expect such an isomorphism at the chain complex level for singular homology!

Idea: Subdivide simplices so that they lie either in $X \setminus A$ or in Y !



For this purpose, consider more generally a family $\mathcal{U} = \{U_j\}$ of subspaces $U_j \subseteq X$ with $\bigcup_j U_j^\circ = X$. Let

$$C_n^{\text{sing}}(X; R) \supseteq C_n^{\mathcal{U}}(X; R) := R\{\sigma^n: \Delta^n \rightarrow X : \sigma^n(\Delta^n) \subseteq U_j \text{ for some } j\}$$

be the *subcomplex of $\mathcal{U}$ -small chains*. We also have inclusions $C_*^{\text{sing}}(U_j; R) \subseteq C_*^{\mathcal{U}}(X; R)$ of subcomplexes for all j .

Proposition II.4.1. The inclusion

$$i_*: C_*^{\mathcal{U}}(X; R) \rightarrow C_*^{\text{sing}}(X; R)$$

has a chain homotopy inverse $r_*: C_*^{\text{sing}}(X; R) \rightarrow C_*^{\mathcal{U}}(X; R)$ with $r_* \circ i_* = \text{id}_{C_*^{\mathcal{U}}(X; R)}$ and $i_* \circ r_* \simeq_{F_*} \text{id}_{C_*^{\text{sing}}(X; R)}$, such that $F_*(C_*^{\text{sing}}(U_j)) \subseteq C_{*+1}^{\text{sing}}(U_j)$ for all j .

Theorem II.4.2. The pair $(H_*^{\text{sing}}, \partial_*)$ satisfies the excision axiom of a homology theory with values in $R\text{-mod}$.

Proof. Let $A \subseteq Y \subseteq X$ with $\overline{A} \subset Y^\circ$. Set $\mathcal{U} = \{Y, X \setminus A\}$. Then $Y^\circ \cup (X \setminus A)^\circ = Y^\circ \cup X \setminus \overline{A} = X$. We obtain

$$r_*: C_*^{\text{sing}}(X; R) \longrightarrow C_*^{\mathcal{U}}(X; R)$$

and $i_* \circ r_* \simeq_{F_*} \text{id}_{C_*^{\text{sing}}(X; R)}$ as in [Proposition II.4.1](#). We have

$$C_*^{\text{sing}}(X; R) \supseteq C_*^{\text{sing}}(Y; R) \subseteq C_*^{\mathcal{U}}(X; R)$$

and hence (since $r_*|_{C_*^{\text{sing}}(Y; R)} = \text{id}_{C_*^{\text{sing}}(Y; R)}$)

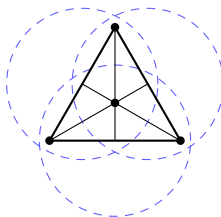
$$\begin{array}{ccc} C_*^{\mathcal{U}}(X; R)/C_*^{\text{sing}}(Y; R) & \xrightleftharpoons[r_*]{i_*} & C_*^{\text{sing}}(X; R)/C_*^{\text{sing}}(Y; R) \\ \cong \downarrow & & \parallel \\ (C_*^{\text{sing}}(X \setminus A; R) + C_*^{\text{sing}}(Y; R))/C_*^{\text{sing}}(Y; R) & & C_*^{\text{sing}}(X, Y; R) \\ \cong \downarrow & & \\ C_*^{\text{sing}}(X \setminus A, Y \setminus A; R) & & \end{array}$$

(Here $+$ denotes a sum that is not direct.) Since $F_*(C_*^{\text{sing}}(Y; R)) \subseteq C_{*+1}^{\text{sing}}(Y; R)$, it follows that

$$C_*^{\text{sing}}(X \setminus A, Y \setminus A; R) \simeq C_*^{\text{sing}}(X, Y; R)$$

and the claim follows from [Lemma II.3.3](#). $\square$

We now prove [Proposition II.4.1](#). **Idea** for constructing r_* : Iterate *barycentric subdivision* until the chain is $\mathcal{U}$ -small:



Problem: This does not define a chain map, since σ^n may need to be subdivided but $\partial\sigma^n$ may not (as in the figure), so that

$$\begin{array}{ccc} C_n^{\text{sing}}(X) & \xrightarrow{\partial_n} & C_{n-1}^{\text{sing}}(X) \\ r_n \downarrow & & \downarrow r_{n-1} \\ C_n^{\mathcal{U}}(X) & \xrightarrow{\partial_n} & C_{n-1}^{\mathcal{U}}(X) \end{array}$$

does not commute. $\Rightarrow$ We need correction terms.

Now for the *formal proof*: We explain barycentric subdivision. Any $p+1$ points $v_0, \dots, v_p \in \Delta^q$ define an affine linear map $\Delta^p \rightarrow \Delta^q$, which we denote (by abuse of notation) by $\sigma = [v_0, \dots, v_p]$. Let $L_p(\Delta^q; R) \subseteq C_p^{\text{sing}}(\Delta^q; R)$ be the free submodule generated by affine linear p -simplices in Δ^q . Each fixed point $v \in \Delta^q$ defines a

“cone operator”

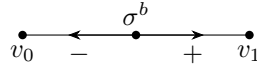
$$\begin{aligned} v: L_p(\Delta^q) &\longrightarrow L_{p+1}(\Delta^q), \\ \sigma &\longmapsto v\sigma = [v, v_0, \dots, v_p]. \end{aligned}$$

For $\sigma = [v_0, \dots, v_p]$, let $\sigma^b = \frac{1}{p+1}(v_0 + \dots + v_p)$ be the barycenter of σ .

Definition II.4.3. The *barycentric subdivision operator* $B_p: L_p(\Delta^q; R) \rightarrow L_p(\Delta^q; R)$ is defined inductively by $B_0(\sigma) = \sigma$ and $B_p(\sigma) = \sigma^b B_{p-1}(\partial_p \sigma)$.

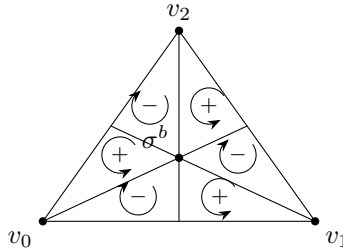
Thus, for $p = 1$ and $\sigma = [v_0, v_1]$, we have

$$\begin{aligned} B_1(\sigma) &= \sigma^b \partial_p \sigma = \sigma^b ([v_1] - [v_0]) \\ &= [\tfrac{1}{2}(v_0 + v_1), v_1] - [\tfrac{1}{2}(v_0 + v_1), v_0]. \end{aligned}$$



For $p = 2$ and $\sigma = [v_0, v_1, v_2]$, we have

$$\begin{aligned} B_2(\sigma) &= \sigma^b B_1(\partial \sigma) = \sigma^b B_1([v_1, v_2] - [v_0, v_2] + [v_0, v_1]) \\ &= \sigma^b ([\tfrac{1}{2}(v_1 + v_2), v_2] - [\tfrac{1}{2}(v_1 + v_2), v_1] - [\tfrac{1}{2}(v_0 + v_2), v_2] \\ &\quad + [\tfrac{1}{2}(v_0 + v_2), v_0] + [\tfrac{1}{2}(v_0 + v_1), v_1] - [\tfrac{1}{2}(v_0 + v_1), v_0]) \\ &= [\sigma^b, \tfrac{1}{2}(v_1 + v_2), v_2] - [\sigma^b, \tfrac{1}{2}(v_1 + v_2), v_1] - [\sigma^b, \tfrac{1}{2}(v_0 + v_2), v_2] \\ &\quad + [\sigma^b, \tfrac{1}{2}(v_0 + v_2), v_0] + [\sigma^b, \tfrac{1}{2}(v_0 + v_1), v_1] - [\sigma^b, \tfrac{1}{2}(v_0 + v_1), v_0], \\ \sigma^b &= \tfrac{1}{3}(v_0 + v_1 + v_2). \end{aligned}$$



We also set $B_p = 0$ for $p < 0$.

Lemma II.4.4. $B_*: L_*(\Delta^q; R) \rightarrow L_*(\Delta^q; R)$ is a chain map.

Proof. We show that $\partial_p \circ B_p = B_{p-1} \circ \partial_p$ by induction on p . Base case: For $p \leq 0$, we have $\partial_p = 0$, so the equality is clear. Inductive step: For $p > 0$, $\sigma = [v_0, \dots, v_p]$, and $v \in \Delta^q$, we have

$$\partial_{p+1}(v\sigma) = \sigma + \sum_{i=0}^p (-1)^{i+1} [v, v_0, \dots, \widehat{v}_i, \dots, v_p] = \sigma - v(\partial_p \sigma).$$

In degree 0, the formula is $\partial_1(vc) = c - \varepsilon(c)[v]$, where ε denotes the sum of the coefficients of a 0-chain. In the following application with $p = 1$, we have $c = B_0(\partial_1 \sigma) = \partial_1 \sigma$ and $\varepsilon(c) = 0$, so the additional term vanishes. It follows that

$$\begin{aligned} \partial_p(B_p(\sigma)) &= \partial_p(\sigma^b B_{p-1}(\partial_p \sigma)) \\ &= B_{p-1}(\partial_p \sigma) - \sigma^b \partial_{p-1}(B_{p-1}(\partial_p \sigma)) \\ &\stackrel{\text{induction hypothesis}}{=} B_{p-1}(\partial_p \sigma) - \sigma^b B_{p-2}(\partial_{p-1}(\partial_p \sigma)) = B_{p-1}(\partial_p \sigma). \end{aligned}$$

□

Lemma II.4.5. The diameter of a simplex in $B_p(\sigma)$ is at most $\frac{p}{p+1} \text{Diam}(\sigma)$.

Proof. We argue by induction on p ; for $p = 0$, the statement is clear. Each simplex of the subdivision is a cone from the barycenter σ^b over a simplex in a subdivided boundary face. By the induction hypothesis, edges within such a face have length at most $\frac{p-1}{p} \text{Diam}(\sigma)$. The remaining edges join σ^b to the barycenter of a face; by convexity, its distance from σ^b is at most $\max_i \|\sigma^b - v_i\|$. We estimate these distances. The extension of $[\sigma^b, v_i]$ meets the i th face at its

barycenter $\sigma_i^b = \frac{1}{p}(v_0 + \dots + \widehat{v}_i + \dots + v_p)$, that is,

$$\sigma^b = \frac{p}{p+1} \sigma_i^b + \frac{1}{p+1} v_i.$$

Thus the segment from v_i to σ^b has $\frac{p}{p+1}$ times the length of the segment from v_i to σ_i^b (take $v_i = 0$ without loss of generality), and hence

$$\|\sigma^b - v_i\| = \frac{p}{p+1} \|\sigma_i^b - v_i\| \leq \frac{p}{p+1} \text{Diam}(\sigma).$$

Since $\frac{p-1}{p} \leq \frac{p}{p+1}$, the required bound holds for all edges and hence for the diameter of each subdivided simplex. □

Lemma II.4.6. Define $h_*: L_*(\Delta^q; R) \rightarrow L_{*+1}(\Delta^q; R)$ inductively by $h_p = 0$ for $p < 0$ and $h_p(\sigma) = \sigma^b(\sigma - h_{p-1}(\partial_p \sigma))$. Then $\text{id}_* \simeq_{h_*} B_*$.

Proof. We show that $\text{id}_p - B_p = h_{p-1} \circ \partial_p + \partial_{p+1} h_p$ by induction on p . Base case: trivial. Inductive step:

$$\begin{aligned} \partial_{p+1}(h_p(\sigma)) &= \partial_{p+1}(\sigma^b(\sigma - h_{p-1}(\partial_p \sigma))) \\ &\stackrel{\text{see above}}{=} \sigma - h_{p-1}(\partial_p \sigma) - \sigma^b(\partial_p(\sigma - h_{p-1}(\partial_p \sigma))) \\ &= \sigma - h_{p-1}(\partial_p \sigma) - \sigma^b(\partial_p(\sigma)) + \sigma^b \partial_p(h_{p-1}(\partial_p \sigma)). \end{aligned}$$

By the induction hypothesis,

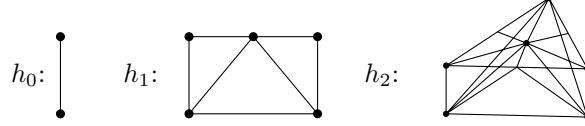
$$\partial_p(h_{p-1}(\partial_p \sigma)) = \partial_p \sigma - B_{p-1}(\partial_p \sigma) - h_{p-2}(\partial_{p-1}(\partial_p \sigma)) = \partial_p \sigma - B_{p-1}(\partial_p \sigma).$$

Substituting above gives

$$\begin{aligned}\partial_{p+1}(h_p(\sigma)) &= \sigma - h_{p-1}(\partial_p \sigma) - \sigma^b(\partial_p(\sigma)) + \sigma^b \partial_p \sigma - \sigma^b B_{p-1}(\partial_p \sigma) \\ &= \text{id}_p(\sigma) - B_p(\sigma) - h_{p-1}(\partial_p \sigma).\end{aligned}$$

□

Illustrations:



Now let X be a topological space. We define a natural transformation in X with component

$$\begin{aligned}B(X)_p: C_p^{\text{sing}}(X; R) &\longrightarrow C_p^{\text{sing}}(X; R), \\ \sigma^p &\longmapsto C_p^{\text{sing}}(\sigma^p; R)(B_p(\text{id}_{\Delta^p}))\end{aligned}$$

and a natural transformation in X with component

$$\begin{aligned}h(X)_p: C_p^{\text{sing}}(X; R) &\longrightarrow C_{p+1}^{\text{sing}}(X; R), \\ \sigma^p &\longmapsto C_{p+1}^{\text{sing}}(\sigma^p; R)(h_p(\text{id}_{\Delta^p})).\end{aligned}$$

Lemma II.4.7. $B(X)_*$ is a chain map and $\text{id}_{C_*^{\text{sing}}(X; R)} \simeq_{h(X)_*} B(X)_*$.

Proof. Both assertions are clear for $p \leq 0$. For $p > 0$, let $f_i: [v_0, \dots, \widehat{v}_i, \dots, v_p] \rightarrow [v_0, \dots, v_p]$ be the i th face map. For $\sigma^p \in C_p^{\text{sing}}(X; R)$, we have

$$\begin{aligned}B_{p-1}(X)(\partial_p \sigma^p) &= \sum_{i=0}^p (-1)^i C_{p-1}^{\text{sing}}(\sigma^p \circ f_i)(B_{p-1}(\text{id}_{\Delta^{p-1}})) \\ &= \sum_{i=0}^p (-1)^i C_{p-1}^{\text{sing}}(\sigma^p)(C_{p-1}^{\text{sing}}(f_i)(B_{p-1}(\text{id}_{\Delta^{p-1}}))) \\ &= \sum_{i=0}^p (-1)^i C_{p-1}^{\text{sing}}(\sigma^p)(B_{p-1}(f_i)) \\ &= C_{p-1}^{\text{sing}}(\sigma^p) \left(B_{p-1} \left(\sum_{i=0}^p (-1)^i f_i \right) \right) \\ &= C_{p-1}^{\text{sing}}(\sigma^p)(B_{p-1}(\partial_p \text{id}_{\Delta^p})) \stackrel{\text{II.4.4}}{=} C_{p-1}^{\text{sing}}(\sigma^p)(\partial_p(B_p(\text{id}_{\Delta^p}))) \\ &= \partial_p(C_p^{\text{sing}}(\sigma^p)(B_p(\text{id}_{\Delta^p}))) = \partial_p(B(X)_p(\sigma^p)).\end{aligned}$$

Similarly, Lemma II.4.6 gives $\text{id}_{C_*^{\text{sing}}(X; R)} \simeq_{h(X)_*} B(X)_*$.

□

Consequently, we also have

$$B^k(X)_* = B(X)_* \circ B^{k-1}(X)_* \simeq B^{k-1}(X)_* \simeq \dots \simeq B(X)_* \simeq \text{id}_{C_*^{\text{sing}}(X; R)}.$$

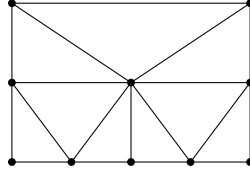
A chain homotopy from $\text{id}_{C_*^{\text{sing}}(X; R)}$ to $B^k(X)_*$ is

$$G_*^k = \sum_{i=0}^{k-1} h(X)_* \circ B^i(X)_*,$$

since

$$\begin{aligned}
\partial_{*+1}G_*^k + G_{*-1}^k\partial_* &= \sum_{i=0}^{k-1} (\partial_{*+1}h(X)_* \circ B^i(X)_* + h(X)_{*-1}B^i(X)_{*-1}\partial_*) \\
&= \sum_{i=0}^{k-1} (\partial_{*+1}h(X)_* + h(X)_{*-1}\partial_*) \circ B^i(X)_* \\
&= \sum_{i=0}^{k-1} (\text{id}_* - B(X)_*) \circ B^i(X)_* \\
&= \text{id}_* - B^k(X)_*.
\end{aligned} \tag{*}$$

Illustration:



Lemma II.4.8. Let $\mathcal{U} = \{U_i\}$ be a family of subspaces $U_i \subseteq X$ with $\bigcup_i U_i^\circ = X$. For every $\sigma^p: \Delta^p \rightarrow X$, there is $k \geq 0$ such that $B^k(X)_p(\sigma^p)$ is $\mathcal{U}$ -small.

Proof. Choose a Lebesgue number δ for $\{(\sigma^p)^{-1}(U_i^\circ)\}$ and $k \in \mathbb{N}$ such that all simplices in $B_p^k(\text{id}_{\Delta^p})$ have diameter less than δ (Lemma II.4.5). Then $B^k(X)_p(\sigma)$ is $\mathcal{U}$ -small. $\square$

Proof of Proposition II.4.1. Let $\mathcal{U} = \{U_i\}$ with $\bigcup_i U_i^\circ = X$. Define

$$\begin{aligned}
F_p: C_p^{\text{sing}}(X; R) &\longrightarrow C_{p+1}^{\text{sing}}(X; R), \\
\sigma^p &\longmapsto G_p^{k(\sigma^p)}(\sigma^p),
\end{aligned}$$

where $k(\sigma^p) \in \mathbb{N}$ is minimal as in Lemma II.4.8. Set

$$r_p(\sigma^p) := \underbrace{B^{k(\sigma^p)}(X)_p(\sigma^p)}_{\text{make the simplex } \mathcal{U}\text{-small}} + \underbrace{G_{p-1}^{k(\sigma^p)}(\partial_p \sigma^p)}_{\text{glue the boundary back together}} - \underbrace{F_{p-1}(\partial_p \sigma^p)}_{\text{make the boundary } \mathcal{U}\text{-small}}.$$

We have $r_*(C_*^{\text{sing}}(X; R)) \subseteq C_*^{\mathcal{U}}(X; R)$, since the difference $G_{p-1}^{k(\sigma^p)}(\partial_p \sigma^p) - F_{p-1}(\partial_p \sigma^p)$ consists only of $\mathcal{U}$ -small simplices. By (*), we have

$$\begin{aligned}
\partial_{p+1}G_p^{k(\sigma^p)}(\sigma^p) + G_{p-1}^{k(\sigma^p)}\partial_p(\sigma^p) &= \sigma^p - B^{k(\sigma^p)}(X)_p(\sigma^p) \\
\iff \partial_{p+1}F_p(\sigma^p) + F_{p-1}\partial_p(\sigma^p) &= \sigma^p - r_p(\sigma^p).
\end{aligned} \tag{**}$$

Therefore

$$\partial_p r_p(\sigma^p) = \partial_p \sigma^p - \partial_p F_{p-1} \partial_p(\sigma^p) = r_{p-1} \partial_p(\sigma^p).$$

(First equality: apply ∂_p to (**). Second equality: apply (**) with $p \mapsto p-1$ to $\partial_p(\sigma^p)$.)

Thus r_* is a chain map and

$$\text{id}_{C_*^{\text{sing}}(X; R)} \simeq_{F_*} i_* \circ r_*.$$

Moreover, $r_* \circ i_* = \text{id}_*$, since $k(\sigma^p) = 0$ for $\sigma_p \in C_p^{\mathcal{U}}(X; R)$, and $F_*(C_*^{\text{sing}}(U_j)) \subseteq C_{*+1}^{\text{sing}}(U_j)$, since every simplex occurring in the chain $F_p(\sigma^p)$ has its image in the image of σ_p . $\square$

Theorem II.4.9. The pair $(H_*^{\text{sing}}, \partial_*)$ satisfies the dimension axiom of a homology theory with values in $R\text{-mod}$ and has coefficient module $H_0^{\text{sing}}(\bullet) = R$.

Proof. “ $\bullet$ ” is terminal in Top . Thus $C_*^{\text{sing}}(\bullet; R) =$

$$\dots \xrightarrow{\text{id}} \overset{3}{R} \xrightarrow{0} \overset{2}{R} \xrightarrow{\text{id}} \overset{1}{R} \xrightarrow{0} \overset{0}{R} \longrightarrow \overset{-1}{0},$$

since, for $\sigma_p: \Delta_p \rightarrow \bullet$, we have

$$\sum_{i=0}^p (-1)^i \sigma_p|_{[v_0, \dots, \widehat{v_i}, \dots, v_p]} = \begin{cases} 0, & p \equiv 1 \pmod{2}, \\ (\Delta^{p-1} \rightarrow \bullet), & p \equiv 0 \pmod{2}. \end{cases}$$

□

Theorem II.4.10. The pair $(H_*^{\text{sing}}, \partial_*)$ is an ordinary homology theory with values in $R\text{-mod}$ and coefficient module $H_0^{\text{sing}}(\bullet) = R$.

