

Topology II

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Warm-Up: The Lefschetz Fixed Point Theorem

Let R be a principal ideal domain, V a finitely generated R -module, and $f: V \rightarrow V$ an endomorphism. Consider

$$\begin{array}{ccccc} V & \longrightarrow & V/\text{tors } V & \xrightarrow{\cong} & R^n \\ f \downarrow & & \downarrow \bar{f} & & \downarrow A \\ V & \longrightarrow & V/\text{tors } V & \xrightarrow{\cong} & R^n. \end{array}$$

Definition W.1. The *trace* of f is

$$\text{tr}(f) := \text{tr}(A) = \sum_{i=1}^n A_{ii} \in R.$$

Clearly, tr is R -linear and $\text{tr}(\text{id}_V) = \text{rk}_R V$.

Proposition W.2.

- (i) $\text{tr}(f)$ is well-defined.
- (ii) For $f, g \in \text{End}_R V$, we have $\text{tr}(f \circ g) = \text{tr}(g \circ f)$.

Proof. (ii) Let $A, B \in M_n(R)$ be matrices representing $\bar{f}, \bar{g}$. Then

$$\text{tr}(A \cdot B) = \sum_{i=1}^n (AB)_{ii} = \sum_{i=1}^n \sum_{j=1}^n A_{ij} B_{ji} = \sum_{i=1}^n \sum_{j=1}^n B_{ij} A_{ji} = \text{tr}(B \cdot A).$$

(i) Choosing a different basis gives $A' = U^{-1}AU$ with $U \in \text{GL}_n(R)$, so

$$\text{tr } A' = \text{tr}(U^{-1}(AU)) \stackrel{(ii)}{=} \text{tr}((AU)U^{-1}) = \text{tr } A. \quad \square$$

Proposition W.3. Suppose the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_0 & \xrightarrow{i} & V_1 & \xrightarrow{p} & V_2 & \longrightarrow & 0 \\ & & f \downarrow & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & V_0 & \xrightarrow{i} & V_1 & \xrightarrow{p} & V_2 & \longrightarrow & 0 \end{array}$$

of finitely generated R -modules has exact rows. Then

$$\text{tr } g = \text{tr } f + \text{tr } h.$$

Proof. First suppose that V_0, V_1, V_2 are free. Then $V_1 \cong V_0 \oplus V_2$ by the splitting lemma (*Topology I*, Lemma I.4.7), and if f, h are represented by A, B , then g is represented by $\begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$. Thus $\text{tr } g = \text{tr } f + \text{tr } h$. Reduce the general case to the free case and use adapted bases. $\square$

Definition. Let X be a topological space such that $H_*^{\text{sing}}(X; R)$ is finitely generated and $H_n(X; R) = 0$ for $n > N$, and let $f: X \rightarrow X$. Then

$$\tau_R(f) = \tau(f) := \sum_{n \geq 0} (-1)^n \text{tr}(H_n(f)) \in R$$

is called the *Lefschetz number* of f . Clearly, $\tau(f) = \tau(g)$ if $f \simeq g$, and $\tau_{\mathbb{Z}}(\text{id}_X) = \chi(X)$.

Theorem W.4. (Lefschetz fixed point theorem) Let X be a finite simplicial complex and $f: X \rightarrow X$ a continuous map with $\tau_R(f) \neq 0$. Then f has a fixed point.

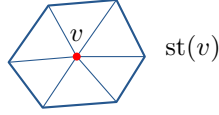
Proof. Suppose that f has no fixed points. Choose a metric d on X . Set

$$\varepsilon := \inf_{x \in X} d(x, f(x)) > 0.$$

Let L be a barycentric subdivision of X such that every *star*

$$\text{st}(\sigma) = \bigcup_{\sigma' \supseteq \sigma} \sigma'$$

has diameter $< \varepsilon/2$.



By simplicial approximation [K., Theorem 3.22], there is a barycentric subdivision K of L and a simplicial map $g: K \rightarrow L$ with $g \simeq f$ and $\sup_{x \in X} d(g(x), f(x))$ arbitrarily small. It follows that $g(\sigma) \cap \sigma = \emptyset$ for every simplex $\sigma \in K$. Consider

$$\begin{aligned} 0 &\longrightarrow B_n^\Delta(K) \longrightarrow Z_n^\Delta(K) \longrightarrow H_n^\Delta(K) \longrightarrow 0, \\ 0 &\longrightarrow Z_n^\Delta(K) \longrightarrow H_n(K^n, K^{n-1}) \longrightarrow B_{n-1}^\Delta(K) \longrightarrow 0. \end{aligned}$$

It follows that

$$\begin{aligned} \text{tr}(g_*: H_n(K^n, K^{n-1}) \rightarrow H_n(K^n, K^{n-1})) &\stackrel{\text{W.3}}{=} \text{tr}(g_*: Z_n^\Delta(K) \rightarrow Z_n^\Delta(K)) \\ &\quad + \text{tr}(g_*: B_{n-1}^\Delta(K) \rightarrow B_{n-1}^\Delta(K)) \\ &= \text{tr}(g_*: B_n^\Delta(K) \rightarrow B_n^\Delta(K)) + \text{tr}(H_n^\Delta(g)) \\ &\quad + \text{tr}(g_*: B_{n-1}^\Delta(K) \rightarrow B_{n-1}^\Delta(K)). \end{aligned}$$

Thus

$$\begin{aligned} \tau(f) = \tau(g) &= \sum_{n \geq 0} (-1)^n \text{tr} H_n^\Delta(g) \\ &= \sum_{n \geq 0} (-1)^n \text{tr}(g_*: H_n(K^n, K^{n-1}) \rightarrow H_n(K^n, K^{n-1})) = 0, \end{aligned}$$

since $g(\sigma) \cap \sigma = \emptyset$ for all $\sigma \in K$. □

Corollary W.5. (Generalized Brouwer fixed point theorem) Let X be a finite simplicial complex with $X \simeq \bullet$. Then every $f: X \rightarrow X$ has a fixed point.

Proof. We have

$$\tau(f) = \sum_{n \geq 0} (-1)^n \operatorname{tr}(H_n(f)) = \operatorname{tr} H_0(f) = \operatorname{tr}(\operatorname{id}_R) = 1. \quad \square$$

Remark. More generally,

$$\tau(f) = \sum_{x \in \operatorname{Fix}(f)} \operatorname{ind}_x f,$$

if f has only finitely many fixed points, all of them isolated. In the local Euclidean model, the fixed point index is the degree of the normalized displacement map on the boundary of a small neighborhood U of x :

$$\begin{array}{ccc} U & \xrightarrow{\operatorname{id} - f} & \mathbb{R}^d \\ \uparrow & & \uparrow \\ \partial U & \xrightarrow{(\operatorname{id} - f)|_{\partial U}} & \mathbb{R}^d \setminus \{0\} \end{array} \quad \operatorname{ind}_x f = \deg \left(y \mapsto \frac{y - f(y)}{\|y - f(y)\|} \right)_{\partial U}.$$

($\implies$ Poincaré–Hopf theorem from differential topology!)

I. Homological Algebra

I.1. Projective Modules

Let R be a (possibly noncommutative) ring with 1. ($R\text{-Mod}$ is the category of left R -modules.)

Definition I.1.1. An R -module P is called *projective* if

$$\begin{array}{ccccc} M & \xrightarrow{p} & N & \longrightarrow & 0 \\ & \nwarrow \tilde{f} & \uparrow f & & \\ & & P & & \end{array}$$

(For every epimorphism $p: M \rightarrow N$ and every $f: P \rightarrow N$, there exists $\tilde{f}: P \rightarrow M$ with $f = p \circ \tilde{f}$.)

Theorem I.1.2. The following statements are equivalent.

- (i) P is projective.
- (ii) Every short exact sequence

$$0 \longrightarrow N \xrightarrow{i} M \xleftarrow[p]{s} P \longrightarrow 0$$

splits. (There exists $s: P \rightarrow M$ with $p \circ s = \text{id}_P$.)

- (iii) There is an R -module Q such that $P \oplus Q$ is free.

Proof. (i) $\Rightarrow$ (ii):

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & P \longrightarrow 0 \\ & & & & & \nwarrow s & \uparrow \text{id}_P \\ & & & & & & P \end{array}$$

(ii) $\Rightarrow$ (iii): Choose an epimorphism $p: F \rightarrow P$ with F free. By (ii), the sequence $0 \rightarrow \ker p \rightarrow F \rightarrow P \rightarrow 0$ splits. By the splitting lemma (*Topology I*, I.4.7), which also holds for noncommutative R , we obtain $F \cong \ker p \oplus P$.

(iii) $\Rightarrow$ (i):

$$\begin{array}{ccccc} M & \longrightarrow & N & \longrightarrow & 0 \\ & \nwarrow g & \uparrow f & & \\ & & P & & \\ & & \uparrow i & & \\ & & P \oplus Q & & \end{array} \quad \text{Set } \tilde{f} := g \circ i.$$

□

Examples.

1.) $R = \mathbb{Z}$: the module $M = \mathbb{Z}/k\mathbb{Z}$ is not projective for $k \geq 2$, since

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot k} \mathbb{Z} \longrightarrow \mathbb{Z}/k\mathbb{Z} \longrightarrow 0$$

does not split.

2.) $R = \mathbb{F}_2 \times \mathbb{F}_2$: the module $P = \mathbb{F}_2 \times \{0\}$ is projective, since $P \oplus (\{0\} \times \mathbb{F}_2) \cong \mathbb{F}_2 \times \mathbb{F}_2$ is free, but P is not free, since $|P| = 2 \neq 4^k$.

3.) $R = C(S^2) = \{f: S^2 \rightarrow \mathbb{R} : f \text{ continuous}\}$. Consider

$$TS^2 = \bigcup_{p \in S^2} T_p S^2 = \{(s, v) : s \in S^2 \subseteq \mathbb{R}^3, v \in \mathbb{R}^3, s \perp v\} \subseteq \mathbb{R}^6,$$

$$p: TS^2 \longrightarrow S^2, \quad (s, v) \longmapsto s.$$

$$P = \Gamma(TS^2) = \{s: S^2 \rightarrow TS^2 : s \text{ continuous, } p \circ s = \text{id}_{S^2}\}.$$

The isomorphism

$$\begin{aligned} P \oplus C(S^2) &\xrightarrow{\cong} C(S^2)^3 \cong C(S^2; \mathbb{R}^3), \\ (s, f) &\longmapsto (x \longmapsto s(x) + f(x) \cdot x) \end{aligned}$$

shows that P is projective. However, P is not free, essentially by the hairy ball theorem. [Rosenberg: K-theory]

I.2. The Fundamental Theorem of Homological Algebra

Definition I.2.1. Let M be an R -module. A *resolution* of M is an exact sequence

$$\cdots \longrightarrow M_2 \longrightarrow M_1 \longrightarrow M_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

(that is, M_* is a chain complex with $H_0(M_*) \cong M$ and $H_k(M_*) = 0$ for $k \geq 1$).

A *projective resolution* of M is a resolution P_* such that P_k is projective for every $k \geq 0$.

Theorem I.2.2.

- (i) Every R -module has a projective resolution.
- (ii) Let $P_{*\geq 0}$ be a chain complex of projective modules. Let $Q_{*\geq 0}$ be a chain complex with $H_k(Q_*) = 0$ for $k > 0$. Let $[P_*, Q_*]$ be the abelian group of chain homotopy classes of chain maps $P_* \rightarrow Q_*$. Then

$$\begin{aligned} [P_*, Q_*] &\xrightarrow{\cong} \text{Hom}_R(H_0(P_*), H_0(Q_*)), \\ [f_*] &\longmapsto H_0(f_*) \end{aligned}$$

is an isomorphism.

Corollary I.2.3. Any two projective resolutions P_*, Q_* of the same R -module M are chain homotopy equivalent.

Proof. By surjectivity, we obtain

$$\begin{array}{ccccccc}
& & & & H_0(Q_*) & & \\
& & & & \wr \parallel & & \\
\cdots & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & Q_0 \longrightarrow M \longrightarrow 0 \\
& & f_2 \uparrow & & f_1 \uparrow & & f_0 \uparrow & \text{id}_M \uparrow \\
\cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 \longrightarrow M \longrightarrow 0 \\
& & & & \wr \parallel & & \\
& & & & H_0(P_*) & &
\end{array}$$

In the other direction, we obtain $g_*: Q_* \rightarrow P_*$. We have

$$H_0(g_* \circ f_*) = H_0(g_*) \circ H_0(f_*) = \text{id}_M \circ \text{id}_M = \text{id}_M$$

and $H_0(f_* \circ g_*) = \text{id}_M$. Injectivity gives $g_* \circ f_* \simeq \text{id}_{P_*}$ and $f_* \circ g_* \simeq \text{id}_{Q_*}$. $\square$

Proof (of Theorem I.2.2). (i) Let M be an R -module. Choose P_0 (for example, free) with $P_0 \xrightarrow{\varepsilon} M \rightarrow 0$, and then construct

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & P_2 & \twoheadrightarrow & \ker d_1 & \hookrightarrow & P_1 \twoheadrightarrow \ker \varepsilon \hookrightarrow P_0 \xrightarrow{\varepsilon} M \longrightarrow 0 \\
& & & \searrow & \text{:=} d_2 & \searrow & \text{:=} d_1
\end{array}$$

inductively.