Proof. Theorems II.2.3, II.3.4, II.4.2, II.4.9. □

Now let M be any R -module. For a pair of spaces (X, A) , let

$$C_*^{\text{sing}}(X, A; M) := (C_*^{\text{sing}}(X, A; R) \otimes_R M, \partial_* \otimes \text{id}_M)$$

and set $H_*^{\text{sing}}(X, A; M) := H_*(C_*^{\text{sing}}(X, A; M))$.

This defines an ordinary homology theory with values in $R\text{-mod}$ and $H_0^{\text{sing}}(\bullet) \cong M$, because:

- Dimension axiom: clear.
- Long exact sequence axiom: $(-) \otimes_R M$ is only right exact, so

$$C_*^{\text{sing}}(A; M) \longrightarrow C_*^{\text{sing}}(X; M) \longrightarrow C_*^{\text{sing}}(X, A; M) \longrightarrow 0,$$

but $C_*^{\text{sing}}(X, A; R)$ is free, that is, the short exact sequence

$$0 \longrightarrow C_*^{\text{sing}}(A; R) \longrightarrow C_*^{\text{sing}}(X; R) \longrightarrow C_*^{\text{sing}}(X, A; R) \longrightarrow 0$$

splits, so we do in fact have

$$0 \longrightarrow C_*^{\text{sing}}(A; M) \longrightarrow C_*^{\text{sing}}(X; M) \longrightarrow C_*^{\text{sing}}(X, A; M) \longrightarrow 0.$$

- Since $P_* \otimes \text{id}_M$ and $F_* \otimes \text{id}_M$ are again chain homotopies, homotopy invariance and excision follow.

Existence Theorem. For every R -module M , there is an ordinary homology theory with values in $R\text{-mod}$ and coefficient module M . □

Later: a uniqueness theorem for so-called *cell complexes*.

II.5. Singular Homology in Degrees Zero and One

Goal: Determine $H_0(X; M)$ and $H_1(X; \mathbb{Z})$. Let M be an R -module and X a space.

Theorem II.5.1. We have a natural isomorphism

$$H_n^{\text{sing}}(X; M) \cong \bigoplus_{X_\alpha \in \pi_0(X)} H_n^{\text{sing}}(X_\alpha; M).$$

Proof. For $\sigma^n \in C_n^{\text{sing}}(X; R)$, the image $\sigma^n(\Delta^n) \subseteq X$ is path connected, so $C_n^{\text{sing}}(X; R) = \bigoplus_{X_\alpha \in \pi_0(X)} C_n^{\text{sing}}(X_\alpha; R)$, and this decomposition is respected by ∂_* and $(-) \otimes_R M$. Naturality is clear. $\square$

Similarly, we see that $H_*^{\text{sing}}(-; M)$ preserves coproducts:

Theorem II.5.2. $(H_*^{\text{sing}}, \partial_*)$ satisfies the *additivity axiom*: The inclusions $X_j \rightarrow \coprod_{i \in I} X_i$ induce an isomorphism

$$\bigoplus_{i \in I} H_n(X_i) \cong H_n(X) \quad \text{for all } n \in \mathbb{Z}.$$

Remark.

- Already seen in Homework 4.4: Every homology theory satisfies *finite additivity*.
- H_*^{sing} does not preserve pushouts in general. However, for

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \text{cofibration} \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array} \quad \sqcap$$

there is a long exact sequence (later).

Theorem II.5.3. $H_0^{\text{sing}}(X; M) \cong \bigoplus_{\pi_0 X} M$.

Proof. By Theorem II.5.1, it remains to show that $H_0(X; M) \cong M$ if X is path connected (in particular, nonempty). We *augment* the chain complex $C_*^{\text{sing}}(X; R)$ by

$$\varepsilon: C_0^{\text{sing}}(X; R) \longrightarrow R, \quad \sum_i r_i \sigma_i \longmapsto \sum_i r_i.$$

Clearly, ε is surjective. Claim: $\ker \varepsilon = \text{im } \partial_1$. “ $\supseteq$ ”: clear. “ $\subseteq$ ”: Let $\varepsilon(\sum_i r_i \sigma_i^0) = 0$, that is, $\sum_i r_i = 0$. Choose $x_0 \in X$ and paths $\gamma_i: I \rightarrow X$ with $\gamma_i(0) = x_0$ and $\gamma_i(1) = \sigma_i(v_0) \in X$. Interpreting γ_i as $\sigma_i^1: \Delta^1 \rightarrow X$,

we have

$$\begin{aligned} \partial_1 \left(\sum_i r_i \sigma_i^1 \right) &= \sum_i r_i (\sigma_i^1|_{[v_1]} - \sigma_i^1|_{[v_0]}) \\ &= \sum_i r_i \sigma_i^1|_{[v_1]} - \left(\sum_i r_i \right) \sigma_i^1|_{[v_0]} = \sum_i r_i \sigma_i^0. \end{aligned}$$

Thus

$$C_1^{\text{sing}}(X; R) \xrightarrow{\partial_1} C_0^{\text{sing}}(X; R) \xrightarrow{\varepsilon} R \longrightarrow 0$$

is exact and remains exact after applying $(-) \otimes_R M$. It follows that

$$H_0^{\text{sing}}(X; M) = C_0^{\text{sing}}(X; M) / \text{im}(\partial_1 \otimes \text{id}_M) = C_0^{\text{sing}}(X; M) / \ker(\varepsilon \otimes \text{id}_M) \cong M.$$

$\square$

Let X be nonempty. Again view paths $f: I \rightarrow X$ as $f: \Delta^1 \rightarrow X$. Then f is a 1-cycle if and only if f is a loop.

Theorem II.5.4. (Hurewicz Theorem in Degree One)

$$\begin{aligned} \text{Hur}: \pi_1(X, x_0) &\longrightarrow H_1^{\text{sing}}(X; \mathbb{Z}), \\ [f] &\longmapsto f + B_1^{\text{sing}}(X; \mathbb{Z}) \end{aligned}$$

is a well-defined group homomorphism and is natural with respect to pointed maps. If X is path connected, then Hur induces an isomorphism

$$\text{Hur}_{\text{ab}}: \pi_1(X, x_0)_{\text{ab}} \xrightarrow{\cong} H_1^{\text{sing}}(X; \mathbb{Z}),$$

that is, $\ker \text{Hur} = [\pi_1(X, x_0), \pi_1(X, x_0)]$.

Proof. Well-definedness: Let

$$[S^1] := \left[\begin{array}{ccc} I \cong \Delta^1 & \longrightarrow & S^1 \\ t & \longmapsto & e^{2\pi i t} \end{array} \right] \in H_1^{\text{sing}}(S^1; \mathbb{Z})$$

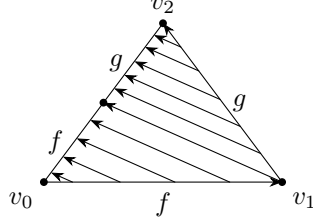
be the *fundamental class* of S^1 . Then $\text{Hur}([f]) = H_1^{\text{sing}}(\bar{f})([S^1])$, where

$$\begin{array}{ccc} I & \xrightarrow{f} & X \\ \downarrow & \nearrow \bar{f} & \\ I/\partial I = S^1 & & \end{array}$$

so well-definedness follows from Theorem II.3.4.

Homomorphism: Let $[f], [g] \in \pi_1(X, x_0)$. Let $\sigma^2: \Delta^2 \rightarrow X$ be given by

$$[v_0, v_1, v_2] \xrightarrow{\text{projection}} [v_0, v_2] \xrightarrow{fg} X :$$



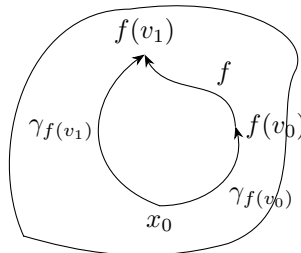
We have $\partial\sigma^2 = g - fg + f \in B_1(X)$, that is, $[fg] = [f] + [g]$ in $H_1^{\text{sing}}(X; \mathbb{Z})$.

Naturality: For $g: (X, x_0) \rightarrow (Y, y_0)$, we have

$$\begin{aligned} H_1^{\text{sing}}(g)(\text{Hur}([f])) &= H_1^{\text{sing}}(g)(H_1^{\text{sing}}(\bar{f})([S^1])) \\ &= H_1^{\text{sing}}(g \circ \bar{f})([S^1]) = \text{Hur}([g \circ f]) = \text{Hur}(\pi_1(g)([f])). \end{aligned}$$

Now let X be path connected. Choose $\gamma_x: I \rightarrow X$, $\gamma_x(0) = x_0$, $\gamma_x(1) = x$ for all $x \in X$, and $\gamma_{x_0} = c_{x_0}$. Define

$$\begin{aligned} \psi: C_1^{\text{sing}}(X; \mathbb{Z}) &\longrightarrow \pi_1(X, x_0)_{\text{ab}}, \\ (f: \Delta^1 \cong I \rightarrow X) &\longmapsto [\gamma_{f(v_0)} f \bar{\gamma}_{f(v_1)}]_{\text{ab}}. \end{aligned}$$



We need to show: For every $\sigma^2 \in C_2^{\text{sing}}(X; \mathbb{Z})$, we have $\partial\sigma^2 \in \ker \psi$. Let f, g, h be the faces of σ^2 . Then

$$\begin{aligned}\psi(\partial\sigma^2) &= \psi(f - g + h) = \psi(f) - \psi(g) + \psi(h) \\ &= [\gamma_{\sigma^2(v_1)} f \bar{\gamma}_{\sigma^2(v_2)} \gamma_{\sigma^2(v_2)} \bar{g} \bar{\gamma}_{\sigma^2(v_0)} \gamma_{\sigma^2(v_0)} h \bar{\gamma}_{\sigma^2(v_1)}]_{\text{ab}} \\ &= [\gamma_{\sigma^2(v_1)} f \bar{g} h \bar{\gamma}_{\sigma^2(v_1)}]_{\text{ab}} = [c_{x_0}]_{\text{ab}},\end{aligned}$$

since the loop $f\bar{g}h$ is homotopic to a constant loop through the interior of σ^2 .

We obtain $\bar{\psi}: H_1^{\text{sing}}(X; \mathbb{Z}) \rightarrow \pi_1(X, x_0)_{\text{ab}}$. We need to show that $\bar{\psi} \circ \text{Hur}_{\text{ab}} = \text{id}_{\pi_1(X, x_0)_{\text{ab}}}$ and $\text{Hur}_{\text{ab}} \circ \bar{\psi} = \text{id}_{H_1^{\text{sing}}(X; \mathbb{Z})}$.

$$\bar{\psi}(\text{Hur}_{\text{ab}}([f]_{\text{ab}})) = \bar{\psi}(f + B_1) = [c_{x_0} f \bar{c}_{x_0}]_{\text{ab}} = [f]_{\text{ab}}.$$

We have an extension $\gamma: C_0^{\text{sing}}(X; \mathbb{Z}) \rightarrow C_1^{\text{sing}}(X; \mathbb{Z})$. For $c \in C_1^{\text{sing}}(X; \mathbb{Z})$, it follows that $\text{Hur}_{\text{ab}}(\psi(c)) = c - \gamma(\partial c) + B_1$. Therefore

$$\text{Hur}_{\text{ab}}(\bar{\psi}(z + B_1)) = z - \underbrace{\gamma(\partial z)}_{=0} + B_1 = z + B_1.$$

□

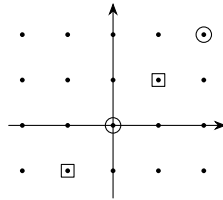
Examples.

•

$$\begin{aligned}H_1^{\text{sing}}(\Sigma_g; \mathbb{Z}) &\cong \pi_1(\Sigma_g, \bullet)_{\text{ab}} \\ &\cong \left\langle a_1, b_1, \dots, a_g, b_g \left| \prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1} \right. \right\rangle_{\text{ab}} \cong \mathbb{Z}^{2g}.\end{aligned}$$

•

$$H_1^{\text{sing}}(K; \mathbb{Z}) = \langle a, b \mid a^2 b^2 \rangle_{\text{ab}} \cong (\mathbb{Z} \oplus \mathbb{Z}) / \langle \begin{pmatrix} 2 \\ 2 \end{pmatrix} \rangle \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$



- For $f: I \rightarrow S^1$, $t \mapsto e^{2\pi i t}$, we have

$$\text{Hur}_{\text{ab}}([f]) = H_1^{\text{sing}}(\text{id}_{S^1})([S^1]) = [S^1],$$

so $H_1^{\text{sing}}(S^1; \mathbb{Z}) = \langle [S^1] \rangle$.

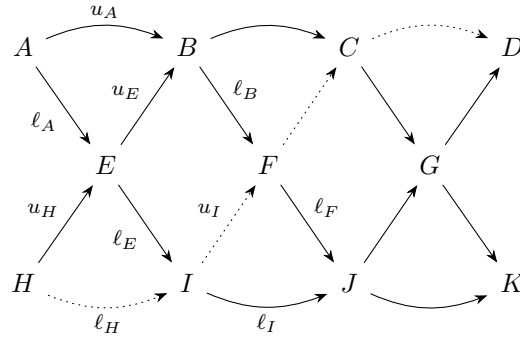
III. Homology: Computations and Applications

Goal: Develop computational methods using only the axioms. Examples and applications.

III.1. Relative and Absolute Homology

Goal: Clarify the relationship between $H_n(X, A)$ and $H_n(X/A)$. Construct the suspension isomorphism and compute the homology of spheres.

Lemma III.1.1. (Braid lemma) Consider



in $R\text{-Mod}$. Suppose the solid strands are exact and the dotted strand is a chain complex. Then the dotted strand is also exact.

Proof. Diagram chase. For example, exactness at I : Let $i \in \ker u_I$. Then $\ell_F(u_I(i)) = 0$, so there is an $e \in E$ with $\ell_E(e) = i$; hence $\ell_B(u_E(e)) = 0$ and there is an $a \in A$ with $u_A(a) = u_E(e)$. Since

$$u_E(e - \ell_A(a)) = u_E(e) - u_A(a) = 0,$$

there is an $h \in H$ with $u_H(h) = e - \ell_A(a)$. Thus

$$\ell_H(h) = \ell_E(u_H(h)) = \ell_E(e - \ell_A(a)) = i \in \text{im } \ell_H. \quad \square$$

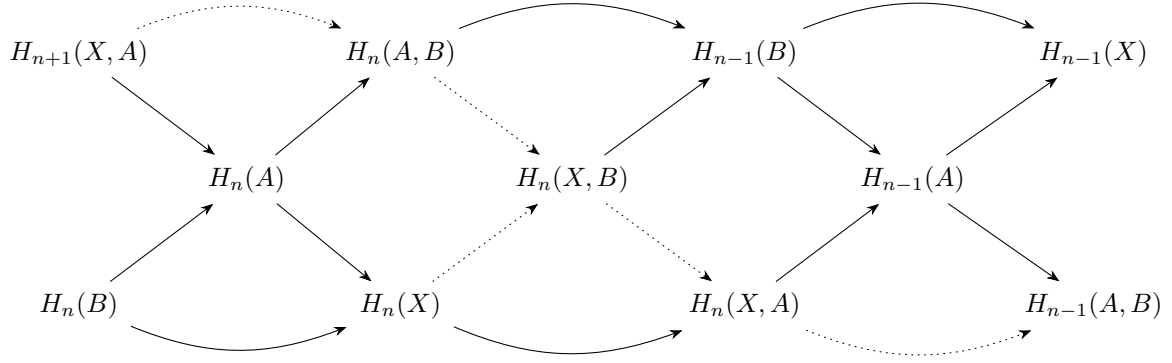
Now let (H_*, ∂_*) be a homology theory with values in $R\text{-Mod}$.

Theorem III.1.2. (Sequence of a triple) For every triple of spaces (X, A, B) , there is a natural exact sequence

$$\cdots \longrightarrow H_n(A, B) \longrightarrow H_n(X, B) \longrightarrow H_n(X, A) \xrightarrow{\partial_n} H_{n-1}(A, B) \longrightarrow \cdots.$$

Proof. Define

$$\partial_n: H_n(X, A) \xrightarrow{\partial_n^{(X, A)}} H_{n-1}(A) \longrightarrow H_{n-1}(A, B).$$



It remains to show that $H_n(A, B) \longrightarrow H_n(X, B) \longrightarrow H_n(X, A)$ is zero.

$$\begin{array}{ccccccc}
 & & H_n(A, B) & & H_n(X, A) & & \\
 & & \searrow & & \nearrow & & \\
 H_n(A) & \xrightarrow{\text{id}} & H_n(A) & \xrightarrow{0} & H_n(A, A) & \xrightarrow{0} & H_{n-1}(A) \xrightarrow{\text{id}} H_{n-1}(A)
 \end{array}
 \quad H_n(A, A) = 0.$$

□

- Consider $H_*: \mathbf{Top}_\bullet \longrightarrow R\text{-Mod}$, $(X, x_0) \longmapsto H_*(X, x_0)$.
- $(\bullet, \bullet)$ is a *zero object* (initial and terminal).
- $H_*(\bullet, \bullet) = 0$ (for example, a homework exercise).

Seek a modification $\tilde{H}_*$ of H_* with $\tilde{H}_*(\bullet) = 0$.

Definition III.1.3. The n th *reduced homology* of $X \neq \emptyset$ is

$$\tilde{H}_n(X) = \ker(H_n(X) \longrightarrow H_n(\bullet)).$$

- Functorial, since

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 & \searrow & \downarrow \\
 & & \bullet
 \end{array}
 \rightsquigarrow
 \begin{array}{ccc}
 H_n(X) & \xrightarrow{H_n(f)} & H_n(Y) \\
 & \searrow & \downarrow \\
 & & H_n(\bullet)
 \end{array}$$

- f null-homotopic $\implies \tilde{H}_*(f) = 0$, since $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \uparrow \\ & & \bullet \end{array}$ in $\mathbf{HoTop}$.

Now let $x_0 \in X$, $r: X \longrightarrow \{x_0\}$, and $j: (X, \emptyset) \longrightarrow (X, x_0)$.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_n(x_0) & \longrightarrow & H_n(X) & \xrightarrow{H_n(j)} & H_n(X, x_0) \longrightarrow 0 \\
 & & & & \nwarrow & & \nwarrow \\
 & & & & H_n(r) & &
 \end{array}$$

$$H_n(j)|_{\ker H_n(r)}: \ker H_n(r) \xrightarrow{\cong} H_n(X, x_0).$$

Thus

$$\boxed{\tilde{H}_n(X) \cong H_n(X, x_0), \quad H_n(X) \cong \tilde{H}_n(X) \oplus H_n(\bullet).}$$

If (H_*, ∂_*) is ordinary, it follows that

$$H_0(X) \cong \tilde{H}_0(X) \oplus H_0(\bullet), \quad H_n(X) \cong \tilde{H}_n(X) \quad \text{for } n \neq 0.$$

Proposition III.1.4. Let (X, A) be a cofibration. Then the collapse map $q: (X, A) \rightarrow (X/A, A/A)$ induces an isomorphism

$$H_n(X, A) \xrightarrow{\cong} \tilde{H}_n(X/A).$$

In preparation:

Lemma III.1.5. Let $\begin{array}{ccc} A & \xrightarrow{f} & Y \\ i \downarrow & \lrcorner & \downarrow j \\ X & \xrightarrow{f} & Z \end{array}$ be a pushout in $\mathbf{Top}$. If i is the inclusion of a closed subspace, then so is j , and $X \setminus A \cong Z \setminus Y$.

Proof. 1. $Z \cong X \amalg Y / \sim$, where $f(a) \sim i(a)$. Thus $j(y_1) = j(y_2)$ can hold for distinct points only if $y_1, y_2 \in \text{im } f$ and there are $a_i \in f^{-1}(y_i)$ with $i(a_1) = i(a_2)$. If i is injective, this cannot occur, so j is also injective. We must show that j is closed. Let $C \subseteq Y$ be closed. Then $j(C)$ is closed, since

$$(\bar{f} \amalg j)^{-1}(j(C)) = i(f^{-1}(C)) \amalg C$$

is closed because i is closed. Since $\bar{f} \amalg j$ is a quotient map, $j(C)$ is closed.