(ii) Surjectivity:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & Q_2 & \xrightarrow{q_2} & Q_1 & \xrightarrow{q_1} & Q_0 \xrightarrow{\varepsilon} H_0(Q_*) \longrightarrow 0 \\
& & f_2 \uparrow & & f_1 \uparrow & & f_0 \uparrow & \varphi \uparrow \\
\cdots & \longrightarrow & P_2 & \xrightarrow{p_2} & P_1 & \xrightarrow{p_1} & P_0 \xrightarrow{\delta} H_0(P_*) \longrightarrow 0
\end{array}$$

- The homomorphism f_0 lifts $\varphi \circ \delta$ along the surjective map ε .
- For $x \in P_1$, we have $p_1(x) \in \ker \delta$, hence

$$\varepsilon(f_0(p_1(x))) = \varphi(\delta(p_1(x))) = \varphi(0) = 0,$$

that is, $\text{im}(f_0 \circ p_1) \subseteq \ker \varepsilon = \text{im } q_1$. Thus there exists a lift f_1 of $f_0 \circ p_1$ along $q_1: Q_1 \rightarrow \text{im } q_1$.

- Continue inductively.

Injectivity: Let $f_*, g_*: P_* \rightarrow Q_*$ be chain maps with $H_0(f_*) = H_0(g_*) =: \varphi$. We need to construct

$$h_*: P_* \longrightarrow Q_{*+1} \quad \text{with} \quad q_{*+1} \circ h_* + h_{*-1} \circ p_* = f_* - g_*.$$

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & Q_2 & \xrightarrow{q_2} & Q_1 & \xrightarrow{q_1} & Q_0 \xrightarrow{\varepsilon} H_0(Q_*) \longrightarrow 0 \\
& & f_2, g_2 \uparrow & & f_1, g_1 \uparrow & & f_0, g_0 \uparrow & \varphi \uparrow \\
\cdots & \longrightarrow & P_2 & \xrightarrow{p_2} & P_1 & \xrightarrow{p_1} & P_0 \xrightarrow{\delta} H_0(P_*) \longrightarrow 0
\end{array}$$

$\swarrow \text{dashed } h_1 \quad \swarrow \text{dashed } h_0$

- We have $\varepsilon \circ (f_0 - g_0) = \varphi \circ \delta - \varphi \circ \delta = 0$, that is, $\text{im}(f_0 - g_0) \subseteq \ker \varepsilon = \text{im } q_1$. We obtain

$$\begin{array}{ccc}
Q_1 & \xrightarrow{q_1} & Q_0 \\
& \nwarrow \text{dashed } h_0 & \uparrow f_0 - g_0 \\
& & P_0
\end{array}$$

- We have

$$\begin{aligned} q_1 \circ (f_1 - g_1 - h_0 \circ p_1) &= f_0 \circ p_1 - g_0 \circ p_1 - q_1 \circ h_0 \circ p_1 \\ &= (f_0 - g_0 - q_1 \circ h_0) \circ p_1 = 0. \end{aligned}$$

We obtain

$$\begin{array}{ccc} Q_2 & \xrightarrow{q_2} & Q_1 \\ & \nwarrow h_1 & \uparrow f_1 - g_1 - h_0 \circ p_1 \\ & & P_1 \end{array}$$

then $q_2 \circ h_1 + h_0 \circ p_1 = f_1 - g_1$.

- Continue inductively.

□

I.3. The Tensor–Hom Adjunction

(For the case where R is a field, see *Introduction to Topology*, Sheet 12, Exercise 3.)

Let R be a ring, M a right R -module, N a left R -module, and X an abelian group. A map $f: M \times N \rightarrow X$ is called *R -balanced* if

- $f(m, n + n') = f(m, n) + f(m, n')$,
- $f(m + m', n) = f(m, n) + f(m', n)$,
- $f(m \cdot r, n) = f(m, r \cdot n)$

for all $m, m' \in M$, $n, n' \in N$, and $r \in R$.

Definition I.3.1. The *tensor product* of M and N is an abelian group $M \otimes_R N$ with an R -balanced map $\otimes: M \times N \rightarrow M \otimes_R N$ satisfying the following universal property:

$$\begin{array}{ccc} M \times N & \xrightarrow{\otimes} & M \otimes_R N \\ & \searrow f & \downarrow \exists! \tilde{f} \\ & & X \end{array}$$

(For every R -balanced map $f: M \times N \rightarrow X$, there is a unique homomorphism of abelian groups $\tilde{f}: M \otimes_R N \rightarrow X$ with $f = \tilde{f} \circ \otimes$.)

Existence: Let $\mathbb{Z}(\cdot): \mathbf{Set} \rightarrow \mathbf{Ab}$ be left adjoint to the forgetful functor $\mathbf{Ab} \rightarrow \mathbf{Set}$, and let $U \leq \mathbb{Z}(M \times N)$ be generated by

$$\{(m, n + n') - (m, n) - (m, n'), \dots, \dots; m, m' \in M, n, n' \in N, r \in R\}.$$

Set $M \otimes_R N := \mathbb{Z}(M \times N)/U$, $m \otimes n := [(m, n)]$. For an R -balanced map $f: M \times N \rightarrow X$, we uniquely obtain

$$\tilde{f} \left(\underbrace{\sum m \otimes n}_{\text{not unique}} \right) := \underbrace{\sum f(m, n)}_{\text{well-defined}}.$$

Uniqueness: abstract nonsense, as usual.

Note. By construction, $M \times N \rightarrow M \otimes_R N$ is functorial in M and N .

Definition I.3.2. Let R, S be rings. An R - S -bimodule is an abelian group M that is both a left R -module and a right S -module, such that $(r \cdot m) \cdot s = r \cdot (m \cdot s)$ for all $m \in M$, $r \in R$, and $s \in S$.

Examples.

- R^n is an $M_n(R)$ - R -bimodule.
- If R is commutative and M is a right R -module, then M is an R - R -bimodule with $r \cdot m := m \cdot r$, since

$$(r \cdot m) \cdot s = (m \cdot r) \cdot s = m \cdot (r \cdot s) = m(s \cdot r) = r \cdot (m \cdot s).$$

- If M is a left R -module, then M is an R - $\mathbb{Z}$ -bimodule, since

$$r \cdot (m \cdot k) = r \cdot (m + \cdots + m) = r \cdot m + \cdots + r \cdot m = (r \cdot m) \cdot k.$$

Let B be an R - S -bimodule, A a right R -module, and C a right S -module. Then

- $A \otimes_R B$ becomes a right S -module via

$$(a \otimes b) \cdot s = a \otimes (b \cdot s),$$

- $\text{Hom}_S(B, C)$ becomes a right R -module via

$$(f \cdot r)(b) = f(r \cdot b).$$

Theorem I.3.3. The functor

$$(\cdot) \otimes_R B: \text{Mod-}R \rightarrow \text{Mod-}S$$

is left adjoint to

$$\text{Hom}_S(B, \cdot): \text{Mod-}S \rightarrow \text{Mod-}R:$$

$$\text{Hom}_S(A \otimes_R B, C) \cong \text{Hom}_R(A, \text{Hom}_S(B, C)),$$

naturally in $A \in \text{Ob}(\text{Mod-}R)$ and $C \in \text{Ob}(\text{Mod-}S)$.

Proof. $f: A \rightarrow \text{Hom}_S(B, C)$ is R -balanced when viewed as a map $f: A \times B \rightarrow C$. By the universal property, there is a unique $\tilde{f}: A \otimes_R B \rightarrow C$ with $\tilde{f}(a \otimes b) = f(a, b)$ for $a \in A$, $b \in B$. For $s \in S$, we have

$$\begin{aligned} \tilde{f}((\sum a \otimes b) \cdot s) &= \sum f(a, b \cdot s) = \sum f(a, b) \cdot s \\ &= \left(\sum f(a, b) \right) \cdot s = \tilde{f} \left(\sum a \otimes b \right) \cdot s. \end{aligned}$$

Naturality likewise follows from the construction of $A \otimes_R B$. □

I.4. Exactness

Definition I.4.1. A (covariant) functor F from left or right R -modules to $\mathbf{Ab}$ is called

- *left exact* if, for every exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3$$

the sequence $0 \rightarrow F(M_1) \rightarrow F(M_2) \rightarrow F(M_3)$ is also exact.

- *right exact* if, for every exact sequence

$$M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

the sequence $F(M_1) \rightarrow F(M_2) \rightarrow F(M_3) \rightarrow 0$ is also exact.

- *exact* if F is both left and right exact.

Contravariant *left exactness*:

$$0 \longleftarrow M_1 \longleftarrow M_2 \longleftarrow M_3 \quad \rightsquigarrow \quad 0 \longrightarrow F(M_1) \longrightarrow F(M_2) \longrightarrow F(M_3).$$

Theorem I.4.2. Let M be a left R -module. Then $\mathrm{Hom}_R(M, \cdot)$ is left exact and $(\cdot) \otimes_R M$ is right exact.

Proof. We observe that

$$\begin{aligned} 0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \text{ exact} &\iff \begin{array}{ccc} M_1 & \longrightarrow & M_2 \\ \downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & M_3 \end{array}, \\ M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0 \text{ exact} &\iff \begin{array}{ccc} M_1 & \longrightarrow & M_2 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & M_3 \end{array}. \end{aligned}$$

By [Theorem I.3.3](#), $\mathrm{Hom}_R(M, \cdot)$ is a right adjoint and $(\cdot) \otimes_R M$ is a left adjoint. By *Introduction to Topology*, Sheet 12, Exercise 4, right adjoints preserve limits and left adjoints preserve colimits. $\square$

Proposition I.4.3. $\mathrm{Hom}_R(M, \cdot)$ is exact $\iff M$ is projective.

Proof.

$$\begin{aligned} &\mathrm{Hom}_R(M, \cdot) \text{ exact} \\ \iff &\left\{ \begin{array}{l} \text{For every surjective } g: A \rightarrow B, \\ g \circ : \mathrm{Hom}_R(M, A) \rightarrow \mathrm{Hom}_R(M, B) \text{ is surjective.} \end{array} \right. \\ \iff &\begin{array}{ccccc} A & \xrightarrow{g} & B & \longrightarrow & 0 \\ & \nwarrow \exists h & \uparrow f & & \\ & & M & & \end{array} \end{aligned}$$

$\square$

Dually, an R -module I is called *injective* if

$$\begin{array}{ccccc} A & \xleftarrow{g} & B & \xleftarrow{\quad} & 0 \\ & \searrow \exists h & \downarrow f & & \\ & & I & & \end{array}$$

$$\begin{aligned} \iff & \left\{ \begin{array}{l} \text{For every injective } g: B \rightarrow A, \\ \circ g: \operatorname{Hom}_R(A, I) \rightarrow \operatorname{Hom}_R(B, I) \text{ is surjective.} \end{array} \right. \\ \iff & \operatorname{Hom}_R(\cdot, I) \text{ is exact.} \end{aligned}$$

(Warning: The proof requires clarifying what adjunction means for contravariant functors...)

Examples. $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ over $\mathbb{Z}$.

Counterexample. $\mathbb{Z}$ over $\mathbb{Z}$.

$$n \geq 2: \quad \begin{array}{ccccc} \mathbb{Z} & \xleftarrow{\cdot n} & \mathbb{Z} & \xleftarrow{\quad} & 0 \\ & \searrow \nexists h & \downarrow \text{id} & & \\ & & \mathbb{Z} & & \end{array} \quad \text{but} \quad \begin{array}{ccccc} \mathbb{Z} & \xleftarrow{\cdot n} & \mathbb{Z} & \xleftarrow{\quad} & 0 \\ & \searrow \cdot \frac{1}{n} & \downarrow & & \\ & & \mathbb{Q} & & \end{array}.$$

Definition I.4.4. A left R -module M is called *flat* if $(\cdot) \otimes_R M$ is exact.

Proposition I.4.5. Projective modules are flat.

Proof. A free module $F = \bigoplus_{i \in I} R$ is flat, since if $f: N \rightarrow M$ is injective, then

$$f \otimes_R F: N \otimes_R F \cong \bigoplus_{i \in I} N \xrightarrow{\bigoplus_{i \in I} f} \bigoplus_{i \in I} M \cong M \otimes_R F$$

is also injective (see [Theorem I.5.1](#) below). For a projective module P , let $P \oplus Q = F$ be free. Then

$$f \otimes_R F: (N \otimes P) \oplus (N \otimes Q) \xrightarrow{(f \otimes P) \oplus (f \otimes Q)} (M \otimes P) \oplus (M \otimes Q)$$

is injective, and hence $f \otimes_R P$ is injective as well. $\square$

Examples and counterexamples.

- $\mathbb{Q}$ is flat but not projective over $\mathbb{Z}$. (Use $\mathbb{Q} = \mathbb{Z}_{\mathbb{Z} \setminus \{0\}}$ and $\mathbb{Q} \subseteq \bigoplus_{i \in I} \mathbb{Z} \not\Rightarrow \mathbb{Q}$.)
- For $n \geq 2$, the module $\mathbb{Z}/n\mathbb{Z}$ is not flat over $\mathbb{Z}$. (Consider $\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}$.)