2. $\bar{f}|_{X \setminus A}: X \setminus A \rightarrow Z \setminus Y$ is a continuous bijection. Let $U \subseteq X \setminus A$ be open. Then $U \subseteq X$ is open because A is closed. Since $\bar{f} \amalg j$ is a quotient map, $\bar{f}(U) \subseteq Z$ is open, and $\bar{f}(U) \subseteq Z \setminus Y$ is also open because $Y \subseteq Z$ is closed. $\square$

Proof of Proposition III.1.4. For $i: A \rightarrow X$, consider the pushout

$$\begin{array}{ccc} A & \xrightarrow{\quad} & \bullet \\ \downarrow & \lrcorner & \downarrow \\ M_i & \xrightarrow{\quad} & C_i. \end{array}$$

By Lemma III.1.5, $M_i \setminus A \cong C_i \setminus \bullet$, and hence

$$(M_i \setminus A, A \times [0, 1)) \cong (C_i \setminus \bullet, CA \setminus \bullet).$$

Since (X, A) is a strong deformation retract of $(M_i \setminus A, A \times [0, 1))$, we have

$$\begin{array}{ccc} (X, A) & \xrightarrow{q} & (X/A, A/A) \\ \simeq \downarrow & & \uparrow \simeq \\ (C_i \setminus \bullet, CA \setminus \bullet) & \longrightarrow & (C_i, CA) \end{array} \quad (\text{Theorem VI.4.6, Introduction to Topology}).$$

Thus $H_*(q)$ factors through isomorphisms by homotopy invariance and excision. $\square$

Theorem III.1.6. Let (X, A) be a nonempty cofibration. Then we have a natural long exact sequence

$$\cdots \rightarrow \tilde{H}_n(A) \rightarrow \tilde{H}_n(X) \rightarrow \tilde{H}_n(X/A) \rightarrow \tilde{H}_{n-1}(A) \rightarrow \cdots$$

Proof. Choose $a_0 \in A$ and consider the sequence of the triple

$$\begin{array}{ccccccc} \cdots & \rightarrow & H_n(A, a_0) & \rightarrow & H_n(X, a_0) & \rightarrow & H_n(X, A) \rightarrow \cdots \\ & & \wr & & \wr & & \wr \quad \text{III.1.4} \\ & & \tilde{H}_n(A) & & \tilde{H}_n(X) & & \tilde{H}_n(X/A). \end{array}$$

$\square$

A pointed space (X, x_0) is called *well-pointed* if (X, x_0) is a cofibration. If (X, x_0) is well-pointed and (Y, y_0) is pointed, then $(X \vee Y, Y)$ is a cofibration, since

$$\begin{array}{ccc} \bullet & \xrightarrow{\quad \Gamma \quad} & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \vee Y \end{array}$$

and:

Lemma III.1.7. Let $\begin{array}{ccc} A & \xrightarrow{f} & Y \\ i \downarrow & \Gamma & \downarrow j \\ X & \xrightarrow{\bar{f}} & Z \end{array}$ be a pushout in $\mathbf{Top}$. If i is a cofibration, then so is j .

Proof. Let $H: Y \times I \rightarrow W$ and $g: Z \rightarrow W$ satisfy $g \circ j = H_0$. Since i is a cofibration, $H \circ (f \times \text{id}_I)$ extends to H' with initial condition $g \circ \bar{f}$.

$$\begin{array}{ccccc} A \times I & \xrightarrow{f \times \text{id}} & Y \times I & & \\ i \times \text{id} \downarrow & \Gamma & \downarrow j \times \text{id} & \searrow H & \\ X \times I & \xrightarrow{\bar{f} \times \text{id}} & Z \times I & \xrightarrow{\exists! H''} & W \\ & \searrow H' & & & \end{array}$$

Then H'' is the desired extension of H relative to g . □

Remark: By the preceding lemma, in the pushout $\begin{array}{ccc} A & \xrightarrow{\quad \Gamma \quad} & Y \\ i \downarrow & & \downarrow j \\ X & \longrightarrow & Z \end{array}$ if (X, A) is a neighborhood deformation retract pair, then so is (Z, Y) .

Consider $Y \xrightleftharpoons[r_Y]{i_Y} X \vee Y$; thus

$$\cdots \xrightarrow{0} \tilde{H}_n(Y) \rightarrow \tilde{H}_n(X \vee Y) \rightarrow \tilde{H}_n(X) \xrightarrow{0} \tilde{H}_{n-1}(Y) \rightarrow \cdots$$

$\xleftarrow{\quad}$
 $\xleftarrow{\quad}$

It follows that

$$\boxed{\tilde{H}_n(X \vee Y) \cong \tilde{H}_n(X) \oplus \tilde{H}_n(Y),}$$

that is, $\tilde{H}_*$ preserves finite coproducts in $\mathbf{Top}_\bullet^{\text{well}}$. (Later: If H_* is additive, $\tilde{H}_*$ also preserves infinite coproducts in $\mathbf{Top}_\bullet^{\text{well}}$.)

Definition III.1.8. Let X be a topological space and $X \subseteq CX$ the inclusion of the base into the cone. Then $SX := CX/X$ is called the *suspension* of X .

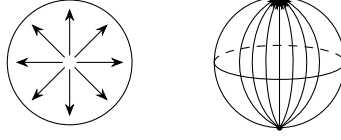
Alternative description:

$$\begin{array}{ccc} X & \xrightarrow{\quad \Gamma \quad} & CX \\ \downarrow & & \downarrow \\ CX & \longrightarrow & SX \end{array} \quad \text{---} \quad \text{Diagram of a cone } CX \text{ with base } X \text{ and apex } \bullet$$

Example: $S(S^{n-1}) = S^n$. More precisely:

$$S(S^{n-1}) = CS^{n-1}/S^{n-1} = D^n/S^{n-1} \xrightarrow[\cong]{\tilde{u}_n} S^n,$$

$$u_n(tx_1, \dots, tx_n) = (ux_1, \dots, ux_n, 2t - 1), \quad u = \sqrt{1 - (2t - 1)^2}.$$



We also have $S(S^{-1}) = S(\emptyset) = C\emptyset/\emptyset = (\emptyset/\emptyset)/\emptyset = \bullet/\emptyset \xrightarrow[\cong]{\tilde{u}_0} \bullet\bullet = S^0$.

Note: $S: \mathbf{Top} \longrightarrow \mathbf{Top}$ is a functor.

Theorem III.1.9. $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX)$, naturally in $X \neq \emptyset$.

Proof. Since (CX, X) is a cofibration, III.1.6 gives

$$\cdots \longrightarrow \tilde{H}_{n+1}(CX) \xrightarrow{0} \tilde{H}_{n+1}(SX) \xrightarrow{\cong} \tilde{H}_n(X) \xrightarrow{0} \tilde{H}_n(CX) \longrightarrow \cdots,$$

because $\tilde{H}_*(CX) \cong \tilde{H}_*(\bullet) = 0$. □

Corollary III.1.10. $H_\ell(D^n, S^{n-1}) \cong \tilde{H}_\ell(S^n) \cong H_{\ell-n}(\bullet)$, that is,

$$H_\ell(S^n) \cong H_{\ell-n}(\bullet) \oplus H_\ell(\bullet).$$

If H_* is ordinary, it follows that

$$H_\ell(D^n, S^{n-1}) \cong \begin{cases} H_0(\bullet) & \text{for } \ell = n, \\ 0 & \text{for } \ell \neq n, \end{cases}$$

$$H_\ell(S^n) \cong \begin{cases} H_0(\bullet) & \text{for } \ell = 0 \text{ or } \ell = n, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } n > 0,$$

$$H_0(S^0) \cong H_0(\bullet) \oplus H_0(\bullet), \quad H_\ell(S^0) = 0 \quad \text{for } \ell \neq 0.$$

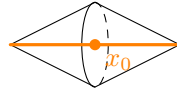
Proof.

$$\begin{aligned} H_\ell(D^n, S^{n-1}) &\stackrel{\text{III.1.4}}{\cong} \tilde{H}_\ell(D^n/S^{n-1}) \stackrel{u_n}{\cong} \tilde{H}_\ell(S^n) \stackrel{\text{Example}}{\cong} \tilde{H}_\ell(S^n(S^0)) \\ &\stackrel{\text{III.1.9}}{\cong} \tilde{H}_{\ell-n}(S^0) \stackrel{\text{III.1.3}}{\cong} H_{\ell-n}(S^0, \bullet) \stackrel{\text{excision}}{\cong} H_{\ell-n}(\bullet). \end{aligned} \quad \square$$

Definition III.1.11. For (X, x_0) ,

$$\Sigma X := SX/[\{x_0\} \times I]$$

is the *reduced suspension*.



$\Sigma: \mathbf{Top}_\bullet \longrightarrow \mathbf{Top}_\bullet$ is a functor. If (X, x_0) is well-pointed, then $\Sigma X \simeq SX$, and hence

$$\boxed{\tilde{H}_{n+1}(\Sigma X) \cong \tilde{H}_n(X)}$$

for well-pointed spaces. Reduced suspension is left adjoint to the *loop space functor* Ω ,

$$\mathrm{Hom}_{\mathbf{Top}_\bullet}(\Sigma(X, x_0), (Y, y_0)) \cong \mathrm{Hom}_{\mathbf{Top}_\bullet}((X, x_0), \Omega(Y, y_0)),$$

$$\Omega: \mathbf{Top}_\bullet \longrightarrow \mathbf{Top}_\bullet, \quad \Omega(X, x_0) = \mathrm{Hom}_{\mathbf{Top}_\bullet}((S^1, \bullet), (X, x_0))$$

with the compact-open topology ($\mathrm{Hom}_{\mathbf{Top}}(X, Y)$ has subbasis

$$\{f: X \longrightarrow Y : f(K) \subseteq U\} \quad \text{for } K \subseteq X \text{ compact, } U \subseteq Y \text{ open}.$$

The adjunction respects homotopy, so we also obtain an adjunction in $\mathbf{HoTop}_\bullet$. For $(X, x_0) = (S^n, \bullet)$, this says

$$\pi_{n+1}(Y, y_0) \cong \pi_n(\Omega(Y, y_0))$$

(compare $\tilde{H}_{n+1}(\Sigma X) \cong \tilde{H}_n(X)$).

A pointer to the homotopy-theoretic construction of homology using “spectra.”

III.2. Equality of Simplicial and Singular Homology

Goal: Show that $H_n^\Delta(X, A; R) \cong H_n^{\mathrm{sing}}(X, A; R)$ for Δ -pairs (X, A) . (We suppress R in the notation.) By III.1.10, we know that $H_n^{\mathrm{sing}}(\Delta^n, \partial\Delta^n) \cong R$.

Proposition III.2.1. $H_n^{\mathrm{sing}}(\Delta^n, \partial\Delta^n) \cong R$ is generated by $[\mathrm{id}_{\Delta^n}]$.

Proof. Induction on n . The base case $n = 0$ is clear. Inductive step: Let $\Lambda = \bigcup_{i=1}^n [v_0, \dots, \hat{v}_i, \dots, v_n] = \partial\Delta^n \setminus [v_1, \dots, v_n]$ be the *zeroth horn* of Δ^n .

$$\begin{aligned} H_n^{\mathrm{sing}}(\Delta^n, \partial\Delta^n) &\stackrel{\text{III.1.2}}{\cong} H_{n-1}^{\mathrm{sing}}(\partial\Delta^n, \Lambda) \stackrel{\text{III.1.4}}{\cong} \tilde{H}_{n-1}^{\mathrm{sing}}(\partial\Delta^n / \Lambda) \\ &\cong \tilde{H}_{n-1}^{\mathrm{sing}}(\Delta^{n-1} / \partial\Delta^{n-1}) \stackrel{\text{III.1.4}}{\cong} H_{n-1}^{\mathrm{sing}}(\Delta^{n-1}, \partial\Delta^{n-1}). \end{aligned}$$

This isomorphism maps $[\mathrm{id}_{\Delta^n}]$ to $[\mathrm{id}_{\Delta^{n-1}}]$. Thus $[\mathrm{id}_{\Delta^n}]$ is a generator by the induction hypothesis. $\square$

Lemma III.2.2. (Five lemma) Suppose the diagram

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{\ell} & E \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \delta \downarrow & & \varepsilon \downarrow \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{\ell'} & E' \end{array}$$

in $R\text{-Mod}$ has exact rows, α is surjective, ε is injective, and β and δ are isomorphisms. Then γ is also an isomorphism.

Proof. Diagram chase. Injectivity of γ : Let $c \in \ker \gamma$. Then $0 = k'(\gamma(c)) = \delta(k(c))$, so $k(c) = 0$ because δ is injective. By exactness at C , there is a $b \in B$ with $j(b) = c$. We have

$$j'(\beta(b)) = \gamma(j(b)) = \gamma(c) = 0,$$

so there is an $a' \in A'$ with $i'(a') = \beta(b)$. Since α is surjective, there is an $a \in A$ with $\alpha(a) = a'$. We have

$$\beta(b - i(a)) = i'(a') - i'(\alpha(a)) = 0.$$

Since β is injective, $b = i(a)$, that is, $c = j(b) = j(i(a)) = 0$ by exactness at B . Surjectivity of γ : exercise. $\square$

Now let (X, A) be a Δ -pair. Consider the chain map

$$C_n^\Delta(X, A) \longrightarrow C_n^{\mathrm{sing}}(X, A),$$

that regards a simplex in X as a singular simplex.

Theorem III.2.3. The induced homomorphism

$$H_n^\Delta(X, A) \xrightarrow{\cong} H_n^{\text{sing}}(X, A)$$

is an isomorphism for all $n \in \mathbb{Z}$.

Proof. First let $A = \emptyset$. Consider

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_n^\Delta(X^{\ell-1}) & \longrightarrow & H_n^\Delta(X^\ell) & \longrightarrow & H_n^\Delta(X^\ell, X^{\ell-1}) \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \alpha \\ \dots & \longrightarrow & H_n^{\text{sing}}(X^{\ell-1}) & \longrightarrow & H_n^{\text{sing}}(X^\ell) & \longrightarrow & H_n^{\text{sing}}(X^\ell, X^{\ell-1}) \longrightarrow \dots \end{array}$$

We must show that α is an isomorphism. Since $C_n^\Delta(X^\ell, X^{\ell-1}) = 0$ for $n \neq \ell$, we have

$$H_n^\Delta(X^\ell, X^{\ell-1}) \cong C_n^\Delta(X^\ell, X^{\ell-1}) \cong \begin{cases} \bigoplus_{\sigma_\alpha^n} R\sigma_\alpha^n & \text{for } n = \ell, \\ 0 & \text{for } n \neq \ell. \end{cases}$$

Moreover, we have

$$\begin{array}{ccc} (\coprod_\alpha \Delta_\alpha^\ell, \coprod_\alpha \partial \Delta_\alpha^\ell) & \xrightarrow{\Phi} & (X^\ell, X^{\ell-1}) \\ q \downarrow & & \downarrow q' \\ (\coprod_\alpha \Delta_\alpha^\ell / \coprod_\alpha \partial \Delta_\alpha^\ell, \bullet) & \xrightarrow{\cong} & (X^\ell / X^{\ell-1}, \bullet) \end{array}$$

and $(X^\ell, X^{\ell-1})$ is a cofibration by Lemma III.1.7, since

$$\begin{array}{ccc} \coprod_\alpha \partial \Delta_\alpha^\ell & \longrightarrow & X^{\ell-1} \\ \downarrow & \lrcorner & \downarrow \\ \coprod_\alpha \Delta_\alpha^\ell & \longrightarrow & X^\ell. \end{array}$$

By Proposition III.1.4, $H_n^{\text{sing}}(\Phi)$ is an isomorphism. Additivity of H_*^{sing} gives

$$\begin{array}{ccccccc} \dots & \longrightarrow & \bigoplus_\alpha H_n^{\text{sing}}(\Delta_\alpha^\ell) & \longrightarrow & \bigoplus_\alpha H_n^{\text{sing}}(\Delta_\alpha^\ell, \partial \Delta_\alpha^\ell) & \longrightarrow & \bigoplus_\alpha H_{n-1}^{\text{sing}}(\partial \Delta_\alpha^\ell) \longrightarrow \dots \\ & & \cong \downarrow & & \downarrow \beta & & \downarrow \cong \\ \dots & \longrightarrow & H_n^{\text{sing}}(\coprod_\alpha \Delta_\alpha^\ell) & \longrightarrow & H_n^{\text{sing}}(\coprod_\alpha \Delta_\alpha^\ell, \coprod_\alpha \partial \Delta_\alpha^\ell) & \longrightarrow & H_{n-1}^{\text{sing}}(\coprod_\alpha \partial \Delta_\alpha^\ell) \longrightarrow \dots \end{array}$$

By the five lemma, β is also an isomorphism. By Proposition III.2.1, we conclude that

$$H_n^{\text{sing}}(X^\ell, X^{\ell-1}) \cong \begin{cases} \bigoplus_{\sigma_\alpha^n} R[\text{id}_{\Delta_\alpha^n}] & \text{for } \ell = n, \\ 0 & \text{for } \ell \neq n. \end{cases}$$

Since α maps the generators to the corresponding generators, α is an isomorphism. Since clearly $H_n^\Delta(X^0) \xrightarrow{\cong} H_n^{\text{sing}}(X^0)$, the five lemma and induction imply that

$$H_n^\Delta(X^\ell) \xrightarrow{\cong} H_n^{\text{sing}}(X^\ell)$$

for all $\ell \geq 0$, and hence

$$H_n^\Delta(X) \cong \text{colim}_\ell H_n^\Delta(X^\ell) \cong \text{colim}_\ell H_n^{\text{sing}}(X^\ell) \cong H_n^{\text{sing}}(X).$$

Use the fact that $\sigma^n(\Delta^n) \subseteq X$ is compact, so $\sigma^n(\Delta^n) \subseteq X^\ell$ for $\ell \gg 0$.

$H_n^\Delta(X, A) \cong H_n^{\text{sing}}(X, A)$ for $A \neq \emptyset$ follows once again from the five lemma. $\square$

Remark: If we define

$$C_*^\Delta(X, A; M) := C_*^\Delta(X, A) \otimes_R M$$

for an R -module M , the same argument gives

$$H_n^\Delta(X, A; M) \cong H_n^{\text{sing}}(X, A; M).$$

Corollary III.2.4. If a pair of spaces (X, A) admits a Δ -pair structure with finitely many simplices outside A and R is a principal ideal domain, then $H_n^{\text{sing}}(X, A; R)$ is finitely presented.

Proof. By the theorem,

$$H_n^{\text{sing}}(X, A; R) \cong Z_n^\Delta(X, A; R) / B_n^\Delta(X, A; R),$$

and by assumption $C_n^\Delta(X, A; R) \cong R^\ell$. Over a principal ideal domain, submodules of free modules of finite rank are again free of finite rank. $\square$

III.3. The Mayer–Vietoris Sequence

Goal: For $Z = U_1 \cup U_2$ or $\begin{array}{ccc} A & \xrightarrow{\quad} & Y \\ \downarrow \cap & \searrow & \downarrow \\ X & \xrightarrow{\quad} & Z \end{array}$ develop a long exact sequence relating $H_*(Z)$ to $H_*(U_1)$ and $H_*(U_2)$, or to $H_*(A)$, $H_*(X)$, and $H_*(Y)$, respectively.

Again let (H_*, ∂_*) be a homology theory with values in $R\text{-Mod}$.

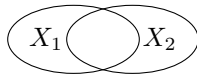
Definition III.3.1. Let $X_1, X_2 \subseteq X$ be subspaces with $X = X_1 \cup X_2$. The triad $(X; X_1, X_2)$ is called *excisive* if the inclusion $(X_1, X_1 \cap X_2) \longrightarrow (X, X_2)$ induces an isomorphism

$$H_n(X_1, X_1 \cap X_2) \cong H_n(X, X_2)$$

for every $n \in \mathbb{Z}$. **Remark:** $(X; X_1, X_2)$ excisive $\iff (X; X_2, X_1)$ excisive. (Diagram chase.)

Example: If X_1, X_2 are open and $X = X_1 \cup X_2$, then $(X; X_1, X_2)$ is excisive: for $Y = X_2$ and $A = X \setminus X_1$, we have $\bar{A} = A \subseteq \dot{Y} = Y$, and the excision axiom gives

$$\begin{array}{ccc} H_n(X \setminus A, Y \setminus A) & \xrightarrow{\cong} & H_n(X, Y) \\ \parallel & & \parallel \\ H_n(X_1, X_1 \cap X_2) & & H_n(X, X_2) \end{array}$$



Theorem III.3.2. (Mayer–Vietoris sequence) If $(X; X_1, X_2)$ is excisive and $A \subseteq X_1 \cap X_2 =: X_0$, there is a natural long exact sequence

$$\dots \xrightarrow{\partial_{n+1}} H_n(X_0, A) \xrightarrow{(H_n(i_1), H_n(i_2))} H_n(X_1, A) \oplus H_n(X_2, A) \xrightarrow{H_n(j_1) - H_n(j_2)} H_n(X, A) \xrightarrow{\partial_n} \dots$$

with $i_k: (X_0, A) \longrightarrow (X_k, A)$ and $j_k: (X_k, A) \longrightarrow (X, A)$, $k = 1, 2$.

Proof. Naturality of the sequence of a triple gives

$$\begin{array}{ccccccc}
 H_n(X_0, A) & \xrightarrow{H_n(i_1)} & H_n(X_1, A) & \xrightarrow{H_n(\ell_1)} & H_n(X_1, X_0) & \xrightarrow{\partial'_n} & H_{n-1}(X_0, A) \\
 \downarrow H_n(i_2) & & \downarrow H_n(j_1) & \nearrow \partial_n & \downarrow \cong & & \downarrow \\
 H_n(X_2, A) & \xrightarrow{H_n(j_2)} & H_n(X, A) & \xrightarrow{H_n(\ell_2)} & H_n(X, X_2) & \xrightarrow{\partial''_n} & H_{n-1}(X_2, A)
 \end{array}$$

Define ∂_n as indicated. Naturality of ∂_n follows from naturality of the sequence of a triple (and is otherwise clear). Exactness: diagram chase.

At $H_{n-1}(X_0, A)$: Let $z \in \ker H_{n-1}(i_1) \cap \ker H_{n-1}(i_2)$. Then there is a $z_1 \in H_n(X_1, X_0)$ with $\partial'_n(z_1) = z$. Regard z_1 as an element of $H_n(X, X_2) \cong H_n(X_1, X_0)$. Then $\partial''_n(z_1) = 0$, so there is a $z_2 \in H_n(X, A)$ with $H_n(\ell_2)(z_2) = z_1$. It follows that

$$\partial_n(z_2) = \partial'_n(H_n(\ell_2)(z_2)) = \partial'_n(z_1) = z \in \text{im } \partial_n.$$

Conversely, let $z \in \text{im } \partial_n$; then $z \in \text{im } \partial'_n$ as well, and hence $H_{n-1}(i_1)(z) = 0$. Moreover, for $z = \partial'_n(H_n(\ell_2)(z_2))$, we have

$$H_{n-1}(i_2)(z) = \partial''_n(H_n(\ell_2)(z_2)) = 0,$$

so $z \in \ker H_{n-1}(i_1) \cap \ker H_{n-1}(i_2)$. Exactness at the other two positions: exercise! $\square$

Recall: A homotopy-commutative square

$$\begin{array}{ccc}
 A & \xrightarrow{f_2} & Y \\
 f_1 \downarrow & & \downarrow g_2 \\
 X & \xrightarrow{g_1} & Z
 \end{array}$$

is called *homotopy cocartesian* (or a *homotopy pushout*) if all vertical arrows in the diagram induced by $g_1 \circ f_1 \simeq_h g_2 \circ f_2$ are “ $\simeq$ ”:

$$\begin{array}{ccccc}
 A \times I & \longrightarrow & M_{f_2} & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & M_{f_1} & \longrightarrow & M_{f_1, f_2} \\
 & & \downarrow & & \downarrow \simeq \\
 A & \longrightarrow & Y & & Z \\
 & \searrow & \downarrow & \searrow & \\
 & & X & \longrightarrow & Z
 \end{array}
 \quad \text{Example).}$$

Theorem III.3.3. (Mayer–Vietoris sequence for homotopy pushouts) Let $\begin{array}{ccc} A & \xrightarrow{f_2} & Y \\ f_1 \downarrow & & \downarrow g_2 \\ X & \xrightarrow{g_1} & Z \end{array}$ be

homotopy cocartesian. Then there is a natural long exact sequence

$$\dots \longrightarrow H_n(A) \xrightarrow{(H_n(f_1), H_n(f_2))} H_n(X) \oplus H_n(Y) \xrightarrow{H_n(g_1) - H_n(g_2)} H_n(Z) \xrightarrow{\partial_n} \dots$$

Proof. We have

$$\begin{array}{ccccc}
A \times (0, 1) & \longrightarrow & \mathring{M}_{f_2} & & \\
\downarrow \simeq & \searrow & \downarrow \simeq & \searrow & \\
& & \mathring{M}_{f_1} & \longrightarrow & M_{f_1, f_2} \\
& & \downarrow \simeq & & \downarrow \simeq \\
A & \longrightarrow & Y & & Z \\
& \searrow & \searrow & & \\
& & X & \longrightarrow & Z
\end{array}$$

and $(M_{f_1, f_2}; \mathring{M}_{f_1}, \mathring{M}_{f_2})$ is excisive. $\square$

In the special case $\begin{array}{ccc} A & \xrightarrow{f} & Y \\ i \downarrow & \lrcorner & \downarrow j \\ X & \xrightarrow{\bar{f}} & Z \end{array}$ with a cofibration i , for $B \subseteq A$ and $B' \subseteq Y$ with $f(B) \subseteq B'$ we have the relative version

$$\cdots \longrightarrow H_n(A, B) \longrightarrow H_n(X, B) \oplus H_n(Y, B') \longrightarrow H_n(Z, B') \longrightarrow \cdots.$$

Setting $A = \bullet \in X_1 \cap X_2$ in [Theorem III.3.2](#), we obtain

$$\cdots \longrightarrow \tilde{H}_n(X_0) \longrightarrow \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_{n-1}(X_0) \longrightarrow \cdots$$

for an excisive triad $(X; X_1, X_2)$ with $X_1 \cap X_2 \neq \emptyset$. The proof of [Theorem III.3.3](#) therefore also gives

$$\cdots \longrightarrow \tilde{H}_n(A) \longrightarrow \tilde{H}_n(X) \oplus \tilde{H}_n(Y) \longrightarrow \tilde{H}_n(Z) \longrightarrow \tilde{H}_{n-1}(A) \longrightarrow \cdots$$

for a homotopy pushout $\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z. \end{array}$

Example: $\begin{array}{ccc} S^1 & \xrightarrow{\quad} & M \\ \downarrow & \lrcorner & \downarrow \\ M & \longrightarrow & K \end{array}$ (Gluing two Möbius bands along their boundaries gives a Klein bottle.)

Let $H_* = H_*^{\text{sing}}(-, -; \mathbb{Z})$.

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & H_2(M) \oplus H_2(M) & \longrightarrow & H_2(K) & \xrightarrow{0} & H_1(S^1) \longrightarrow H_1(M) \oplus H_1(M) \longrightarrow \\
& & \wr & & \wr & & \wr \\
& & 0 & & 0 & & \mathbb{Z} \oplus \mathbb{Z}
\end{array}$$

$$\begin{array}{ccccccc}
& & & & x \mapsto (x, x) & & \\
& \longrightarrow & H_1(K) & \xrightarrow{0} & H_0(S^1) & \longrightarrow & H_0(M) \oplus H_0(M) \longrightarrow H_0(K) \\
& & \wr & & \wr & & \wr \\
& & \mathbb{Z} \oplus \mathbb{Z}/2 & & \mathbb{Z} & & \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}.
\end{array}$$

Example: $\begin{array}{ccc} S^{n-1} & \xrightarrow{f} & Y \\ i \downarrow & \lrcorner & \downarrow j \\ D^n & \xrightarrow{\bar{f}} & Z \end{array}$ (Cell attachment). Since $\tilde{H}_*(D^n) \cong 0$, the Mayer–Vietoris sequence

has the form

$$\cdots \longrightarrow \tilde{H}_\ell(S^{n-1}) \xrightarrow{\tilde{H}_\ell(f)} \tilde{H}_\ell(Y) \xrightarrow{\tilde{H}_\ell(j)} \tilde{H}_\ell(Z) \xrightarrow{\partial} \tilde{H}_{\ell-1}(S^{n-1}) \xrightarrow{\tilde{H}_{\ell-1}(f)} \tilde{H}_{\ell-1}(Y) \longrightarrow \cdots.$$

Thus $\tilde{H}_\ell(j)$ is an isomorphism if and only if $\tilde{H}_{\ell-1}(f)$ is injective and $\tilde{H}_\ell(f) = 0$. If H_* is ordinary, this holds automatically for $\ell \notin \{n, n-1\}$.

- $\ell = n$: Cell attachment can only enlarge the ℓ th homology.
- $\ell = n - 1$: The ℓ th homology can only shrink.

We now fulfill an earlier promise:

Theorem III.3.4. Let (H_*, ∂_*) be additive and let $\{(X_i, x_i)\}_{i \in I}$ be well-pointed spaces. Then the maps $X_i \rightarrow \bigvee_{i \in I} X_i$ induce isomorphisms

$$\bigoplus_{i \in I} \tilde{H}_*(X_i) \cong \tilde{H}_*\left(\bigvee_{i \in I} X_i\right).$$

Proof. Additivity, the five lemma, and [Proposition III.1.4](#) give

$$\bigoplus_{i \in I} \tilde{H}_*(X_i) \cong \bigoplus_{i \in I} H_*(X_i, x_i) \cong H_*\left(\coprod_i X_i, \coprod_i \{x_i\}\right) \cong \tilde{H}_*\left(\bigvee_{i \in I} X_i\right). \quad \square$$

III.4. The Degree of a Map

Goal: Classify all maps $f: S^n \rightarrow S^n$ up to homotopy and determine the induced morphisms in homology.

Definition III.4.1. The *degree* of $f: S^n \rightarrow S^n$ is the unique integer $\deg f \in \mathbb{Z}$ such that

$$\tilde{H}_n^{\text{sing}}(f)(z) = (\deg f) \cdot z \quad \text{for all } z \in \tilde{H}_n^{\text{sing}}(S^n) \cong \mathbb{Z}.$$

Observation:

- $f \simeq g \implies \deg f = \deg g$.
- f null-homotopic $\implies \deg f = 0$ (for example, if f is not surjective).
- For $f, g: S^n \rightarrow S^n$, we have $\deg(f \circ g) = (\deg f) \cdot (\deg g)$.
- f a homotopy equivalence $\implies \deg f = \pm 1$.
- $\deg(\text{id}_{S^n}) = 1$.

Theorem III.4.2. Let (H_*, ∂_*) be a homology theory with values in $R\text{-Mod}$ and $f: S^n \rightarrow S^n$. Then

$$\tilde{H}_n(f)(z) = (\deg f) \cdot z \quad \text{for all } z \in \tilde{H}_n(S^n).$$

(We give the proof in the remainder of this section.)

Proposition III.4.3. Let $n \geq 0$, and let $r_n: S^n \rightarrow S^n$ be reflection across the hyperplane $\{x_{n+1} = 0\}$, that is,

$$r_n(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n, -x_{n+1}).$$

Then $\tilde{H}_n(r_n) = -\text{id}_{\tilde{H}_n(S^n)}$ and, in particular, $\deg r_n = -1$.

Proof. Induction on n . Base case:

$$r_0: (S^0; \{+1\}, \{-1\}) \rightarrow (S^0; \{-1\}, \{+1\}),$$

$$i_-: \{-1\} \rightarrow S^0, \quad i_+: \{+1\} \rightarrow S^0, \quad \sigma := H_0(\{+1\} \rightarrow \{-1\}).$$

Naturality of the Mayer–Vietoris sequence gives

$$\begin{array}{ccc} H_0(\{+1\}) \oplus H_0(\{-1\}) & \xrightarrow[\cong]{H_0(i_+) - H_0(i_-)} & H_0(S^0) \\ \tau: a \oplus b \mapsto \sigma^{-1}(b) \oplus \sigma(a) \downarrow & & \downarrow H_0(r_0) \\ H_0(\{+1\}) \oplus H_0(\{-1\}) & \xrightarrow[\cong]{H_0(i_+) - H_0(i_-)} & H_0(S^0). \end{array}$$

Let $D = \{a \oplus \sigma(a) : a \in H_0(\{+1\})\}$. With $\delta = (H_0(i_+) - H_0(i_-))|_D$, we have $\delta: D \xrightarrow{\cong} \tilde{H}_0(S^0)$ and $\tau|_D = \text{id}_D$, so

$$\tilde{H}_0(r_0) = -\delta \circ \text{id}_D \circ \delta^{-1} = -\text{id}_{\tilde{H}_0(S^0)}.$$

Inductive step: Let $\eta: \tilde{H}_n(S^n) \rightarrow \tilde{H}_{n-1}(S^{n-1})$ be the natural suspension isomorphism. Then $r_n = S(r_{n-1})$ and

$$\tilde{H}_n(r_n) = \eta^{-1} \circ \tilde{H}_{n-1}(r_{n-1}) \circ \eta \stackrel{\text{inductive hypothesis}}{=} \eta^{-1} \circ (-\text{id}_{\tilde{H}_{n-1}(S^{n-1})}) \circ \eta = -\text{id}_{\tilde{H}_n(S^n)}. \quad \square$$

Remark. Any two reflections in $\mathbb{R}^{n+1}$ are homotopic, so $\tilde{H}_n(r) = -\text{id}$ for every reflection r .

Theorem III.4.4. For every $f: S^{2n} \rightarrow S^{2n}$, there is an $x \in S^{2n}$ with $f(x) = \pm x$.

Proof. Otherwise, there are homotopies $\text{id}_{S^{2n}} \simeq_F f$ and $f \simeq_G -\text{id}_{S^{2n}}$,

$$F(x, t) = \frac{(1-t)x + tf(x)}{\|(1-t)x + tf(x)\|}, \quad G(x, t) = \frac{(1-t)(-x) + tf(x)}{\|(1-t)(-x) + tf(x)\|}.$$

Thus $\text{id}_{S^{2n}} \simeq -\text{id}_{S^{2n}} = r^1 \circ \dots \circ r^{2n+1}$, where r^i is reflection across the i th coordinate hyperplane, so

$$1 = \deg \text{id}_{S^{2n}} = \deg(r^1 \circ \dots \circ r^{2n+1}) = (-1)^{2n+1} = -1,$$

a contradiction. $\square$

Theorem III.4.5. (Hairy ball theorem). Every tangent vector field on an even-dimensional sphere has a zero: Given $v: S^{2n} \rightarrow \mathbb{R}^{2n+1}$ with $\langle v(x), x \rangle = 0$ for all $x \in S^{2n}$, there is an $x_0 \in S^{2n}$ with $v(x_0) = 0$.

Proof. Otherwise, there is an $x_0 \in S^{2n}$ with $v(x_0)/\|v(x_0)\| = \pm x_0$, that is,

$$0 = \frac{\langle v(x_0), x_0 \rangle}{\|v(x_0)\|} = \langle \pm x_0, x_0 \rangle = \pm 1,$$

a contradiction. $\square$

Thus, at any given time, there is a place on Earth with no wind.

Proposition III.4.6. Let $f_k: S^1 \rightarrow S^1$, $z \mapsto z^k$. Then $\tilde{H}_1(f_k) = k \cdot \text{id}_{\tilde{H}_1(S^1)}$ (in particular, $\deg f_k = k$; compare the naturality of the Hurewicz isomorphism).

Proof. We have $f_{-k}(z) = \overline{f_k(z)} = r_1(f_k(z))$. By Proposition III.4.3, we may assume without loss of generality that $k \geq 0$. Consider

$$\begin{array}{ccc} & \bigvee_{i=1}^k S^1 & \\ g_k \nearrow & & \searrow \bigvee_{i=1}^k \text{id}_{S^1} \\ S^1 & \xrightarrow{f_k} & S^1. \end{array}$$

Also, let $q_j: \bigvee_{i=1}^k S^1 \rightarrow S^1$ be the map that collapses all circles except the j th. Then $q_j \circ g_k \simeq \text{id}_{S^1}$, that is,

$$(\tilde{H}_1(q_1), \dots, \tilde{H}_1(q_k)) \circ \tilde{H}_1(g_k) = (\text{id}_{\tilde{H}_1(S^1)}, \dots, \text{id}_{\tilde{H}_1(S^1)}).$$

For the j th inclusion $i_j: S^1 \rightarrow \bigvee_{i=1}^k S^1$, we have $(\bigvee_{i=1}^k \text{id}_{S^1}) \circ i_j = \text{id}_{S^1}$, that is,

$$\tilde{H}_1\left(\bigvee_{i=1}^k \text{id}_{S^1}\right) \circ (\tilde{H}_1(i_1) \oplus \dots \oplus \tilde{H}_1(i_k)) = \tilde{H}_1(\text{id}_{S^1}) \oplus \dots \oplus \tilde{H}_1(\text{id}_{S^1}).$$

Since

$$\left(\bigoplus_i \tilde{H}_1(i_i)\right) \circ \left(\prod_j \tilde{H}_1(q_j)\right) = \text{id}_{\tilde{H}_1(\bigvee S^1)}$$

(see the discussion following Theorem III.1.6), we have

$$\begin{aligned} \tilde{H}_1(f_k) &= \tilde{H}_1\left(\bigvee_i \text{id}_{S^1}\right) \circ \tilde{H}_1(g_k) \\ &= \text{id}_{\tilde{H}_1(S^1)} + \dots + \text{id}_{\tilde{H}_1(S^1)} = k \cdot \text{id}_{\tilde{H}_1(S^1)}. \end{aligned} \quad \square$$

By naturality of the suspension isomorphism, we therefore also have

$$\tilde{H}_n(S^{n-1}(f_k)) = k \cdot \text{id}_{\tilde{H}_n(S^n)}$$

and, in particular, $\deg(S^{n-1}(f_k)) = k$. Setting $[X, Y] := \text{Hom}_{\text{HoTop}}(X, Y)$, it follows that

$$\deg: [S^n, S^n] \longrightarrow \mathbb{Z}$$

is surjective. Theorem III.4.2 follows once we show that $\deg$ is injective. **We need a connection to differential topology.**

Proposition III.4.7. Let M be a smooth manifold, let $f: M \rightarrow \mathbb{R}^k$ be continuous, and let $\delta: M \rightarrow \mathbb{R}$ be positive and continuous. Then there is a smooth map $\bar{f}: M \rightarrow \mathbb{R}^k$ such that $\|f(x) - \bar{f}(x)\| < \delta(x)$ for all $x \in M$.

Proof. For $x \in M$, let $U_x \subseteq M$ be a neighborhood of x such that $\delta(y) > \frac{1}{2}\delta(x)$ and $\|f(y) - f(x)\| < \frac{1}{2}\delta(x)$ for all $y \in U_x$. Since M is second countable, M is separable (*Introduction to Topology*, I.1), and the cover $(U_x)_{x \in M}$ has a countable subcover: There are countably many points $x_1, x_2, \dots \in M$ such that $U_{x_1}, U_{x_2}, \dots$ cover all of M . Choose a subordinate smooth partition of unity, $\{\varphi_i: M \rightarrow [0, 1]\}$ (that is, for each $x \in M$, we have $\varphi_i(x) = 0$ for all but finitely many i , $\sum_{i \geq 1} \varphi_i(x) = 1$, $\text{supp } \varphi_i \subseteq U_{x_i}$). Define

$$\bar{f}: M \rightarrow \mathbb{R}^k, \quad \bar{f}(x) = \sum_{i \geq 1} \varphi_i(x) f(x_i).$$

Then $\bar{f}$ is smooth and

$$\begin{aligned} \|f(x) - \bar{f}(x)\| &= \left\| \sum_{i \geq 1} \varphi_i(x) f(x) - \sum_{i \geq 1} \varphi_i(x) f(x_i) \right\| \\ &\leq \sum_{i \geq 1} \varphi_i(x) \|f(x) - f(x_i)\| \leq \sum_{i \geq 1} \varphi_i(x) \frac{1}{2} \delta(x_i) < \delta(x). \end{aligned} \quad \square$$

Proposition III.4.8. Let $f: M \rightarrow S^n$ be continuous, with M smooth. Then there is a smooth map $\tilde{f}: M \rightarrow S^n$ with $f \simeq \tilde{f}$.

Proof. By the preceding proposition, $f: M \rightarrow S^n \subseteq \mathbb{R}^{n+1}$ has a smooth approximation $\tilde{f}: M \rightarrow \mathbb{R}^{n+1}$ with $\|f(x) - \tilde{f}(x)\| < 1$ for all $x \in M$. Let $r: \mathbb{R}^{n+1} \setminus \{0\} \rightarrow S^n$ be the radial retraction. Then $r \circ \tilde{f} =: \tilde{f} \simeq_h f$ with $h(x, t) = r(tf(x) + (1-t)\tilde{f}(x))$. $\square$

Remark. The proposition holds more generally for $f: M \rightarrow N$ with N smooth (Whitney embedding theorem $N \subseteq \mathbb{R}^n$, existence of tubular neighborhoods).

Recall (differential topology seminar). Let $f: M^n \rightarrow N^n$ be smooth, and let $y \in N$ be a regular value (that is, for all $x \in M$ with $f(x) = y$, we have $df_x: T_x M \xrightarrow{\cong} T_y N$). Let M^n be compact, let N^n be connected, and let M^n and N^n be oriented (that is, we have chosen locally compatible orientations on all tangent spaces of M^n and N^n). Then

$$\deg^{\text{diff}}(f) := \sum_{x \in f^{-1}(y)} \text{sgn } df_x \in \mathbb{Z}$$

with $\text{sgn } df_x \in \{\pm 1\}$ ($+/ -$ if df_x preserves/reverses orientation),

- is independent of the choice of y , and
- if f is smoothly homotopic to g , then $\deg^{\text{diff}}(f) = \deg^{\text{diff}}(g)$.

Theorem III.4.9. Equip S^n with the standard orientation ($S^n = \partial D^{n+1}$), and let $f: S^n \rightarrow S^n$ be smooth. Then $\deg^{\text{diff}}(f) = \deg(f)$.

Lemma III.4.10. Let $A \in \text{GL}_n \mathbb{R}$. Using $S^n \cong \mathbb{R}^n \cup \{\infty\}$, A induces a map $f_A: S^n \rightarrow S^n$ of degree $\text{sgn } \det A \in \{\pm 1\}$.