Remark. We have

$$\begin{array}{llll} \text{Always:} & \text{free} \Rightarrow & \text{projective} \Rightarrow & \text{flat} \Rightarrow & \text{torsion-free} \\ R \text{ a principal} & & \Leftarrow & & \Leftarrow \\ \text{ideal domain:} & & & & \\ R \text{ a principal ideal domain,} & & \Leftarrow & & \Leftarrow \\ M \text{ finitely generated:} & & & & \Leftarrow \end{array}$$

I.5. Computing Tensor Products

Theorem I.5.1. (Rules for computing tensor products)

- (i) $R \otimes_R N \cong N$ (isomorphism of left R -modules).
- (ii) $M \otimes_R R \cong M$ (isomorphism of right R -modules).
- (iii) $M \otimes_R (\bigoplus_{i \in I} N_i) \cong \bigoplus_{i \in I} M \otimes_R N_i$.
- (iv) $(\bigoplus_{i \in I} M_i) \otimes_R N \cong \bigoplus_{i \in I} M_i \otimes_R N$.

Proof. (i) + (ii): The map $r \otimes n \mapsto r \cdot n$ is well-defined, surjective, and injective, since $\sum_i r_i \cdot n_i = 0$ gives

$$\begin{aligned} \sum_i r_i \otimes n_i &= \sum_i 1 \otimes r_i \cdot n_i \\ &= 1 \otimes \left(\sum_i r_i \cdot n_i \right) = 1 \otimes 0 = 1 \otimes \underbrace{0}_{\in R} \cdot \underbrace{0}_{\in N} = (1 \cdot 0) \otimes 0 = 0 \otimes 0 = 0. \end{aligned}$$

(iii) + (iv):

$$m \otimes \left(\sum_{i \in I} n_i \right) \mapsto \sum_{i \in I} (m \otimes n_i).$$

□

Examples.

- In $M \otimes_R N$, we always have $m \otimes 0 = 0 \otimes n = 0$.
- For a finite abelian group G , we have $G \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, since for $g \otimes q \in G \otimes_{\mathbb{Z}} \mathbb{Q}$ we have

$$g \otimes q = g \otimes |G| \cdot \frac{q}{|G|} = \underbrace{(g + \cdots + g)}_{|G| \text{ times}} \otimes \frac{q}{|G|} = 0 \otimes \frac{q}{|G|} = 0.$$

- We have $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Z}/m \cong \mathbb{Z}/\gcd(n, m)$. Applying $(\cdot) \otimes_{\mathbb{Z}} \mathbb{Z}/m$ to

$$\mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

gives

$$\mathbb{Z}/m \xrightarrow{n} \mathbb{Z}/m \longrightarrow \underbrace{\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}}_{\text{coker}(\mathbb{Z}/m \xrightarrow{n} \mathbb{Z}/m)} \longrightarrow 0,$$

that is, we need to show that

$$\mathbb{Z}/m \xrightarrow{n} \mathbb{Z}/m \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow 0 \quad \text{is exact for } d = \gcd(n, m).$$

To this end, suppose that $[k] \mapsto 0$, that is, $k = \ell \cdot d = \ell(r \cdot n + s \cdot m) \equiv \ell \cdot r \cdot n \pmod{m}$, so $[\ell \cdot r] \cdot n = [k]$. □

•

$$\begin{aligned} (\mathbb{Z}^3 \oplus \mathbb{Z}/6) \otimes (\mathbb{Z}/9 \oplus \mathbb{Z}/2) &\cong (\mathbb{Z}^3 \oplus \mathbb{Z}/6) \otimes \mathbb{Z}/18 \\ &\cong \underbrace{(\mathbb{Z}/18)^3 \oplus \mathbb{Z}/6}_{\text{invariant factors}} \\ &\cong \underbrace{(\mathbb{Z}/9)^3 \oplus \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^4}_{\text{elementary divisors}}. \end{aligned}$$

I.6. The Tor Functor

Definition I.6.1. Let M be a right R -module, N a left R -module, and $P_* \rightarrow M$ a projective resolution. The i th *Tor group* of M and N is

$$\mathrm{Tor}_i^R(M, N) := H_i(P_* \otimes_R N) \quad \text{for } i \geq 0.$$

Proposition I.6.2. $\mathrm{Tor}_i^R(M, N)$ is well-defined up to isomorphism.

Proof. Let $Q_* \rightarrow M$ be another projective resolution. By [Corollary I.2.3](#), there exists an equivalence $Q_* \simeq P_*$, which is unique up to chain homotopy by [I.2.2\(ii\)](#). Since $(\cdot) \otimes_R N: R\text{-chain} \rightarrow \mathbb{Z}\text{-chain}$ preserves chain homotopies (apply $(\cdot) \otimes_R N$ to $\partial_{*+1}h_* + h_{*-1}\partial_* = f_* - g_*$), there is a distinguished isomorphism

$$f_{P_*, Q_*}: H_i(P_* \otimes_R N) \xrightarrow{\cong} H_i(Q_* \otimes_R N). \quad (*)$$

□

The distinguished isomorphism f_{P_*, Q_*} allows the functorial definition

$$\mathrm{Tor}_i^R(M, N) := \left\{ x \in \prod_{P_* \rightarrow M} H_i(P_* \otimes_R N) : f_{P_*, Q_*}(\mathrm{pr}_{P_*}(x)) = \mathrm{pr}_{Q_*}(x) \right\}.$$

- $g: N \rightarrow N'$ induces $\mathrm{id}_* \otimes g: P_* \otimes_R N \rightarrow P_* \otimes_R N'$ and hence $\mathrm{Tor}_i^R(\mathrm{id}, g): \mathrm{Tor}_i^R(M, N) \rightarrow \mathrm{Tor}_i^R(M, N')$.
- $f: M \rightarrow M'$ induces

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P'_2 & \longrightarrow & P'_1 & \longrightarrow & P'_0 & \longrightarrow & M' & \longrightarrow & 0 \\ & & f_2 \uparrow & & f_1 \uparrow & & f_0 \uparrow & & f \uparrow & & \\ \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

uniquely up to chain homotopy, and thus determines a distinguished chain homotopy class

$$f_* \otimes \mathrm{id}_N: P_* \otimes_R N \rightarrow P'_* \otimes_R N$$

and hence a well-defined homomorphism

$$\mathrm{Tor}_i^R(f, \mathrm{id}): \mathrm{Tor}_i^R(M, N) \longrightarrow \mathrm{Tor}_i^R(M', N).$$

Theorem I.6.3.

1.) For a projective resolution $Q_* \rightarrow N$, we have

$$\mathrm{Tor}_i^R(M, N) \cong H_i(M \otimes_R Q_*) \quad \text{for } i \geq 0.$$

2.) If R is commutative, there is a natural isomorphism

$$\mathrm{Tor}_i^R(M, N) \cong \mathrm{Tor}_i^R(N, M) \quad \text{for } i \geq 0.$$

Proof. [[Lück, Algebraische Topologie](#), Lemma 6.12].