Proof. By Gaussian elimination, A is a product of elementary matrices. For

$$A = \text{diag}(1, \dots, 1, \lambda, 1, \dots, 1)$$

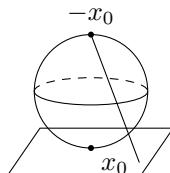
we have $f_A \simeq f_B$ with $B = \text{diag}(1, \dots, 1, \pm 1, 1, \dots, 1)$; apply Proposition III.4.3. Likewise, apply it to the reflection

$$\begin{pmatrix} I & & \\ & 0 & 1 \\ & 1 & 0 \\ & & I \end{pmatrix}.$$

For a shear matrix $A = I + \lambda E_{ij}$, $i \neq j$, we have $f_A \simeq \text{id}_{S^n}$. $\square$

Lemma III.4.11. Let $f: S^n \rightarrow S^n$ be such that there is an $x_0 \in S^n$ with $f^{-1}(x_0) = \{x_0\}$ and an open neighborhood $U \subseteq S^n$ of x_0 on which f is smooth, with df_{x_0} invertible. Then $\deg f = \text{sgn } \det df_{x_0}$.

Proof. Identify $S^n \cong \mathbb{R}^n \cup \{\infty\}$ by stereographic projection.



Then, for $A \in \mathrm{GL}_n \mathbb{R}$, we have $\mathrm{Jac}_0(s \circ f_A \circ s^{-1}) = A$. Setting $A = (\mathrm{Jac}_0(s \circ f \circ s^{-1}))^{-1}$, we obtain

$$\begin{aligned} \mathrm{Jac}_0(s \circ f \circ f_A \circ s^{-1}) &= \mathrm{Jac}_0(s \circ f \circ s^{-1} \circ s \circ f_A \circ s^{-1}) \\ &= \mathrm{Jac}_0(s \circ f \circ s^{-1}) \mathrm{Jac}_0(s \circ f_A \circ s^{-1}) = A^{-1}A = I_n, \end{aligned}$$

that is, $d(f \circ f_A)_{x_0} = \mathrm{id}_{T_{x_0} S^n}$. Since $f = (f \circ f_A) \circ f_{A^{-1}}$ and $\deg f_{A^{-1}} \stackrel{\text{Lemma}}{=} \mathrm{sgn} \det A^{-1} = \mathrm{sgn} \det df_{x_0}$, it remains to show that $\deg f = 1$ if $df_{x_0} = \mathrm{id}$. We show, in fact, that $f \simeq \mathrm{id}_{S^n}$. By Taylor's theorem, $f(x) = x + g(x)$ in the chart $s = (x_1, \dots, x_n)$ around x_0 , and $\|g(x)\| < \frac{1}{2}\|f(x)\|$ for $0 < \|x\| \leq 2\varepsilon$. Define $h: S^n \times I \rightarrow S^n$ by

$$h(x, t) = \begin{cases} f(x), & 2\varepsilon \leq \|x\|, \\ f(x) - t(2 - \frac{\|x\|}{\varepsilon})g(x), & \varepsilon \leq \|x\| \leq 2\varepsilon, \\ f(x) - tg(x), & \|x\| \leq \varepsilon. \end{cases}$$

Set $f_1(x) := h(x, 1)$. Since $\|g(x)\| < \frac{1}{2}\|f(x)\|$ for $\|x\| \leq 2\varepsilon$, we still have $f_1^{-1}(x_0) = \{x_0\}$. Choose a disk neighborhood of x_0 such that the complementary disk D satisfies $f_1(D) \subseteq D$ and $f_1|_{\partial D} = \mathrm{id}$. Then $f_1|_D \simeq_{\partial D} \mathrm{id}_D$ via $h'(x, t) = tx + (1-t)f_1(x)$, and hence $f_1 \simeq \mathrm{id}_{S^n}$ on the whole sphere. $\square$

Proof of Theorem III.4.9. Let $y \in S^n$ be a regular value, let $f^{-1}(y) = \{x_1, \dots, x_k\}$, and let $\mathring{D}^n \cong D_i \ni x_i$ be disk neighborhoods such that $f|_{D_i}$ is a diffeomorphism onto its image $D \subseteq S^n$. We obtain

$$\begin{array}{ccc} S^n & \xrightarrow{f} & S^n \\ \downarrow & & \downarrow \simeq \\ S^n / (S^n \setminus \bigcup_{i=1}^k D_i) & \longrightarrow & S^n / (S^n \setminus D) \\ \downarrow \cong \text{use orientation} & & \downarrow \cong \text{use orientation} \\ \bigvee_{i=1}^k S^n & \xrightarrow{\bigvee_i \overline{f|_{D_i}}} & S^n. \end{array}$$

Thus, as in the proof of Proposition III.4.6,

$$\deg f = \sum_{i=1}^k \deg(\overline{f|_{D_i}}) \stackrel{\text{preceding lemma}}{=} \sum_{i=1}^k \mathrm{sgn} \det d(\overline{f|_{D_i}})_{x_i} = \sum_{i=1}^k \mathrm{sgn} df_{x_i} = \deg^{\mathrm{diff}}(f). \quad \square$$

Recall (differential topology seminar, Milnor, *Topology from the Differentiable Viewpoint*, p. 51):

Theorem III.4.12. (Hopf). Let M^n be a connected, compact, oriented smooth manifold without boundary, and let $f, g: M^n \rightarrow S^n$ be smooth. Then f and g are smoothly homotopic if and only if $\deg^{\mathrm{diff}} f = \deg^{\mathrm{diff}} g$.

Corollary III.4.13. The map $\deg: [S^n, S^n] \rightarrow \mathbb{Z}$ is a bijection.

Proof. It remains to show that if $f, g: S^n \rightarrow S^n$ and $\deg f = \deg g$, then $f \simeq g$. We have $f \simeq \tilde{f}$, $g \simeq \tilde{g}$ with $\tilde{f}, \tilde{g}: S^n \rightarrow S^n$ smooth and

$$\deg^{\mathrm{diff}} \tilde{f} = \deg \tilde{f} = \deg f = \deg g = \deg \tilde{g} = \deg^{\mathrm{diff}} \tilde{g}.$$

Hopf's theorem gives $f \simeq \tilde{f} \simeq \tilde{g} \simeq g$. $\square$

Theorem III.4.14. Let $n \geq 1$. Then $\pi_k(S^n, \cdot) = 1$ for $k < n$, and $\deg: \pi_n(S^n, \cdot) \xrightarrow{\cong} \mathbb{Z}$ is a group isomorphism.

Proof. For $n = 1$, the assertion follows from $\pi_1(S^1) \cong \mathbb{Z}$ and Proposition III.4.6. In what follows, let $n \geq 2$. For $k < n$, a map $S^k \rightarrow S^n$ is homotopic to a smooth map, which is therefore not surjective. (Choosing the partition of unity appropriately makes the homotopy preserve the basepoint.) Hence $\pi_k(S^n, \cdot) = 1$. Since, for $[f], [g] \in \pi_n(S^n, \cdot)$,

$$[f] \cdot [g] = [f \diamond g: S^n \longrightarrow S^n \vee S^n \xrightarrow{f \vee g} S^n],$$

we again obtain

$$\deg([f] \cdot [g]) = \deg(f \diamond g) = \deg f + \deg g = \deg[f] + \deg[g],$$

that is, $\deg$ is a homomorphism. Since $\deg[\text{id}_{S^n}] = 1$, it is surjective. If $[f] \in \ker \deg$, the corollary implies that f is null-homotopic. Since $\pi_1(S^n, \cdot) = 1$ and $(S^n, \cdot)$ is well-pointed, f is also based null-homotopic. $\square$

An alternative without differential topology

Theorem III.4.7. (Simplicial approximation). Let X, Y be simplicial complexes, with X finite. Then, for every continuous map $f: X \rightarrow Y$, there are an $r \geq 0$ and a simplicial map $g: X^{[r]} \rightarrow Y$ with $f \simeq g$ ($X^{[r]}$: the r th barycentric subdivision).

K., Theorem 3.22. $\square$

Proposition III.4.8. Let $f: S^n \rightarrow S^n$ satisfy $\deg f = 0$. Then f is null-homotopic.

Proof. Use $S^n \cong \partial \Delta^{n+1} =: K$ to give the sphere a simplicial structure. We may assume that $f: K^{[r]} \rightarrow K$ is simplicial. Fix a vertex $v \in K$ with opposite face τ . “Stretch τ over the sphere” using

$$\Phi: (K, \overline{K \setminus \tau}) \longrightarrow (K, v),$$

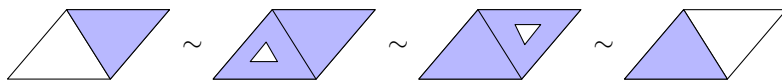
such that $\Phi \simeq \text{id}_K$ and $\Phi|_{\hat{\tau}}: \hat{\tau} \xrightarrow{\cong} K \setminus \{v\}$. Thus $f \simeq g := \Phi \circ f$. We call $\sigma \in K^{[r]}$ *ordinary* if $g(\sigma) = \{v\}$, and *special* otherwise. We obtain

$$g: K^{[r]} \longrightarrow \bigvee_{\sigma \text{ special}} S^n \xrightarrow{\bigvee_{\sigma} g_{\sigma}} K$$

with $g_{\sigma}: (S^n, \cdot) \xrightarrow{\cong} (K, v)$ for special σ . As in the proof of III.4.6, we obtain

$$\deg g = \sum_{\sigma \text{ special}} \deg(g_{\sigma})$$

with $\deg g_{\sigma} = \pm 1$. Since $\deg g = 0$, there are equally many “+1” and “−1” terms in the sum. We can make “+1” simplices adjacent to “−1” simplices by moves of the form



Finally, on adjacent special simplices with opposite signs, g is null-homotopic (to v). $\square$

Theorem III.4.9. Let $n \geq 1$. Then $\pi_k(S^n, \cdot) = 1$ for $k < n$, and $\deg: \pi_n(S^n) \xrightarrow{\cong} \mathbb{Z}$ is a group isomorphism.

Proof. By simplicial approximation, for $k < n$, every map $S^k \rightarrow S^n$ is homotopic to a map that is not surjective and hence is based null-homotopic, that is, $\pi_k(S^n) = 1$. For $[f], [g] \in \pi_n(S^n, \cdot)$, we have

$$[f] \cdot [g] = [S^n \longrightarrow S^n \vee S^n \xrightarrow{f \vee g} S^n]$$

and hence $\deg([f] \cdot [g]) = \deg f + \deg g$, that is, $\deg$ is a group homomorphism. It is surjective because $\deg[\text{id}_{S^n}] = 1$, and injective by Proposition III.4.8, since the null-homotopy constructed there is based at v . $\square$

Corollary III.4.10. For $n \geq 1$, $\deg: [S^n, S^n] \xrightarrow{\cong} \mathbb{Z}$ is a bijection.

Proof. Given $f: S^n \rightarrow S^n$, choose $r: I \rightarrow \text{SO}(n+1)$, $r_0 = \text{id}$, $r_1 f(\cdot) = \cdot$. Then $f \simeq_{r \circ f} \bar{f} := r_1 f$, and $\bar{f}$ is based. $\square$

Proof of Theorem III.4.2. Let $f: S^n \rightarrow S^n$. By the corollary, $f \simeq S^{n-1}(f_k)$ with $k = \deg f$. Hence

$$\tilde{H}_n(f) = \tilde{H}_n(S^{n-1}(f_k)) = \eta^{-1} \circ \tilde{H}_1(f_k) \circ \eta \stackrel{\text{III.4.6}}{=} k \cdot \text{id}_{\tilde{H}_n(S^n)}$$

where the $(n-1)$ -fold natural suspension isomorphism is $\eta: \tilde{H}_n(S^n) \xrightarrow{\cong} \tilde{H}_1(S^1)$. $\square$

III.5. Applications

Goal: We present applications of homology. Set $H_*(-) = H_*^{\text{sing}}(-; \mathbb{Z}/2)$.

The fundamental theorem of algebra

Theorem III.5.1. Let $p \in \mathbb{C}[z]$ satisfy $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then p is constant.

Proof. Let $f_t: S^1 \rightarrow S^1$, $f_t(z) = p(tz)/|p(tz)|$. Since

$$h(s, z) = \frac{p((1-s)tz)}{|p((1-s)tz)|}$$

is a null-homotopy, we have $\deg f_t = 0$. For $p(z) = \sum_{k=0}^n a_k z^k$, $a_n \neq 0$, let $r_0 > 0$ be such that

$$|a_n z^n| > \left| \sum_{k=0}^{n-1} a_k z^k \right| \quad \text{for } |z| = r_0.$$

Set $p_s(z) = a_n z^n + s \sum_{k=0}^{n-1} a_k z^k$. Then $g(s, z) = p_s(r_0 z)/|p_s(r_0 z)|$ is a homotopy

$$\frac{a_n}{|a_n|} z^n \simeq g_0 \simeq g_1 = f_{r_0}.$$

It follows that $n = \deg g_0 = \deg g_1 = \deg f_{r_0} = 0$. $\square$

Invariance of dimension

Theorem III.5.2. $\mathbb{R}^n \cong \mathbb{R}^m \implies n = m$.

Proof. The case where one of the dimensions is zero follows from the fact that $\mathbb{R}^0$ is a point. For $n, m \geq 1$, we have

$$S^{n-1} \simeq \mathbb{R}^n \setminus \{0\} \xrightarrow[\cong]{f} \mathbb{R}^m \setminus \{f(0)\} \simeq S^{m-1},$$

so $\tilde{H}_{n-1}(S^{n-1}) \cong \tilde{H}_{n-1}(S^{m-1})$. Corollary III.1.10 gives $n - 1 = m - 1$. $\square$

No retractions onto the boundary

Theorem III.5.3. The disk D^n does not retract onto its boundary.

Proof. Otherwise, there is an $r: D^n \rightarrow S^{n-1}$ such that

$$\begin{array}{ccc} & D^n & \\ i \nearrow & & \searrow r \\ S^{n-1} & \xrightarrow{\text{id}} & S^{n-1} \end{array} \xrightarrow{\tilde{H}_{n-1}} \begin{array}{ccc} & 0 & \\ \nearrow & & \searrow \\ \mathbb{Z}/2 & \xrightarrow{\text{id}} & \mathbb{Z}/2, \end{array}$$

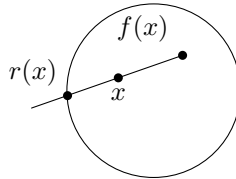
a contradiction. $\square$

Remark. More generally, a compact manifold does not retract onto its boundary (Topology II).

Brouwer's fixed point theorem

Theorem III.5.4. Every map $f: D^n \rightarrow D^n$ has a fixed point.

Proof. This is clear for $n = 0$. For $n \geq 1$, if the map has no fixed point, we obtain $r: D^n \rightarrow S^{n-1}$:



with $r|_{S^{n-1}} = \text{id}_{S^{n-1}}$. This contradicts the preceding theorem. $\square$

Remark. The theorem implies the existence of “Nash equilibria” in game theory.

The Borsuk–Ulam theorem

Theorem III.5.5. (Borsuk–Ulam). For every $f: S^n \rightarrow \mathbb{R}^n$, there is an $x \in S^n$ with $f(x) = f(-x)$. Thus, on every great circle of Earth, there are always antipodal points with the same temperature. On Earth, there are always antipodal points with the same temperature and the same atmospheric pressure.

Theorem III.5.6. (Antipodal theorem). Let $g: S^n \rightarrow S^m$ satisfy $g(-x) = -g(x)$ for all $x \in S^n$. Then $n \leq m$.

Theorem III.5.6 $\implies$ Theorem III.5.5. Suppose there were an $f: S^n \rightarrow \mathbb{R}^n$ with $f(x) \neq f(-x)$ for all $x \in S^n$. Then $g: S^n \rightarrow S^{n-1}$,

$$g(x) := \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|},$$

satisfies $g(-x) = -g(x)$, a contradiction. $\square$

Proof of Theorem III.5.6. $\mathbb{Z}/2 = \langle t \mid t^2 \rangle$ acts on S^k by $t \cdot x = -x$. We have $S^k/(\mathbb{Z}/2) = \mathbb{RP}^k$, and $p: S^k \rightarrow \mathbb{RP}^k$ is a covering map (universal for $k \geq 2$).

$$\begin{array}{ccc} & & S^k \\ & \nearrow \bar{\sigma}_+ & \downarrow p \\ \Delta_q & \xrightarrow{\sigma} & \mathbb{RP}^k, \end{array} \quad t \cdot \bar{\sigma}_\pm = \bar{\sigma}_\mp.$$

Set $C_* := C_*^{\text{sing}}(-; \mathbb{Z}/2)$. We obtain

$$\tau_*: C_*(\mathbb{RP}^k) \longrightarrow C_*(S^k), \quad \tau_*(\sigma) = \bar{\sigma}_+ + \bar{\sigma}_-.$$

This is a chain map (the *transfer*). We obtain a short exact sequence

$$0 \longrightarrow C_*(\mathbb{RP}^k) \xrightarrow{\tau_*} C_*(S^k) \xrightarrow{p_*} C_*(\mathbb{RP}^k) \longrightarrow 0.$$

Let $g: S^n \rightarrow S^m$ satisfy $g(-x) = -g(x)$. We obtain $\bar{g}: \mathbb{RP}^n \rightarrow \mathbb{RP}^m$ and

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_*(\mathbb{RP}^n) & \xrightarrow{\tau_*} & C_*(S^n) & \xrightarrow{p_*} & C_*(\mathbb{RP}^n) \longrightarrow 0 \\ & & \bar{g}_* \downarrow & & g_* \downarrow & & \downarrow \bar{g}_* \\ 0 & \longrightarrow & C_*(\mathbb{RP}^m) & \xrightarrow{\tau_*} & C_*(S^m) & \xrightarrow{p_*} & C_*(\mathbb{RP}^m) \longrightarrow 0. \end{array} \quad (*)$$

To check commutativity in $(*)$, consider

$$\begin{array}{ccccc} \Delta_q & \xrightarrow{\sigma} & \mathbb{RP}^n & \xleftarrow{p_n} & S^n & \xleftarrow[\bar{\sigma}_+]{\bar{\sigma}_-} \Delta_q \\ & & \bar{g} \downarrow & & \downarrow g \\ & & \mathbb{RP}^m & \xleftarrow{p_m} & S^m \end{array}$$

and

$$p_m \circ (g \circ \bar{\sigma}_\pm) = (p_m \circ g) \circ \bar{\sigma}_\pm = (\bar{g} \circ p_n) \circ \bar{\sigma}_\pm = \bar{g} \circ (p_n \circ \bar{\sigma}_\pm) = \bar{g} \circ \sigma.$$

The property of g and uniqueness of lifts imply commutativity.

We have the long exact sequence (here in dimension $m > 0$)

$$\begin{aligned}
\underbrace{H_{m+1}(\mathbb{RP}^m)}_{0 \text{ (III.2.3)}} &\xrightarrow{0} H_m(\mathbb{RP}^m) \xrightarrow{H_m(\tau_*)} H_m(S^m) \xrightarrow{H_m(p)} H_m(\mathbb{RP}^m) \longrightarrow \cdot \\
&\longrightarrow H_{m-1}(\mathbb{RP}^m) \longrightarrow \dots \longrightarrow 0 \longrightarrow H_1(\mathbb{RP}^m) \longrightarrow \cdot \\
&\longrightarrow H_0(\mathbb{RP}^m) \xrightarrow{0} H_0(S^m) \xrightarrow{\cong} H_0(\mathbb{RP}^m) \longrightarrow 0.
\end{aligned}$$

We have

$$\begin{aligned}
H_m(\tau_*) \circ H_m(p) &= H_m(\tau_* \circ p_*) \\
&= \text{id}_{H_m(S^m)} + H_m(S^m \rightarrow S^m, x \mapsto -x) \\
&= 2 \cdot \text{id}_{H_m(S^m)} = 0.
\end{aligned}$$

Indeed, the antipodal map is a homeomorphism and induces an automorphism of $H_m(S^m) \cong \mathbb{Z}/2$, hence the identity. Since $H_m(\tau_*)$ is injective, it follows that $H_m(p) = 0$. Therefore, $H_m(\tau_*)$ is an isomorphism. Moreover,

$$H_k(\mathbb{RP}^m) \xrightarrow[\cong]{\partial} H_{k-1}(\mathbb{RP}^m) \quad \text{for } 1 \leq k \leq m.$$

Naturality in Theorem II.2.2 gives, for $n > m$ and $1 \leq i \leq m$:

$$\begin{array}{ccc}
H_i(\mathbb{RP}^n) & \xrightarrow[\cong]{\partial} & H_{i-1}(\mathbb{RP}^n) \\
H_i(\bar{g}) \downarrow & & \downarrow H_{i-1}(\bar{g}) \\
H_i(\mathbb{RP}^m) & \xrightarrow[\cong]{\partial} & H_{i-1}(\mathbb{RP}^m).
\end{array}$$

Since $H_0(\bar{g})$ is an isomorphism, induction shows that $H_i(\bar{g})$ is an isomorphism for $0 \leq i \leq m$. Thus we obtain

$$\begin{array}{ccc}
H_m(\mathbb{RP}^n) & \xrightarrow{H_m(\tau_*)} & H_m(S^n) \\
H_m(\bar{g}) \downarrow \cong & & \downarrow H_m(g) \\
H_m(\mathbb{RP}^m) & \xrightarrow[\cong]{H_m(\tau_*)} & H_m(S^m),
\end{array}$$

but $H_m(S^n) = 0$ for $n > m > 0$, whereas $H_m(S^m) \cong \mathbb{Z}/2$ for $m > 0$, a contradiction. For $m = 0$, the theorem is clear. $\square$

The ham sandwich theorem

Theorem III.5.7. Let $A_1, \dots, A_n \subseteq \mathbb{R}^n$ be Borel sets of finite Lebesgue measure, $\lambda(A_i) < \infty$ for $i = 1, \dots, n$. Then there is an affine hyperplane in $\mathbb{R}^n$ that cuts each A_i into two subsets of equal measure.

Proof. Consider $\mathbb{R}^n \cong \mathbb{R}^n \times \{1\} \subseteq \mathbb{R}^{n+1}$. For $x \in \mathbb{R}^{n+1} \setminus \{0\}$, let

$$\begin{aligned}
H_x^+ &= \{y \in \mathbb{R}^{n+1} : \langle x, y \rangle \geq 0\} \cap (\mathbb{R}^n \times \{1\}), \\
H_x^- &= \{y \in \mathbb{R}^{n+1} : \langle x, y \rangle \leq 0\} \cap (\mathbb{R}^n \times \{1\}), \\
H_x &= H_x^+ \cap H_x^-.
\end{aligned}$$

Set $f_i: S^n \rightarrow \mathbb{R}$, $f_i(x) = \lambda(A_i \cap H_x^+)$. Then $f_i(-x) = \lambda(A_i \cap H_x^-)$. If $f: S^n \rightarrow \mathbb{R}^n$, $f = (f_1, \dots, f_n)$, is continuous, the assertion follows from Borsuk–Ulam. Thus, let (x_j) be a sequence in S^n with $\lim_{j \rightarrow \infty} x_j = x \in S^n$. The characteristic functions $\chi_{A_i \cap H_{x_j}^+}$ converge to $\chi_{A_i \cap H_x^+}$ outside $A_i \cap H_x$. Hence

$$\lim_{j \rightarrow \infty} \chi_{A_i \cap H_{x_j}^+} = \chi_{A_i \cap H_x^+} \quad \lambda\text{-almost everywhere.}$$

Since $\chi_{A_i \cap H_{x_j}^+} \leq \chi_{A_i}$ and $\int \chi_{A_i} d\lambda = \lambda(A_i) < \infty$, dominated convergence gives

$$\lim_{j \rightarrow \infty} f_i(x_j) = \lim_{j \rightarrow \infty} \int \chi_{A_i \cap H_{x_j}^+} d\lambda = \int \chi_{A_i \cap H_x^+} d\lambda = \lambda(A_i \cap H_x^+) = f_i(x). \quad \square$$

Borsuk–Ulam II

Theorem III.5.8. Let $\{A_1, \dots, A_{n+1}\}$ be a closed cover of S^n . Then there are an $x \in S^n$ and an $i \in \{1, \dots, n+1\}$ such that $\{\pm x\} \subseteq A_i$.

Proof. For $y \in S^n$, let $d_i(y) = d(y, A_i) = \inf_{a \in A_i} \|y - a\|$. Set $f = (d_1, \dots, d_n)$. By Borsuk–Ulam, there is

an $x \in S^n$ with $f(x) = f(-x)$. If $d_i(x) = 0$ for some i , then $\pm x \in A_i$, since A_i is closed. If, however, $d_i(x) > 0$ for $i = 1, \dots, n$, then $\pm x \notin A_1 \cup \dots \cup A_n$, so $\pm x \in A_{n+1}$. $\square$

Remark. The number $n+1$ is optimal. Consider the radial projection of an inscribed $\partial\Delta^{n+1}$ onto S^n .

IV. Cell Complexes and Cellular Homology

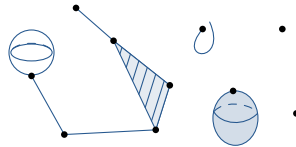
Recall: attaching a cell

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{f} & Y \\ i \downarrow & \lrcorner & \downarrow j \\ D^n & \xrightarrow{\bar{f}} & Z \end{array}$$

Mayer–Vietoris sequence:

$$\cdots \longrightarrow \tilde{H}_k(S^{n-1}) \xrightarrow{\tilde{H}_k(f)} \tilde{H}_k(Y) \xrightarrow{\tilde{H}_k(j)} \tilde{H}_k(Z) \xrightarrow{\partial} \tilde{H}_{k-1}(S^{n-1}) \xrightarrow{\tilde{H}_{k-1}(f)} \tilde{H}_{k-1}(Y) \longrightarrow \cdots$$

Suppose $X = \bigcup_{n=0}^{\infty} X^n$, with X^0 discrete and

$$\begin{array}{ccc} \coprod_{i \in I_n} S^{n-1} & \xrightarrow{\coprod f_i^n} & X^{n-1} \\ \downarrow & \lrcorner & \downarrow \\ \coprod_{i \in I_n} D^n & \xrightarrow{\coprod \bar{f}_i^n} & X^n \end{array}$$


For $i \in I_n$, $j \in I_{n-1}$, consider

$$f_{ij}^n: S^{n-1} \xrightarrow{f_i^n} X^{n-1} \xrightarrow{\text{collapse}} X^{n-1} / (X^{n-1} \setminus \text{im } \bar{f}_j^n) \xrightarrow{\cong} S^{n-1}.$$

The maps $(f_{ij}^n)_j$ form a decomposition of the attaching map f_i^n . The Mayer–Vietoris sequence suggests that, for an ordinary homology theory H_* , the homology $H_*(X)$ is already determined by the morphisms $\tilde{H}_*(f_{ij}^n)$. Thus:

- It seems possible to compute $H_*(X)$ combinatorially from the integers $\deg f_{ij}^n$.
- $H_*(X)$ is independent of $H_*!$ (Thus determined solely by the Eilenberg–Steenrod axioms.)

IV.1. Cell Complexes

Goal: Construct and study the category of cell complexes.

Definition IV.1.1. A *cell complex*, or *CW complex*, is a topological space X with a filtration

$$\emptyset = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \cdots \subseteq X,$$

such that, for every $n \geq 0$, there is a family of maps $Q_i^n: D^n \rightarrow X^n$ with $i \in I_n$ and $Q_i^n|_{S^{n-1}} =: q_i^n: S^{n-1} \rightarrow X^{n-1}$, and:

1.) The square

$$\begin{array}{ccc} \coprod_{i \in I_n} S^{n-1} & \xrightarrow{\coprod q_i^n} & X^{n-1} \\ \downarrow & \lrcorner & \downarrow j^n \\ \coprod_{i \in I_n} D^n & \xrightarrow{\coprod Q_i^n} & X^n \end{array}$$

is a pushout.

2.) $X = \operatorname{colim}_n X^n$.

Observations / terminology:

- $n = 0$:

$$\begin{array}{ccc} \emptyset & \xrightarrow{\quad} & \emptyset \\ \downarrow & \lrcorner & \downarrow \\ \coprod_{i \in I_0} \bullet & \xrightarrow{\cong} & X^0 \end{array}$$

that is, X^0 is discrete.

- The n -skeleton X^n is obtained from X^{n-1} by attaching n -cells. By [Lemma III.1.7](#), $j^n: X^{n-1} \rightarrow X^n$ is a cofibration. The pushout construction implies that j^n is closed and

$$X^n \setminus X^{n-1} \cong \coprod_{i \in I_n} \mathring{D}^n.$$

The path components e_i^n of $X^n \setminus X^{n-1}$ are called (*open*) n -cells. Q_i^n is called a *characteristic map* for the i th n -cell.

- $\mathbf{Top} \rightarrow \mathbf{Set}$ is a left adjoint and hence cocontinuous, that is, (2) implies $X = \bigcup_{n \geq -1} X^n$ as a set. Moreover, (2) implies that X carries the *final topology* with respect to $(X^n)_n$, that is:

$$A \subseteq X \text{ is closed} \iff A \cap X^n \subseteq X^n \text{ is closed for all } n.$$

Indeed, this topology is the finest one for which X is a cocone

$$\begin{array}{ccc} X^{n-1} & \xrightarrow{j^n} & X^n \\ & \searrow i_{n-1} \quad \swarrow i_n & \\ & X & \end{array}$$

at all, and the coarsest one for which this cocone is universal:

$$\begin{array}{ccc} X^{n-1} & \xrightarrow{j^n} & X^n \\ & \searrow \quad \swarrow & \\ & X' & \\ & \downarrow \text{id} & \\ & X & \end{array}$$

- **Important:** The filtration (X^n) is part of the structure of a cell complex; the characteristic

maps Q_i^n (equivalently, the pushouts $\begin{array}{ccc} \coprod S^{n-1} & \rightarrow & X^{n-1} \\ \downarrow & & \downarrow \\ \coprod D^n & \rightarrow & X^n \end{array}$) are not; only their existence is

required.

Theorem IV.1.2. Cell complexes are T_4 (normal and Hausdorff).

Proof. Exercise. □

Theorem IV.1.3. Let X be a cell complex and $C \subseteq X$ a subset.

- (i) C is closed $\iff C \cap \overline{e_i^n}$ is compact for all $i \in I_n, n \geq 0$.
- (ii) C is compact $\iff (C \text{ is closed and } C \cap e_i^n \neq \emptyset \text{ for only finitely many } i \text{ and } n)$.

Proof. (i) “ $\Rightarrow$ ”: $C \cap \overline{e_i^n} \subseteq \overline{e_i^n}$ is closed and hence compact, since $\overline{e_i^n} = Q_i^n(D^n)$ is compact. “ $\Leftarrow$ ”: Follows by induction, since

$$\left(\coprod_i Q_i^n \right) \amalg j^n: \left(\coprod_i D^n \right) \amalg X^{n-1} \longrightarrow X^n$$

is a quotient map and $C \cap Q_i^n(D^n) = C \cap \overline{e_i^n}$ is compact in a Hausdorff space and hence closed.

(ii) “ $\Leftarrow$ ”: C is closed,

$$C \subseteq C \cap \overline{e_{i_1}^{n_1}} \cup \dots \cup C \cap \overline{e_{i_k}^{n_k}}$$

the right-hand side is compact, so C is compact. “ $\Rightarrow$ ”: Consult the literature. □

Remark: CW complex: “C” stands for “closure finite”: each closed cell $\overline{e_i^n}$ meets only finitely many other cells. “W” stands for “weak topology”: weak in the sense that the intersections with the skeleta already determine whether a set is open / closed.

Example IV.1.4. $X = S^n, X^{-1} = \emptyset, X^0 = X^1 = \dots = X^{n-1} = \bullet, X^n = S^n$,

$$\begin{array}{ccc} \emptyset & \xrightarrow{\quad} & X^{k-1} \\ \downarrow \lrcorner & & \downarrow \\ \emptyset & \longrightarrow & X^k \end{array} \quad (k = 1, \dots, n-1), \quad \begin{array}{ccc} S^{n-1} & \xrightarrow{\quad} & X^{n-1} = \bullet \\ \downarrow \lrcorner & & \downarrow \\ D^n & \longrightarrow & X^n \end{array} \quad \text{and} \quad \text{Diagram of } S^n \text{ with a point}$$

Example IV.1.5. $X = \Sigma_g, X^{-1} = \emptyset, X^0 = \bullet, X^1 = \bigvee_{i=1}^{2g} S^1, X^2 = \Sigma_g$.

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & X^1 \\ \downarrow \lrcorner & & \downarrow \\ D^2 & \longrightarrow & X^2 \end{array} \quad f: S^1 \longrightarrow \bigvee_{i=1}^{2g} S^1, \quad \text{Diagram of } S^1 \text{ with } 2g \text{ arcs } a_i, b_i \quad \longmapsto \quad \prod_{i=1}^g [a_i, b_i].$$

Example IV.1.6. $X = \mathbb{RP}^d, X^n = \{[x_0 : \dots : x_n : 0 : \dots : 0]\} \cong \mathbb{RP}^n$, one cell in each dimension:

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{q^n} & X^{n-1} \cong \mathbb{RP}^{n-1} \\ \downarrow \lrcorner & & \downarrow \\ D^n & \xrightarrow{Q^n} & X^n \cong \mathbb{RP}^n \end{array}$$

$$q^n(x_1, \dots, x_n) = [x_1 : \dots : x_n] \quad (\text{universal covering for } n \geq 3),$$

$$Q^n(x_1, \dots, x_n) = \left[x_1 : \dots : x_n : \sqrt{1 - \|(x_1, \dots, x_n)\|^2} \right].$$

Example IV.1.7. $X = \mathbb{CP}^d$ has one $2n$ -cell in each even dimension,

$$X^{2n} = X^{2n+1} = \{[z_0 : \cdots : z_n : 0 : \cdots : 0]\} \cong \mathbb{CP}^n,$$

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{q^{2n}} & X^{2n-1} \cong \mathbb{CP}^{n-1} \\ \downarrow & \lrcorner & \downarrow \\ D^{2n} & \xrightarrow{Q^{2n}} & X^{2n} \cong \mathbb{CP}^n \end{array}$$

$$q^{2n}(z_1, \dots, z_n) = [z_1 : \cdots : z_n],$$

$$Q^{2n}(z_1, \dots, z_n) = \left[z_1 : \cdots : z_n : \sqrt{1 - \|(z_1, \dots, z_n)\|^2} \right].$$

Remark: $S^3 \xrightarrow{q^4} \mathbb{CP}^1 \cong S^2$ is the Hopf bundle.

Definition IV.1.8. Let A be a Hausdorff space. We define a *relative cell complex*, or *relative CW complex*, as before, with $X^{-1} = A$.

Definition IV.1.9. A map $f: (X, A) \rightarrow (Y, B)$ of relative cell complexes is called *cellular* if $f(X^n) \subseteq Y^n$ for $n \geq -1$.

$\rightsquigarrow$ categories CW and relCW.

Definition IV.1.10. Let X be a cell complex. A subspace $A \subseteq X$ is called a *subcomplex* if A is a closed union of open cells. In this case, A is itself a cell complex with $A^n := A \cap X^n$.

$\rightsquigarrow$ category $\text{CW}^{(2)}$ of CW pairs.

We have a functor $\text{CW}^{(2)} \rightarrow \text{relCW}$ that takes the filtration $(X^n)_{n=-\infty}^\infty$ to the filtration $(X^n \cup A)_{n=-1}^\infty$.

Definition IV.1.11. Let (X, A) be a relative CW complex. We define $\dim(X, A) = n$ if $X = X^n$ and $X \neq X^{n-1}$. We say that (X, A) is of *finite type* if it has only finitely many cells in each dimension. We call (X, A) *finite* if it has only finitely many cells in total.

Stability of the category CW:

- If X, Y are cell complexes, then so is $X \amalg Y$ (clear).

Theorem IV.1.12. Suppose X or Y is locally compact. Then $X \times Y$ is a cell complex with filtration

$$(X \times Y)^n = \bigcup_{p+q=n} X^p \times Y^q.$$

Proof. The n -cells in $X \times Y$ are products $e_k^p \times e_l^q$; use $D^n \cong D^p \times D^q$. We have quotient maps

$$f: \coprod_{n \geq 0} \coprod_{i \in I_n^X} D^n \longrightarrow X, \quad g: \coprod_{n \geq 0} \coprod_{i \in I_n^Y} D^n \longrightarrow Y.$$

To show: $f \times g$ is a quotient map. Without loss of generality, X is locally compact. Then

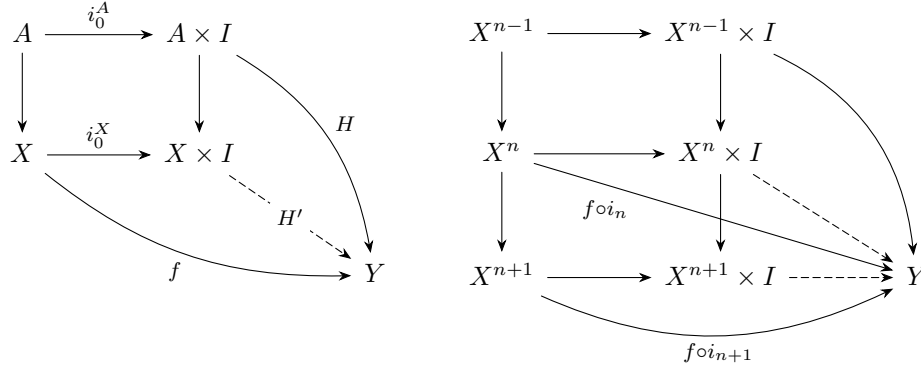
$$f \times g = (\text{id}_X \times g) \circ (f \times \text{id}_{\coprod_{n,i} D^n}),$$

and both maps are quotient maps by Problem Sheet 4/5, Exercise 2, *Introduction to Topology*. $\square$

- If (X, A) is a CW pair, then X/A is a cell complex.
- Hence so are CX , SX , ΣX .
- If $p: Y \rightarrow X$ is a covering and X is a cell complex, then so is Y .

Theorem IV.1.13. A relative CW complex (X, A) is a closed cofibration.

Proof. By Lemma III.1.7, each $j^n: X^{n-1} \rightarrow X^n$ is a closed cofibration. Thus (X^n, A) is a closed cofibration and (X, A) is closed. It remains to show that (X, A) is a cofibration.



We obtain $H': \text{colim}_n (X^n \times I) \rightarrow Y$. Since I is compact, $(-) \times I$ preserves not only pushouts (as we know), but also arbitrary colimits, that is,

$$\text{colim}_n (X^n \times I) = X \times I.$$

□

Corollary IV.1.14. For a relative cell complex (X, A) and any homology theory H_* , there is a long exact sequence

$$\cdots \longrightarrow \tilde{H}_n(A) \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_n(X/A) \longrightarrow \tilde{H}_{n-1}(A) \longrightarrow \cdots$$

Corollary IV.1.15. Let $\emptyset \neq X$ be a cell complex and $x_0 \in X^0$. Then (X, x_0) is well-pointed.

Theorem IV.1.16. Let X be a path-connected cell complex with exactly one 0-cell $x_0 \in X$. Choose characteristic maps

$$(Q_i^n, q_i^n): (D^n, S^{n-1}) \longrightarrow (X^n, X^{n-1})$$

for $i \in I_n$ and $n = 1, 2$. Then

$$\pi_1(X, x_0) = \langle [Q_i^1], i \in I_1 \mid [q_j^2], j \in I_2 \rangle.$$

Proof. Consider

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_1(X^{n-1}, x_0) & \xrightarrow{\pi_1(i_n)} & \pi_1(X^n, x_0) & \longrightarrow & \cdots \\ & & & \searrow & \downarrow & & \\ & & & & \pi_1(X, x_0) & & \end{array}$$

Claim: $\text{colim}_n \pi_1(X^n, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$.

Surjectivity: Let $[\gamma] \in \pi_1(X, x_0)$. Then $\gamma(I)$ is compact, so $\gamma(I) \subseteq X^n$ for sufficiently large n by [Theorem IV.1.3\(ii\)](#). Injectivity: Let $[\gamma] \in \pi_1(X^n, x_0)$ and let $H: I \times I \rightarrow X$ be a null-homotopy of γ in X . Then also $H(I \times I) \subseteq X^m$ for sufficiently large m , that is, $[\gamma] = 1$ in $\pi_1(X^m, x_0)$.

For $[\gamma] \in \pi_1(X^n, x_0)$, the image $\gamma(I)$ lies in a finite subcomplex of X^n , so

$$\pi_1(X^{n-1}, x_0) \cong \pi_1(X^n, x_0) \quad \text{for } n \geq 3$$

by [Theorem II.5.2 \(Introduction to Topology\)](#). Thus

$$\pi_1(X, x_0) = \operatorname{colim}_n \pi_1(X^n, x_0) = \pi_1(X^2, x_0).$$

Since X^2 is a presentation complex, the claim follows. $\square$

Attaching cell complexes. Let (X, A) be a CW pair, Y a CW complex, and $f: A \rightarrow Y$ a cellular map (the *attaching map*). Then

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ i \downarrow & \ulcorner & \downarrow j \\ X & \xrightarrow{\bar{f}} & Z \end{array} \quad (*)$$

in [Top](#) defines a cell complex Z with filtration

$$Z^n = \bar{f}(X^n) \cup j(Y^n)$$

(obtained by *attaching* X to Y).

- $\{n\text{-cells in } Z\} = \{n\text{-cells in } Y\} \cup \{n\text{-cells in } X \setminus A\}$.
- Characteristic maps are obtained by composing with $\bar{f}$ and j .
- The filtration satisfies

$$\begin{aligned} \operatorname{colim}_n Z^n &= \operatorname{colim}_n ((\bar{f} \amalg j)(X^n \amalg Y^n)) \\ &= (\bar{f} \amalg j)(\operatorname{colim}_n (X^n \amalg Y^n)) \\ &= (\bar{f} \amalg j)(X \amalg Y) = Z, \end{aligned}$$

since taking quotients is a left adjoint and hence cocontinuous.