□

Proposition I.6.3. There is a natural isomorphism

$$\mathrm{Tor}_0^R(M, N) \cong M \otimes_R N.$$

Proof. Let $P_* \rightarrow M$ be a projective resolution. Then

$$P_1 \otimes_R N \longrightarrow P_0 \otimes_R N \longrightarrow M \otimes_R N \longrightarrow 0$$

is exact, so

$$\begin{aligned} M \otimes_R N &\cong \mathrm{coker}(P_1 \otimes_R N \longrightarrow P_0 \otimes_R N) \\ &\cong H_0(P_* \otimes_R N) = \mathrm{Tor}_0^R(M, N). \end{aligned}$$

□

The Tor functors describe the failure of the functor $(\cdot) \otimes_R N$ to be left exact ($\mathrm{Tor}_i^R(\cdot, N)$ are the “left derived functors” of $(\cdot) \otimes_R N$).

Theorem I.6.4. (Long exact Tor sequence)

- (i) Let $0 \rightarrow M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow 0$ be a short exact sequence of right R -modules, and let N be a left R -module. Then there is a natural long exact sequence

$$\begin{aligned} \cdots &\longrightarrow \mathrm{Tor}_2^R(M_0, N) \longrightarrow \mathrm{Tor}_2^R(M_1, N) \longrightarrow \mathrm{Tor}_2^R(M_2, N) \longrightarrow \\ &\longrightarrow \mathrm{Tor}_1^R(M_0, N) \longrightarrow \mathrm{Tor}_1^R(M_1, N) \longrightarrow \mathrm{Tor}_1^R(M_2, N) \longrightarrow \\ &\longrightarrow M_0 \otimes_R N \longrightarrow M_1 \otimes_R N \longrightarrow M_2 \otimes_R N \longrightarrow 0 \end{aligned}$$

- (ii) Let $0 \rightarrow M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow 0$ be a short exact sequence of left R -modules, and let N be a right R -module. Then there is a natural long exact sequence

$$\begin{aligned} \cdots &\longrightarrow \mathrm{Tor}_2^R(N, M_0) \longrightarrow \mathrm{Tor}_2^R(N, M_1) \longrightarrow \mathrm{Tor}_2^R(N, M_2) \longrightarrow \\ &\longrightarrow \mathrm{Tor}_1^R(N, M_0) \longrightarrow \mathrm{Tor}_1^R(N, M_1) \longrightarrow \mathrm{Tor}_1^R(N, M_2) \longrightarrow \\ &\longrightarrow N \otimes_R M_0 \longrightarrow N \otimes_R M_1 \longrightarrow N \otimes_R M_2 \longrightarrow 0 \end{aligned}$$

Lemma I.6.5. (Horseshoe lemma) Let $0 \rightarrow M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow 0$ be a short exact sequence of right R -modules, and let $P_*^0 \rightarrow M_0$ and $P_*^2 \rightarrow M_2$ be projective resolutions:

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \\ & & P_1^0 & & P_1^2 & & \\ & & \downarrow & & \downarrow & & \\ & & P_0^0 & & P_0^2 & & \\ & & \downarrow \varepsilon^0 & & \downarrow \varepsilon^2 & & \\ 0 & \longrightarrow & M_0 & \xrightarrow{\varphi} & M_1 & \xrightarrow{\psi} & M_2 \longrightarrow 0 \\ & & \downarrow & & & & \downarrow \\ & & 0 & & & & 0 \end{array}$$

Then there exist a projective resolution $P_*^1 \rightarrow M_1$ and chain maps $i_*: P_*^0 \rightarrow P_*^1, p_*: P_*^1 \rightarrow P_*^2$, such that

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & P_1^0 & \xrightarrow{i_1} & P_1^1 & \xrightarrow{p_1} & P_1^2 \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & P_0^0 & \xrightarrow{i_0} & P_0^1 & \xrightarrow{p_0} & P_0^2 \longrightarrow 0 \\
& & \varepsilon^0 \downarrow & & \downarrow \varepsilon^1 & & \downarrow \varepsilon^2 \\
0 & \longrightarrow & M_0 & \xrightarrow{\varphi} & M_1 & \xrightarrow{\psi} & M_2 \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

commutes and has exact rows (and columns).

Note. By [Theorem I.1.2\(ii\)](#), the rows in the projective resolutions then split degreewise, and $P_*^1 \cong P_*^0 \oplus P_*^2$ degreewise by the splitting lemma (*Topology I*, I.4.7).

Proof (of [Lemma I.6.5](#)). Set $P_0^1 := P_0^0 \oplus P_0^2$, let i_0 and p_0 be the canonical inclusion and projection, and set $\varepsilon^1|_{P_0^0} := \varphi \circ \varepsilon^0$. Let

$$\begin{array}{ccc}
M_1 & \xrightarrow{\psi} & M_2 \longrightarrow 0 \\
& \nwarrow \chi & \uparrow \varepsilon^2 \\
& & P_0^2
\end{array}$$

and set $\varepsilon^1|_{P_0^2} := \chi$. Then the bottom two squares commute. The (proof of the) five lemma (*Topology I*, III.2.2) shows that ε^1 is surjective. By the snake lemma (*Topology I*, II.2.1), the first row in

$$\begin{array}{ccccccc}
0 & \longrightarrow & \ker \varepsilon^0 & \longrightarrow & \ker \varepsilon^1 & \longrightarrow & \ker \varepsilon^2 \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & P_0^0 & \xrightarrow{i_0} & P_0^1 & \xrightarrow{p_0} & P_0^2 \longrightarrow 0 \\
& & \varepsilon^0 \downarrow & & \downarrow \varepsilon^1 & & \downarrow \varepsilon^2 \\
0 & \longrightarrow & M_0 & \xrightarrow{\varphi} & M_1 & \xrightarrow{\psi} & M_2 \longrightarrow 0
\end{array}$$

is again exact, since $\text{coker } \varepsilon^0 = 0$.

We obtain

$$\begin{array}{ccccccc}
& P_1^0 & & & & P_1^2 & \\
& \downarrow & & & & \downarrow & \\
0 & \longrightarrow & \ker \varepsilon^0 & \longrightarrow & \ker \varepsilon^1 & \longrightarrow & \ker \varepsilon^2 \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & P_0^0 & \xrightarrow{i_0} & P_0^1 & \xrightarrow{p_0} & P_0^2 \longrightarrow 0 \\
& & \varepsilon^0 \downarrow & & \downarrow \varepsilon^1 & & \downarrow \varepsilon^2 \\
0 & \longrightarrow & M_0 & \xrightarrow{\varphi} & M_1 & \xrightarrow{\psi} & M_2 \longrightarrow 0
\end{array}$$

and hence, by induction, the assertion of the lemma. $\square$

Proof (of Theorem I.6.4). (i) By the horseshoe lemma, we have a short exact sequence of chain complexes

$$0 \longrightarrow P_*^0 \longrightarrow P_*^1 \longrightarrow P_*^2 \longrightarrow 0 ,$$

whose terms are projective resolutions of M_0 , M_1 , and M_2 . By I.1.2(ii), the short exact sequence splits in each degree. Thus

$$0 \longrightarrow P_*^0 \otimes_R N \longrightarrow P_*^1 \otimes_R N \longrightarrow P_*^2 \otimes_R N \longrightarrow 0$$

is again exact, and we obtain the long exact sequence from *Topology I*, II.2.2.