We call $(*)$ a *cellular pushout*. By [Lemma III.1.5](#), (Z, Y) is again a CW pair, and $\bar{f}: X \rightarrow Z$ is cellular by construction.

Theorem IV.1.17. (Seifert–van Kampen: CW version) Let

$$\begin{array}{ccc} A & \rightarrow & Y \\ \downarrow & \ulcorner & \downarrow \\ X & \rightarrow & Z \end{array}$$

be a cellular pushout, let A, X, Y be path-connected, and let a_0, x_0, y_0, z_0 be compatible basepoints. Then

$$\begin{array}{ccc} \pi_1(A, a_0) & \longrightarrow & \pi_1(Y, y_0) \\ \downarrow & \ulcorner & \downarrow \\ \pi_1(X, x_0) & \longrightarrow & \pi_1(Z, z_0). \end{array}$$

Proof. By [Theorem IV.1.13](#), (X, A) is a cofibration, so the pushout version of the theorem applies. $\square$

Theorem IV.1.18. (Mayer–Vietoris sequence: CW version) Let

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ i \downarrow & \lrcorner & \downarrow j \\ X & \xrightarrow{\bar{f}} & Z \end{array}$$

be cellular, and let $B \subseteq A$, $B' \subseteq Y$ with $f(B) \subseteq B'$. Then we have a natural long exact sequence

$$\cdots \longrightarrow H_n(A, B) \xrightarrow{(H_n(i), H_n(f))} H_n(X, B) \oplus H_n(Y, B') \xrightarrow{H_n(\bar{f}) - H_n(j)} H_n(Z, B') \longrightarrow \cdots.$$

Proof. IV.1.13 and the discussion following III.3.3. $\square$

If $A \neq \emptyset$, set $B = B' = \bullet$ to obtain

$$\begin{aligned} \cdots \longrightarrow \tilde{H}_n(A) &\xrightarrow{(\tilde{H}_n(i), \tilde{H}_n(f))} \tilde{H}_n(X) \oplus \tilde{H}_n(Y) \\ &\xrightarrow{\tilde{H}_n(\bar{f}) - \tilde{H}_n(j)} \tilde{H}_n(Z) \xrightarrow{\tilde{\partial}_n} \tilde{H}_{n-1}(A) \longrightarrow \cdots, \end{aligned}$$

the *reduced Mayer–Vietoris sequence* of a cellular pushout.

IV.2. Cellular Homology and the Euler Characteristic

Goal: Compute the homology of cell complexes; define and determine the Euler characteristic. Let (H_*, ∂_*) be a homology theory with values in $R\text{-mod}$.

Definition IV.2.1. Let (X, A) be a relative cell complex. The *cellular chain complex* of (X, A) with respect to (H_*, ∂_*) is

$$C_n^{\text{CW}}(X, A; H_*) := H_n(X^n, X^{n-1})$$

with differentials

$$\partial_n^{\text{CW}}: C_n^{\text{CW}}(X, A; H_*) \longrightarrow C_{n-1}^{\text{CW}}(X, A; H_*)$$

given by

$$\partial_n: H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}, X^{n-2})$$

from the exact sequence of the triple (X^n, X^{n-1}, X^{n-2}) . We have the diagram

$$\begin{array}{ccccc} & & H_{n-1}(X^{n-1}) & & \\ & \nearrow & \downarrow & & \\ H_n(X^n, X^{n-1}) & \longrightarrow & H_{n-1}(X^{n-1}, X^{n-2}) & \longrightarrow & H_{n-2}(X^{n-2}, X^{n-3}) \\ & & \downarrow & \nearrow & \\ & & H_{n-2}(X^{n-2}) & & \end{array}$$

so $\partial_{n-1}^{\text{CW}} \circ \partial_n^{\text{CW}} = 0$.

The *cellular homology* of (X, A) with respect to (H_*, ∂_*) is

$$H_*^{\text{CW}}(X, A) := \underbrace{H_*}_{\text{homology of the chain complex}} \left(C_*^{\text{CW}} \left(X, A; \underbrace{H_*}_{\text{given homology theory}} \right) \right).$$

Theorem IV.2.2. Let (X, A) be a relative cell complex and (H_*, ∂_*) an ordinary homology theory. If (X, A) is infinite, also assume that (H_*, ∂_*) is additive. Then we have natural isomorphisms

$$H_n(X, A) \xrightarrow{\cong} H_n^{\text{CW}}(X, A) \quad \text{for } n \geq 0.$$

Proof. Choose pushouts

$$\begin{array}{ccc} \coprod_{i \in I_n} S^{n-1} & \xrightarrow{\coprod q_i^n} & X^{n-1} \\ \text{cofibration} \downarrow & \lrcorner & \downarrow j^n \\ \coprod_{i \in I_n} D^n & \xrightarrow{\coprod Q_i^n} & X^n. \end{array}$$

$$\begin{aligned} H_k(X^n, X^{n-1}) &\cong H_k\left(\coprod_{i \in I_n} D^n, \coprod_{i \in I_n} S^{n-1}\right) \\ &\cong \bigoplus_{i \in I_n} H_k(D^n, S^{n-1}) \cong \bigoplus_{i \in I_n} H_{k-n}(\bullet), \end{aligned}$$

by the five lemma and additivity. H_* is ordinary, so

$$H_k(X^n, X^{n-1}) \cong \begin{cases} \bigoplus_{i \in I_n} H_0(\bullet), & k = n, \\ 0, & k \neq n. \end{cases}$$

Consider the exact sequence of the triple (X^n, X^{n-1}, A) :

$$\cdots \rightarrow H_{k+1}(X^n, X^{n-1}) \rightarrow H_k(X^{n-1}, A) \xrightarrow[\text{if } k \notin \{n, n-1\}]{\cong} H_k(X^n, A) \rightarrow H_k(X^n, X^{n-1}) \rightarrow \cdots$$

$$k > n \implies H_k(X^n, A) \cong H_k(X^{n-1}, A) \cong \cdots \cong H_k(A, A) \cong 0,$$

$$k < n \implies H_k(X^n, A) \cong H_k(X^{n+1}, A) \cong H_k(X^{n+2}, A) \cong \cdots,$$

that is, $H_k(X^n, A) \cong \text{colim}_m H_k(X^m, A)$.

Claim: $\text{colim}_m H_k(X^m, A) \cong H_k(X, A)$.

- Clear if X is finite-dimensional.
- Otherwise, use $T(X^\bullet) = \text{hocolim}_m X^m$:

$$\begin{array}{c} \text{Mapping telescope} \\ \begin{array}{c} \text{Diagram: A sequence of cylinders (mapping telescopes) connected by arrows, representing the mapping telescope of the sequence } X^0 \rightarrow X^1 \rightarrow X^2 \rightarrow X^3 \rightarrow \cdots \\ A \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow X^3 \longrightarrow \cdots \end{array} \end{array}$$

“Filtered homotopy colimit.”

And

$$H_k(T(X^\bullet), A) \cong \text{colim}_m H_k(X^m, A) \quad \text{and} \quad T(X^\bullet) \simeq_A X.$$

(Compare: For $f: A \rightarrow X$,

$$H_k(\text{hocolim}(A \xrightarrow{f} X)) = H_k(M_f)$$

and

$$\text{colim } H_k(A \xrightarrow{f} X) = \text{colim}(H_k(A) \xrightarrow{H_k(f)} H_k(X)) = H_k(X)$$

and $M_f \simeq X$.)

Thus $H_k(X^n, A) \cong H_k(X, A)$ for $k < n$. Now consider again

$$\begin{array}{ccccccc}
& & H_n(X^{n-1}, A) = 0 & & & & \\
& & \downarrow & & & & \\
& & H_n(X^n, A) & \longrightarrow & H_n(X^{n+1}, A) & \longrightarrow & H_n(X^{n+1}, X^n) = 0 \\
& \nearrow \partial_{n+1} & \downarrow i_n & & & & \\
H_{n+1}(X^{n+1}, X^n) & \xrightarrow{\partial_{n+1}^{\text{CW}}} & H_n(X^n, X^{n-1}) & \xrightarrow{\partial_n^{\text{CW}}} & H_{n-1}(X^{n-1}, X^{n-2}) & & \\
& & \downarrow \partial_n & \nearrow i_{n-1} & & & \\
& & H_{n-1}(X^{n-1}, A) & & & & \\
& \nearrow & & & & & \\
H_{n-1}(X^{n-2}, A) = 0 & & & & & &
\end{array}$$

Claim: i_n induces

$$\text{coker } \partial_{n+1} \xrightarrow{\cong} H_n^{\text{CW}}(X, A) = \frac{\ker \partial_n^{\text{CW}}}{\text{im } \partial_{n+1}^{\text{CW}}}.$$

Proof: Diagram chase.

We conclude that

$$H_n^{\text{CW}}(X, A) \cong \text{coker } \partial_{n+1} \cong H_n(X^{n+1}, A) \cong H_n(X, A),$$

and this isomorphism is natural with respect to cellular maps. $\square$

Corollary IV.2.3. Let H_* be ordinary and $H_0(\bullet)$ free of finite rank over a principal ideal domain R . Let the pair of spaces (X, A) be homotopy equivalent to a relative cell complex of finite type; if this complex is infinite, also assume that H_* is additive. Then $H_n(X, A)$ is a finitely presented R -module for all $n \geq 0$.

A natural boundary map

$$\partial_n^{\text{CW}}: H_n^{\text{CW}}(X, A) \longrightarrow H_{n-1}^{\text{CW}}(A)$$

is obtained from the following proposition using [Theorem II.2.2](#).

Proposition IV.2.4. For a CW pair (X, A) , we have a natural short exact sequence

$$0 \longrightarrow C_*^{\text{CW}}(A; H_*) \longrightarrow C_*^{\text{CW}}(X; H_*) \longrightarrow C_*^{\text{CW}}(X, A; H_*) \longrightarrow 0.$$

Proof. Clear if $A = \emptyset$. Otherwise,

$$\begin{array}{ccc}
A^n/A^{n-1} & \xhookrightarrow{i_n} & X^n/X^{n-1} \\
\cong \downarrow & & \downarrow \cong \\
\bigvee_i S^n & \longrightarrow & \bigvee_j S^n
\end{array}$$

is a cofibration with retraction $r_n: X^n/X^{n-1} \rightarrow A^n/A^{n-1}$. Since

$$\frac{X^n/X^{n-1}}{A^n/A^{n-1}} \cong \frac{X^n \cup A}{X^{n-1} \cup A},$$

III.1.6 gives short exact sequences:

$$\begin{array}{ccccccc} 0 & \rightarrow & \tilde{H}_n(A^n/A^{n-1}) & \rightarrow & \tilde{H}_n(X^n/X^{n-1}) & \rightarrow & \tilde{H}_n\left(\frac{X^n \cup A}{X^{n-1} \cup A}\right) \rightarrow 0 \\ & & \cong \downarrow \text{III.1.4} & & \downarrow \cong & & \downarrow \cong \\ & & C_n^{\text{CW}}(A; H_*) & & C_n^{\text{CW}}(X; H_*) & & C_n^{\text{CW}}(X, A; H_*) \end{array}$$

Naturality follows from III.1.4 and III.1.6. □

Warning! Standard abuse of notation: ∂_n^{CW} denotes both

$$H_n^{\text{CW}}(X, A) \rightarrow H_{n-1}^{\text{CW}}(A) \quad \text{and} \quad C_n^{\text{CW}}(X, A) \rightarrow C_{n-1}^{\text{CW}}(X, A).$$

Proposition IV.2.5. Under the assumptions of IV.2.2, we have

$$\begin{array}{ccc} H_n(X, A) & \xrightarrow{\partial_n} & H_{n-1}(A) \\ \cong \downarrow & & \downarrow \cong \\ H_n^{\text{CW}}(X, A) & \xrightarrow{\partial_n^{\text{CW}}} & H_{n-1}^{\text{CW}}(A). \end{array}$$

Proof. Use $H_n^{\text{CW}}(X, A) \cong \text{coker}(\partial_{n+1})$ as in IV.2.2 and the naturality of ∂_n . □

Thus, altogether we have a natural isomorphism of long exact sequences:

$$\begin{array}{ccccccccccc} \cdots & \rightarrow & H_n(A) & \rightarrow & H_n(X) & \rightarrow & H_n(X, A) & \rightarrow & H_{n-1}(A) & \rightarrow & \cdots \\ & & \cong \downarrow & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ \cdots & \rightarrow & H_n^{\text{CW}}(A) & \rightarrow & H_n^{\text{CW}}(X) & \rightarrow & H_n^{\text{CW}}(X, A) & \rightarrow & H_{n-1}^{\text{CW}}(A) & \rightarrow & \cdots \end{array}$$

The Euler characteristic. Euler's polyhedron formula (1758): For a convex polyhedron with E vertices, K edges, and F faces,

$$E - K + F = 2.$$

More generally:

Definition IV.2.6. Let X be a finite cell complex. Then

$$\chi(X) = \sum_{n \geq 0} (-1)^n |\pi_0(X^n \setminus X^{n-1})|$$

is called the *Euler characteristic* of X .

Definition IV.2.7. For a topological space X , the n th Betti number is

$$\begin{aligned} b_n(X) &:= \text{rank}_{\mathbb{Z}} H_n(X; \mathbb{Z}) \\ &= \dim_{\mathbb{Q}}(H_n(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}) = \dim_{\mathbb{Q}} H_n(X; \mathbb{Q}). \end{aligned}$$

Theorem IV.2.8. (Euler–Poincaré formula) Let X be a finite cell complex. Then

$$\chi(X) = \sum_{n \geq 0} (-1)^n b_n(X).$$

Proof. Set $C_* := C_*^{\text{CW}}(X; H_*^{\text{sing}}(-; \mathbb{Q}))$. We show in general that:

$$\sum_{n \geq 0} (-1)^n \dim_{\mathbb{Q}} C_n = \sum_{n \geq 0} (-1)^n \dim_{\mathbb{Q}} H_n(C_*). \quad (*)$$

Consider

$$0 \longrightarrow B_n \longrightarrow Z_n \longrightarrow H_n(C_*) \longrightarrow 0$$

and

$$0 \longrightarrow Z_n \longrightarrow C_n \xrightarrow{\partial_n} B_{n-1} \longrightarrow 0.$$

It follows that $\dim Z_n = \dim B_n + \dim H_n(C_*)$ and $\dim C_n = \dim Z_n + \dim B_{n-1}$, so

$$\begin{aligned} \sum_{n \geq 0} (-1)^n \dim C_n &= \sum_{n \geq 0} (-1)^n (\dim Z_n + \dim B_{n-1}) \\ &= \sum_{n \geq 0} (-1)^n \dim H_n(C_*) + \underbrace{\sum_{n \geq 0} (-1)^n (\dim B_n + \dim B_{n-1})}_{=0}. \end{aligned}$$

□

Corollary IV.2.9. Let X, Y be finite cell complexes with $X \simeq Y$. Then $\chi(X) = \chi(Y)$.

Examples: If X is contractible, then $\chi(X) = \chi(\bullet) = 1$ (a necessary test for contractibility).

- $\chi(S^1) = 0$.
- $\chi(\Sigma_g) = 1 - 2g + 1 = 2 - 2g$.
- $\chi(N_g) = 1 - g + 1 = 2 - g$.
- $\chi(\mathbb{RP}^2) = \chi(N_1) = 1$ (not contractible; the test is not sufficient).
- For products,

$$\begin{aligned} \chi(X \times Y) &= |I_0^X| |I_0^Y| - (|I_1^X| |I_0^Y| + |I_0^X| |I_1^Y|) \\ &\quad + (|I_2^X| |I_0^Y| + |I_1^X| |I_1^Y| + |I_0^X| |I_2^Y|) \mp \cdots \\ &= \left(\sum_{n \geq 0} (-1)^n |I_n^X| \right) \left(\sum_{n \geq 0} (-1)^n |I_n^Y| \right) = \chi(X) \chi(Y). \end{aligned}$$

- $\chi(S^1 \times X) = 0$, $\chi(T^d) = 0$.
- Let $Y \rightarrow X$ be an n -sheeted covering: $\chi(Y) = n\chi(X)$.
- Thus, if $\chi(X) \neq 0$, any self-covering of X is trivial.

Theorem IV.2.10. (Additivity of χ) Let

$$\begin{array}{ccc} A & \xrightarrow{\quad} & Y \\ \downarrow \scriptstyle \Gamma & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

be a cellular pushout of finite cell complexes. Then

$$\chi(Z) = \chi(X) + \chi(Y) - \chi(A).$$

Proof. Consider the Mayer–Vietoris sequence for $H_*^{\text{sing}}(-; \mathbb{Q})$ as a chain complex with vanishing homology. By (*),

$$\begin{aligned} \sum_{n \geq 0} (-1)^n \dim H_n(Z) - \sum_{n \geq 0} (-1)^n (\dim H_n(X) + \dim H_n(Y)) \\ + \sum_{n \geq 0} (-1)^n \dim H_n(A) = 0. \end{aligned}$$

The claim follows from IV.2.8. □

IV.3. Computing Cellular Homology

Goal: Compute the differentials of the cellular chain complex as integer matrices. Examples. Let (X, A) be a relative cell complex. Choose pushouts:

$$\begin{array}{ccc} \coprod_{i \in I_n} S^{n-1} & \xrightarrow{\coprod q_i^n} & X^{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} D^n & \xrightarrow{\coprod Q_i^n} & X^n \end{array}$$

Definition IV.3.2. For $n \geq 1$, let $i \in I_n$ and $j \in I_{n-1}$. The *incidence number* $\text{inc}_{i,j}^n \in \mathbb{Z}$ is the degree of the map

$$S^{n-1} \xrightarrow{q_i^n} X^{n-1} \longrightarrow \frac{X^{n-1}}{X^{n-1} \setminus e_j^{n-1}} \xrightarrow{(\overline{Q_j^{n-1}})^{-1}} D^{n-1}/S^{n-2} \xrightarrow{u_{n-1}} S^{n-1}.$$

- Well-defined only up to sign.
- For $n = 1$, this means

$$\text{inc}_{i,j}^1 = \begin{cases} 1, & q_i^1(1) = Q_j^0(D^0), \quad q_i^1(-1) \neq Q_j^0(D^0), \\ -1, & q_i^1(-1) = Q_j^0(D^0), \quad q_i^1(1) \neq Q_j^0(D^0), \\ 0, & \text{otherwise.} \end{cases}$$

- Let $\text{INC}_n = (\text{inc}_{i,j}^n)$ be the n th *incidence matrix*. It may be infinite, but by Theorem IV.1.3(ii), each row has only finitely many nonzero entries.
- Let (H_*, ∂_*) be a homology theory, assumed to be additive if (X, A) is infinite.
- Let

$$\nu_n : H_n(X^n, X^{n-1}) \xrightarrow{\cong} \bigoplus_{i \in I_n} H_0(\bullet).$$

No dimension axiom is required here.

Theorem IV.3.2. For $n \geq 1$, we have

$$\begin{array}{ccc} C_n^{\text{CW}}(X, A; H_*) & \xrightarrow{\partial_n^{\text{CW}}} & C_{n-1}^{\text{CW}}(X, A; H_*) \\ \nu_n \downarrow \cong & & \cong \downarrow \nu_{n-1} \\ \bigoplus_{i \in I_n} H_0(\bullet) & \xrightarrow{\cdot \text{INC}_n} & \bigoplus_{j \in I_{n-1}} H_0(\bullet). \end{array}$$

Proof. The following diagram commutes:

$$\begin{array}{ccccccc}
& & H_n(X^n, X^{n-1}) & \xrightarrow{\partial} & H_{n-1}(X^{n-1}) & & \\
& & \downarrow \cong & \searrow \partial & \downarrow & \searrow & \\
& \tilde{H}_n(D^n/S^{n-1}) & \xrightarrow{\tilde{H}_n(\overline{Q}_i^n)} & \tilde{H}_n(X^n/X^{n-1}) & & H_{n-1}(X^{n-1}, x_0) & \longrightarrow & H_{n-1}(X^{n-1}, X^{n-2}) \\
& \downarrow \tilde{H}_n(u_n) \cong & & \downarrow \partial & \swarrow \cong & & \swarrow \cong & \\
& & & \tilde{H}_{n-1}(X^{n-1}) & \longrightarrow & \tilde{H}_{n-1}(X^{n-1}/X^{n-2}) & \longrightarrow & \tilde{H}_{n-1}(X^{n-1}/(X^{n-1} \setminus e_j^{n-1})) \\
& & & \uparrow \tilde{H}_{n-1}(q_i^n) & & & & \uparrow \cong \tilde{H}_{n-1}(\overline{Q}_j^{n-1}) \\
& \tilde{H}_n(S^n) & \xrightarrow[\cong]{\text{desuspension}} & \tilde{H}_{n-1}(S^{n-1}) & \xrightarrow{\cdot \text{inc}_{i,j}^n} & \tilde{H}_{n-1}(S^{n-1}) & \xleftarrow[\tilde{H}_{n-1}(u_{n-1})]{\cong} & \tilde{H}_{n-1}(D^{n-1}/S^{n-2}) \\
& \downarrow (n-1)\text{-fold desuspension} \cong & & \downarrow \cong & & \downarrow (n-1)\text{-fold desuspension} \cong & & \\
& & & \tilde{H}_0(S^0) & & \tilde{H}_0(S^0) & & \\
& & & \downarrow \cong & & \downarrow \cong & & \\
& & & H_0(\bullet) & \xrightarrow{\cdot \text{inc}_{i,j}^n} & H_0(\bullet) & &
\end{array}$$

The square labeled with the degree uses [Theorem III.4.2](#). With

$$\iota_i : H_0(\bullet) \longrightarrow \bigoplus_{I_n} H_0(\bullet), \quad \pi_j : \bigoplus_{I_{n-1}} H_0(\bullet) \longrightarrow H_0(\bullet)$$

we obtain

$$\pi_j \circ \nu_{n-1} \circ \partial_n^{\text{CW}} \circ \nu_n^{-1} \circ \iota_i = \cdot \text{inc}_{i,j}^n, \quad \text{hence} \quad \nu_{n-1} \circ \partial_n^{\text{CW}} \circ \nu_n^{-1} = \cdot \text{INC}_n.$$

□

Remark. In particular, $\text{INC}_{n+1} \text{INC}_n = 0$, which is not so obvious if one proves $\pi_n(S^n, \bullet) \cong \mathbb{Z}$ using homotopy theory and thus defines the degree without homology.