(ii) Let $P_* \rightarrow N$ be a projective resolution. Then

$$0 \longrightarrow P_* \otimes M_0 \longrightarrow P_* \otimes M_1 \longrightarrow P_* \otimes M_2 \longrightarrow 0$$

is exact, because projective modules are flat by Proposition I.4.5. Again, we conclude using *Topology I*, II.2.2. $\square$

Examples.

- 1.) If M (or N) is projective, then $0 \rightarrow M \xrightarrow{\text{id}} M \rightarrow 0$ is a projective resolution, so $\text{Tor}_i^R(M, N) = 0$ for $i \geq 1$.
- 2.) In particular, $\text{Tor}_i^K(M, N) = 0$ for $i \geq 1$ if K is a field, since vector spaces are free.
- 3.) Let G be a group. Then $\mathbb{Z}G$ becomes the *group ring* of G with multiplication

$$\left(\sum_{g \in G} \lambda_g g \right) \cdot \left(\sum_{g' \in G} \mu_{g'} g' \right) = \sum_{g \in G} \sum_{g' \in G} (\lambda_g \mu_{g'}) (gg').$$

Regard $\mathbb{Z}$ as a right $\mathbb{Z}G$ -module with

$$k \cdot \left(\sum_{g \in G} \lambda_g g \right) = \sum_{g \in G} k \lambda_g$$

(the “trivial $\mathbb{Z}G$ -module $\mathbb{Z}$ ”). For a left $\mathbb{Z}G$ -module M , we have

$$\begin{aligned} \mathbb{Z} \otimes_{\mathbb{Z}G} M &\cong M / \langle gm - m : g \in G, m \in M \rangle =: M_G \quad (\text{“coinvariants”}), \\ k \otimes m &\mapsto [km], \\ 1 \otimes m &\longleftarrow [m]. \end{aligned}$$

This is well-defined because $1 \otimes (gm - m) = 1 \otimes gm - 1 \otimes m = 1g \otimes m - 1 \otimes m = 0$.

The abelian groups $H_i(G; M) := \text{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, M)$ form the *group homology* of G with coefficients in M . For

$$0 \longrightarrow I \longrightarrow \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

$$g \longmapsto 1$$

(I : “augmentation ideal”, ε : “augmentation”) the long exact Tor sequence ends in

$$0 \longrightarrow H_1(G; M) \longrightarrow I \otimes_{\mathbb{Z}G} M \longrightarrow M \longrightarrow M_G \longrightarrow 0 ,$$

that is, $H_1(G; M) \cong \ker(I \otimes_{\mathbb{Z}G} M \rightarrow M)$. If M is a trivial $\mathbb{Z}G$ -module (that is, every $g \in G$ acts as the identity), then $M_G = M$, so

$$\begin{aligned} H_1(G; M) &\cong I \otimes_{\mathbb{Z}G} M \cong I \otimes_{\mathbb{Z}G} (\mathbb{Z} \otimes_{\mathbb{Z}G} M) \\ &\cong (I \otimes_{\mathbb{Z}G} \mathbb{Z}) \otimes_{\mathbb{Z}G} M \cong I_G \otimes_{\mathbb{Z}} M. \end{aligned}$$

Lemma I.6.6. There is a natural isomorphism $I_G \cong_{\mathbb{Z}} I/I^2$.

Proof. We have $I_G = I \otimes_{\mathbb{Z}G} \mathbb{Z} \cong I/\langle gx - x : x \in I \rangle$ and I is generated by the elements $x = g' - 1$ for $g' \in G$. Since

$$gx - x = g(g' - 1) - (g' - 1) = gg' - g - g' + 1 = (g - 1)(g' - 1) \in I^2$$

the claim follows. $\square$

Lemma I.6.7. There is a natural isomorphism $I/I^2 \cong G_{\text{ab}}$.

Proof. Consider

$$\begin{aligned} \varphi : G &\longrightarrow I/I^2, \\ g &\longmapsto g - 1 + I^2. \end{aligned}$$

We have

$$\begin{aligned} \varphi(gh) &= gh - 1 + I^2 = gh - 1 - (g - 1)(h - 1) + I^2 \\ &= (g - 1) + (h - 1) + I^2 = \varphi(g) + \varphi(h). \end{aligned}$$

This gives $\bar{\varphi} : G_{\text{ab}} \rightarrow I/I^2$. Consider

$$\begin{aligned} \psi : I &\longrightarrow G_{\text{ab}}, \\ g - 1 &\longmapsto g[G, G]. \end{aligned}$$

We have

$$\begin{aligned} \psi((g - 1)(h - 1)) &= \psi(gh - g - h + 1) \\ &= \psi((gh - 1) - (g - 1) - (h - 1)) \\ &= ghg^{-1}h^{-1}[G, G] = [G, G]. \end{aligned}$$

This gives $\bar{\psi} : I/I^2 \rightarrow G_{\text{ab}}$, and $\bar{\psi} \circ \bar{\varphi} = \text{id}_{G_{\text{ab}}}$, $\bar{\varphi} \circ \bar{\psi} = \text{id}_{I/I^2}$ by construction. $\square$

We have therefore proved:

Theorem I.6.8. For a trivial $\mathbb{Z}G$ -module M , we have

$$H_1(G; M) \cong G_{\text{ab}} \otimes_{\mathbb{Z}} M.$$

Using the usual convention $H_i(G) := H_i(G; \mathbb{Z})$, we thus have $H_0(G) \cong \mathbb{Z}$ and $H_1(G) \cong G_{\text{ab}}$.

Remark. Let X be path-connected and $G = \pi_1(X, \cdot)$. Choose a generator so that $\langle [S^n] \rangle = H_n^{\text{sing}}(S^n)$. This gives

$$\begin{aligned} \text{Hur}_n : \pi_n(X, \cdot) &\longrightarrow H_n^{\text{sing}}(X), \\ [f : S^n \rightarrow X] &\longmapsto f_*[S^n]. \end{aligned}$$

Then

$$\begin{aligned} &\pi_1(X, \cdot) \xrightarrow{\text{Hur}_1} H_1^{\text{sing}}(X) \\ \bullet \quad &\parallel \qquad \qquad \qquad \parallel \qquad \qquad \text{(Hurewicz, Theorem } \textit{Topology I}, \text{ II.5.4).} \\ &G \longrightarrow G_{\text{ab}} \\ \\ \bullet \quad &\pi_2(X, \cdot) \xrightarrow{\text{Hur}_2} H_2^{\text{sing}}(X) \longrightarrow H_2G \longrightarrow 0 \quad \text{(Hopf).} \end{aligned}$$

(Hurewicz: Hur_2 is an isomorphism if $G = 1$.)