Examples IV.3.3. Let (H_*, ∂_*) be ordinary, with $H_0(\bullet) \cong \mathbb{Z}$. We use CW structures as in [Examples IV.1.4–IV.1.7](#).

1)

$$C_*^{\text{CW}}(S^n; H_*) : \quad \cdots \longrightarrow 0 \longrightarrow \underset{\text{degree } n}{\mathbb{Z}} \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \underset{\text{degree } 0}{\mathbb{Z}} \longrightarrow 0 \quad (n \geq 1).$$

All differentials vanish, including for $n = 1$. Thus

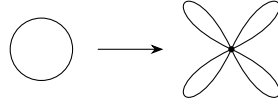
$$H_\ell^{\text{CW}}(S^n) \cong H_\ell(S^n) \cong \begin{cases} \mathbb{Z}, & \ell = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

2)

$$C_*^{\text{CW}}(\Sigma_g; H_*) : \quad \cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot \text{INC}_2} \mathbb{Z}_2^{2g} \xrightarrow{0} \mathbb{Z} \longrightarrow 0.$$

The pushout for the 2-cell is

$$\begin{array}{ccc} S^1 & \xrightarrow{w} & \bigvee_{i=1}^{2g} S^1 \\ \downarrow & & \downarrow \\ D^2 & \longrightarrow & \Sigma_g \end{array} \quad w = \prod_{i=1}^g [a_i, b_i].$$



Each 1-cell is traversed in both directions. Hence all the maps in [Definition IV.3.2](#) are null-homotopic, that is, $\text{INC}_2 = (0, \dots, 0)$. Thus

$$H_\ell(\Sigma_g) \cong \begin{cases} \mathbb{Z}, & \ell = 0, 2, \\ \mathbb{Z}^{2g}, & \ell = 1, \\ 0, & \text{otherwise.} \end{cases}$$

3)

$$C_*^{\text{CW}}(\mathbb{RP}^d; H_*) \cong \left(\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{d} \mathbb{Z}_{d-1} \longrightarrow \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0 \right).$$

The ℓ -cell is attached to the $(\ell - 1)$ -skeleton by $S^{\ell-1} \rightarrow \mathbb{RP}^{\ell-1}$. We need to determine

$$\deg(S^{\ell-1} \rightarrow \mathbb{RP}^{\ell-1}/\mathbb{RP}^{\ell-2} \cong S^{\ell-1}) = 1 + (-1)^\ell.$$

For this, use the factorization

$$\begin{array}{ccc} S^{\ell-1} & \xrightarrow{\quad} & S^{\ell-1} \\ \searrow \text{collapse the equator} & & \nearrow \text{id} \vee (-\text{id}) \\ & S^{\ell-1} \vee S^{\ell-1} & \end{array}$$

and consequently

$$H_\ell(\mathbb{RP}^d) \cong \begin{cases} \mathbb{Z}, & \ell = 0 \text{ or } (\ell = d \text{ and } d \text{ odd}), \\ \mathbb{Z}/2, & 0 < \ell < d \text{ and } \ell \text{ odd}, \\ 0, & \text{otherwise.} \end{cases}$$

4)

$$C_*^{\text{CW}}(\mathbb{CP}^d) : \quad 0 \longrightarrow \mathbb{Z}_{2d} \longrightarrow 0 \longrightarrow \mathbb{Z}_{2d-2} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

It follows that

$$H_\ell(\mathbb{CP}^d) \cong \begin{cases} \mathbb{Z}, & 0 \leq \ell \leq 2d \text{ and } \ell \text{ even}, \\ 0, & \text{otherwise.} \end{cases}$$

IV.4. Uniqueness of Ordinary Homology

Goal. All ordinary and additive homology theories with fixed $H_0(\bullet) \cong M$ are equivalent on $\text{CW}^{(2)}$.

Definition IV.4.1. A *cellular homology theory* with values in $R\text{-Mod}$ is a family of functors $(H_n)_{n \in \mathbb{Z}}$,

$$H_n : \text{CW}^{(2)} \longrightarrow R\text{-Mod},$$

together with a family of natural transformations

$$\partial_n : H_n \longrightarrow H_{n-1} \circ J,$$

where $J : \text{CW}^{(2)} \rightarrow \text{CW}^{(2)}$ is defined by $J(X, A) = (A, \emptyset)$ and

$$J(f : (X, A) \rightarrow (Y, B)) = f|_A : (A, \emptyset) \rightarrow (B, \emptyset)$$

such that the following hold:

- 1) **Homotopy invariance.** If $f, g : (X, A) \rightarrow (Y, B)$ satisfy $f \simeq_h g$ with $h_t(A) \subseteq B$ for all $t \in I$, then $H_n(f) = H_n(g)$; here f, g , and h_t for all $t \in I$ are cellular.
- 2) **Long exact sequence.** The pair (X, A) has the inclusions

$$i : (A, \emptyset) \longrightarrow (X, \emptyset), \quad j : (X, \emptyset) \longrightarrow (X, A),$$

and these induce the long exact sequence

$$\cdots \longrightarrow H_n(A, \emptyset) \xrightarrow{H_n(i)} H_n(X, \emptyset) \xrightarrow{H_n(j)} H_n(X, A) \xrightarrow{\partial_n(X, A)} H_{n-1}(A, \emptyset) \longrightarrow \cdots$$

- 3) **Excision.** Let $X = A \cup B$ for subcomplexes $A, B \subseteq X$. Then $H_n(A, A \cap B) \cong H_n(X, B)$ for all $n \in \mathbb{Z}$.

The cellular homology theory is also called *ordinary* if it satisfies the *dimension axiom*:

- 4) $H_n(\bullet, \emptyset) = 0$ for $n \neq 0$.

Applicable to cellular and noncellular theories:

Definition IV.4.2. Let (H_*, ∂_*^H) and (K_*, ∂_*^K) be homology theories with values in $R\text{-Mod}$. A *natural transformation of homology theories* is a family of natural transformations $\omega_* : H_* \rightarrow K_*$ such that, for (X, A) and $n \in \mathbb{Z}$, the diagram

$$\begin{array}{ccc} H_n(X, A) & \xrightarrow{\partial_n^H} & H_{n-1}(A) \\ \omega_n(X, A) \downarrow & & \downarrow \omega_{n-1}(A) \\ K_n(X, A) & \xrightarrow{\partial_n^K} & K_{n-1}(A) \end{array}$$

commutes. If each ω_n is a natural isomorphism, ω_* is called an *equivalence of homology theories*.

- This gives the categories **HomologyTheories** and **CW-HomologyTheories**.
- Equivalences are the isomorphisms in these categories.
- Let $F : \text{CW}^{(2)} \rightarrow \text{Top}^{(2)}$ be the forgetful functor and (H_*, ∂_*) a homology theory. Then $(H_* \circ F, \partial_* \circ F)$ is a cellular homology theory. For excision: If $X = A \cup B$ with subcomplexes, the pushout

$$\begin{array}{ccc} A \cap B & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}$$

is cellular, hence homotopy cocartesian. Thus we have the Mayer–Vietoris isomorphism $H_n(X, B) \cong H_n(A, A \cap B)$; see the [proof of III.3.3](#).

Theorem IV.4.3. Let (H_*, ∂_*) be an ordinary and additive homology theory with values in $R\text{-Mod}$. Then there is an equivalence of homology theories

$$\omega_* : H_* \circ F \xrightarrow{\cong} H_*^{\text{CW}}.$$

Proof. Theorem IV.2.2 and Proposition IV.2.5. □

Moreover, $\omega_0(\bullet) = \text{id}_{H_0(\bullet)}$ by construction.

Theorem IV.4.4. Let H_* and K_* be ordinary additive homology theories with values in $R\text{-Mod}$. Let $f : H_0(\bullet) \rightarrow K_0(\bullet)$ be an R -module homomorphism. Then there is a natural transformation

$$\omega_*^f : H_* \circ F \longrightarrow K_* \circ F$$

of cellular homology theories with $\omega_0^f(\bullet) = f$. If f is an isomorphism, then ω_*^f is an equivalence.

Proof. By Theorem IV.4.3, it remains to construct a natural transformation

$$\tilde{\omega}_*^f : H_*^{\text{CW}} \longrightarrow K_*^{\text{CW}}$$

with $\tilde{\omega}_0^f(\bullet) = f$ such that the $\tilde{\omega}_n^f$ are natural isomorphisms if f is an isomorphism. In fact, we construct a natural transformation

$$\tilde{\Omega}_*^f : C_*^{\text{CW}}(-; H_*) \longrightarrow C_*^{\text{CW}}(-; K_*)$$

with $\tilde{\Omega}_0^f(\bullet) = f$ that is a natural isomorphism if f is an isomorphism. We then set

$$\tilde{\omega}_n^f(X, A) = H_n(\tilde{\Omega}_*^f(X, A)).$$

Composition with functors takes natural transformations and isomorphisms to natural transformations and isomorphisms, respectively. The additional naturality condition

$$\partial_n^{\text{CW}, K} \circ \tilde{\omega}_n^f = \tilde{\omega}_{n-1}^f \circ \partial_n^{\text{CW}, H}$$

follows from the naturality assertion in Theorem II.2.2 for

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_*^{\text{CW}}(A; H_*) & \longrightarrow & C_*^{\text{CW}}(X; H_*) & \longrightarrow & C_*^{\text{CW}}(X, A; H_*) \longrightarrow 0 \\ & & \tilde{\Omega}_*^f(A) \downarrow & & \downarrow \tilde{\Omega}_*^f(X) & & \downarrow \tilde{\Omega}_*^f(X, A) \\ 0 & \longrightarrow & C_*^{\text{CW}}(A; K_*) & \longrightarrow & C_*^{\text{CW}}(X; K_*) & \longrightarrow & C_*^{\text{CW}}(X, A; K_*) \longrightarrow 0. \end{array}$$

We now construct $\tilde{\Omega}_*^f$. Let (X, A) be a CW pair. Regard (X, A) as a relative cell complex and choose pushouts

$$\begin{array}{ccc} \coprod_{i \in I_n} S^{n-1} & \xrightarrow{\coprod q_i^n} & X^{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} D^n & \xrightarrow{\coprod Q_i^n} & X^n. \end{array}$$

Using the isomorphisms $\nu_n^{H_*}$ and $\nu_n^{K_*}$ from Theorem IV.3.2, let

$$\tilde{\Omega}_n^f : C_n^{\text{CW}}(X, A; H_*) \xrightarrow{\nu_n^{H_*}} \bigoplus_{i \in I_n} H_0(\bullet) \xrightarrow{\bigoplus f} \bigoplus_{i \in I_n} K_0(\bullet) \xrightarrow{(\nu_n^{K_*})^{-1}} C_n^{\text{CW}}(X, A; K_*).$$

This defines a chain map because

$$\begin{array}{ccccccc}
C_n^{\text{CW}}(X, A; H_*) & \xrightarrow{\nu_n^{H_*}} & \bigoplus_{i \in I_n} H_0(\bullet) & \xrightarrow{\bigoplus f} & \bigoplus_{i \in I_n} K_0(\bullet) & \xrightarrow{(\nu_n^{K_*})^{-1}} & C_n^{\text{CW}}(X, A; K_*) \\
\downarrow \partial_n^{\text{CW}, H} & & \downarrow \cdot \text{INC}_n & & \downarrow \cdot \text{INC}_n & & \downarrow \partial_n^{\text{CW}, K} \\
C_{n-1}^{\text{CW}}(X, A; H_*) & \xrightarrow{\nu_{n-1}^{H_*}} & \bigoplus_{i \in I_{n-1}} H_0(\bullet) & \xrightarrow{\bigoplus f} & \bigoplus_{i \in I_{n-1}} K_0(\bullet) & \xrightarrow{(\nu_{n-1}^{K_*})^{-1}} & C_{n-1}^{\text{CW}}(X, A; K_*).
\end{array}$$

We have:

- $\tilde{\Omega}_*^f(X, A)$ is an isomorphism for every CW pair (X, A) if and only if f is an isomorphism.
- $\tilde{\Omega}_0^f(\bullet) = f$.

Thus it remains to show:

- (i) $\tilde{\Omega}_*^f(X, A)$ is independent of the choice of (Q_i^n, q_i^n) .
- (ii) $\tilde{\Omega}_*^f(X, A)$ is natural with respect to cellular maps $g : (X, A) \rightarrow (Y, B)$.

For (i): Let (P_i^n, p_i^n) be a second choice. We have

$$\begin{array}{ccc}
D^n / S^{n-1} & \xrightarrow{(\overline{P_i^n})^{-1} \circ \overline{Q_i^n}} & D^n / S^{n-1} \\
u_n \downarrow \cong & & \cong \downarrow u_n \\
S^n & \xrightarrow{\quad} & S^n
\end{array}$$

with degree ± 1 .

Consider

$$\begin{array}{ccccc}
& \bigoplus_{i \in I_n} H_0(\bullet) & \xrightarrow{\bigoplus f} & \bigoplus_{i \in I_n} K_0(\bullet) & \\
& \swarrow \bigoplus \tilde{H}_n(\overline{Q_i^n}) & \downarrow \bigoplus (\pm \text{id}) & \downarrow \bigoplus (\pm \text{id}) & \searrow \bigoplus \tilde{K}_n(\overline{Q_i^n}) \\
\tilde{H}_n(X^n / X^{n-1}) & & & & \tilde{K}_n(X^n / X^{n-1}) \\
& \nwarrow \bigoplus \tilde{H}_n(\overline{P_i^n}) & \downarrow \bigoplus (\pm \text{id}) & \downarrow \bigoplus (\pm \text{id}) & \nearrow \bigoplus \tilde{K}_n(\overline{P_i^n}) \\
& \bigoplus_{i \in I_n} H_0(\bullet) & \xrightarrow{\bigoplus f} & \bigoplus_{i \in I_n} K_0(\bullet) &
\end{array}$$

The middle square commutes by [Theorem III.4.2](#). Here the isomorphisms $H_0(\bullet) \cong \tilde{H}_n(S^n)$ and $K_0(\bullet) \cong \tilde{K}_n(S^n)$ are implicit.

For (ii): For cells $e_i^n \subseteq X$ and $f_j^n \subseteq Y$, let $\text{inc}_{i,j}^n(g) \in \mathbb{Z}$ be the degree of

$$D^n/S^{n-1} \xrightarrow{\overline{Q}_i^n(X)} X^n/X^{n-1} \xrightarrow{\overline{g}} Y^n/Y^{n-1} \longrightarrow Y^n/(Y^n \setminus f_j^n) \xrightarrow{(\overline{Q}_j^n(Y))^{-1}} D^n/S^{n-1}.$$

Set $\text{INC}_n(g) = (\text{inc}_{i,j}^n(g))$. Then, as in [Theorem IV.3.2](#), we obtain

$$(\cdot \text{INC}_n(g)) \circ \nu_n^X = \nu_n^Y \circ C_n^{\text{CW}}(g; H_*),$$

and from this, as above,

$$\begin{array}{ccc} C_*^{\text{CW}}(X, A; H_*) & \xrightarrow{C_*^{\text{CW}}(g)} & C_*^{\text{CW}}(Y, B; H_*) \\ \tilde{\Omega}_*^f(X, A) \downarrow & & \downarrow \tilde{\Omega}_*^f(Y, B) \\ C_*^{\text{CW}}(X, A; K_*) & \xrightarrow{C_*^{\text{CW}}(g)} & C_*^{\text{CW}}(Y, B; K_*). \end{array}$$

□

- We have just seen: An isomorphism $f : H_0(\bullet) \xrightarrow{\cong} K_0(\bullet)$ induces

$$C_*^{\text{CW}}(X, A; H_*) \cong C_*^{\text{CW}}(X, A; K_*),$$

naturally in (X, A) .

- We saw earlier: For every R -module M , there is an ordinary, additive homology theory $H_* = H_*^{\text{sing}}(-; M)$.

Thus there is a functor

$$C_*^{\text{CW}}(-; M) : \text{CW}^{(2)} \longrightarrow R\text{-Chain},$$

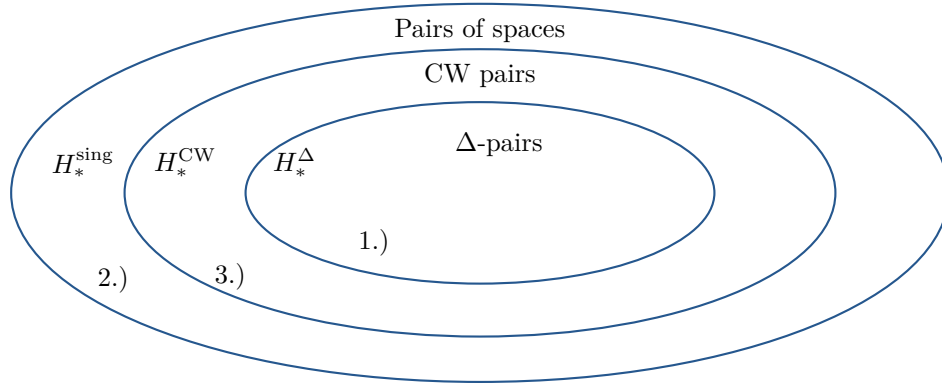
such that $H_n^{\text{CW}} = H_n(C_*^{\text{CW}}(-; M))$, together with ∂_n^{CW} from

$$0 \longrightarrow C_*^{\text{CW}}(A; M) \longrightarrow C_*^{\text{CW}}(X; M) \longrightarrow C_*^{\text{CW}}(X, A; M) \longrightarrow 0,$$

is a cellular homology theory, without reference to a given homology theory.

We call $(H_*^{\text{CW}}, \partial_*^{\text{CW}})$ *cellular homology* and usually omit the superscript CW.

Logical structure of the course:



The constructions proceed in this order: 1. simplicial homology, 2. singular homology, 3. cellular homology.

IV.5. What Comes Next?

- **Homological algebra:** free or projective resolutions of modules over rings that need not be commutative, the fundamental theorem. Applications:
 - How is $H_n(X, A; R)$ related to $H_n(X, A; \mathbb{Z}) \otimes_{\mathbb{Z}} R$? The universal coefficient theorem.
 - How can one compute $H_n(X \times Y; R)$ from $H_n(X; R)$ and $H_n(Y; R)$? When do we have

$$H_n(X \times Y; R) \cong \bigoplus_{p+q=n} H_p(X; R) \otimes_R H_q(Y; R)?$$

- **Cohomology:**

$$C^*(X; R) := \text{Hom}_R(C_*(X; R), R), \quad d^n(\varphi) = \varphi \circ d_{n+1}.$$

A map $f : X \rightarrow Y$ induces

$$f^* : C^*(Y; R) \longrightarrow C^*(X; R), \quad \varphi \longmapsto \varphi \circ f_*.$$

Advantage: The map

$$C^p(X) \times C^q(Y) \longrightarrow C^{p+q}(X \times Y), \quad (c^p, c^q) \longmapsto c^p \times c^q,$$

induces

$$H^p(X) \times H^q(Y) \longrightarrow H^{p+q}(X \times Y).$$

For $X = Y$, we obtain the *cup product*

$$\smile : H^p(X) \times H^q(X) \longrightarrow H^{p+q}(X \times X) \xrightarrow{\Delta^*} H^{p+q}(X).$$

This makes $H^*(X) := \bigoplus_{n \geq 0} H^n(X)$ a graded ring and $H_*(X)$ a right $H^*(X)$ -module. For example,

$$H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[a], \quad H^*(\mathbb{CP}^\infty; \mathbb{Z}) \cong \mathbb{Z}[a] \quad (\deg a = 2).$$

- **Poincaré duality:** For orientable compact manifolds M of dimension n without boundary, we have

$$H_\ell(M) \cong H^{n-\ell}(M).$$

In particular, the universal coefficient theorem gives $b_\ell(M) = b_{n-\ell}(M)$.