Proposition I.6.9.

- (i) $\mathrm{Tor}_i^R\left(\bigoplus_{j \in I} M_j, N\right) \cong \bigoplus_{j \in I} \mathrm{Tor}_i^R(M_j, N).$
- (ii) $\mathrm{Tor}_i^R\left(M, \bigoplus_{j \in I} N_j\right) \cong \bigoplus_{j \in I} \mathrm{Tor}_i^R(M, N_j).$

Proof. This follows from

$$\left(\bigoplus_{j \in I} P_*^j\right) \otimes_R N \cong \bigoplus_{j \in I} (P_*^j \otimes_R N).$$

□

Theorem I.6.10. (Computing Tor over principal ideal domains) Let R be a principal ideal domain, M a right R -module, and N a left R -module. Then:

- (i) $\mathrm{Tor}_i^R(M, N) = 0$ for $i \geq 2$.
- (ii) If M, N are finitely generated, the inclusions $\mathrm{tors}(M) \subseteq M$, $\mathrm{tors}(N) \subseteq N$ induce an isomorphism

$$\mathrm{Tor}_1^R(\mathrm{tors}(M), \mathrm{tors}(N)) \cong \mathrm{Tor}_1^R(M, N).$$

- (iii) For $r, s \in R \setminus \{0\}$, we have

$$\mathrm{Tor}_1^R(R/(r), R/(s)) \cong R/(\mathrm{gcd}(r, s)).$$

Proof. (i) Over a principal ideal domain, submodules of free modules are free. Thus we can choose

$$0 \longrightarrow \ker \pi \longrightarrow F \xrightarrow{\pi} M \longrightarrow 0$$

with F free.

- (ii) Since $M \cong F \oplus \mathrm{tors}(M)$, $N \cong F' \oplus \mathrm{tors}(N)$ with F, F' free, this follows from [Proposition I.6.9](#), (i), and Example 1.

- (iii) Consider $0 \rightarrow R \xrightarrow{r} R \rightarrow R/(r) \rightarrow 0$, so

$$\mathrm{Tor}_1^R(R/(r), R/(s)) \cong \ker(R/(s) \xrightarrow{r} R/(s)).$$

Set $d := \mathrm{gcd}(r, s)$ and $r = r'd$, $s = s'd$. Then $r' \in (R/(s'))^*$, so

$$\begin{aligned} \ker(R/(s) \xrightarrow{r} R/(s)) &= \ker(R/(s) \xrightarrow{d} R/(s)) \\ &= \{xs' \in R/(s) : x \in R/(s)\} \cong R/(d), \end{aligned} \quad xs' \longleftarrow x.$$

□

I.7. The Universal Coefficient Theorem

Theorem I.7.1. Let R be a principal ideal domain and C_* a chain complex of projective R -modules with $C_n = 0$ for $n < 0$. Then, for every R -module M , there is a natural short exact sequence

$$0 \rightarrow H_n(C_*) \otimes_R M \xrightarrow{\alpha_n} H_n(C_* \otimes_R M) \xrightarrow{\beta_n} \text{Tor}_1^R(H_{n-1}(C_*), M) \rightarrow 0$$

$$[z] \otimes m \longmapsto [z \otimes m]$$

for all $n \geq 0$. The short exact sequence splits via

$$s_n : \text{Tor}_1^R(H_{n-1}(C_*), M) \rightarrow H_n(C_* \otimes_R M)$$

and s_n is natural in M , but (in general) not in C_* .

Corollary I.7.2.

$$H_n(C_* \otimes_R M) \cong (H_n(C_*) \otimes_R M) \oplus \text{Tor}_1^R(H_{n-1}(C_*), M)$$

is determined up to isomorphism by $H_*(C_*)$ and M . □

Proof (of Theorem I.7.1). Consider the short exact sequences

$$(i) \quad 0 \longrightarrow Z_n \xrightarrow{\quad} C_n \xrightarrow{\partial_n} B_{n-1} \longrightarrow 0$$

$\quad \quad \quad \swarrow \scriptstyle r_n \quad \quad \quad \swarrow$

and

$$(ii) \quad 0 \longrightarrow B_n \xrightarrow{i_n} Z_n \longrightarrow H_n(C_*) \longrightarrow 0.$$

The short exact sequence (i) splits because, over a principal ideal domain, projective modules are free and submodules of free modules are free, hence projective. We obtain

$$0 \longrightarrow Z_* \otimes_R M \longrightarrow C_* \otimes_R M \longrightarrow B_{*-1} \otimes_R M \longrightarrow 0$$

and this is a short exact sequence of chain complexes if we regard $Z_* \otimes_R M$ and $B_{*-1} \otimes_R M$ as chain complexes with zero differentials. By *Topology I*, II.2.2, there is a long exact sequence

$$\cdots \rightarrow B_n \otimes_R M \xrightarrow{\delta_{n+1}} Z_n \otimes_R M \rightarrow H_n(C_* \otimes_R M) \rightarrow B_{n-1} \otimes_R M \xrightarrow{\delta_n} Z_{n-1} \otimes_R M \rightarrow \cdots$$

The long exact Tor sequence associated to (ii) ends in

$$0 \rightarrow \text{Tor}_1^R(H_n(C_*), M) \rightarrow B_n \otimes_R M \xrightarrow{i_n \otimes \text{id}_M} Z_n \otimes_R M \rightarrow H_n(C_*) \otimes_R M \rightarrow 0.$$

The construction of the connecting homomorphism in the snake lemma gives $\delta_{n+1} = i_n \otimes \text{id}_M$. Thus

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{coker } \delta_{n+1} & \longrightarrow & H_n(C_* \otimes_R M) & \longrightarrow & \ker \delta_n \longrightarrow 0 \\ & & \wr & & & & \wr \\ & & H_n(C_*) \otimes_R M & & & & \text{Tor}_1^R(H_{n-1}(C_*), M) \end{array}$$

is exact. A left inverse of α_n is given by

$$\begin{aligned} H_n(C_* \otimes_R M) &\longrightarrow H_n(C_*) \otimes_R M, \\ [c \otimes m] &\longmapsto [r_n(c)] \otimes m. \end{aligned}$$

Here r_n is not natural, whereas the dependence on m is natural. □

Theorem I.7.3. Let R be a principal ideal domain, $H_* = H_*^{\text{sing}}(-; R)$, (X, A) a pair of spaces, and M an R -module. Then there is a natural short exact sequence

$$0 \rightarrow H_n(X, A) \otimes_R M \rightarrow H_n(X, A; M) \rightarrow \text{Tor}_1^R(H_{n-1}(X, A), M) \rightarrow 0,$$

which splits (not naturally in (X, A)). □

Example. Recall (*Topology I*, IV.3.3):

$$H_k(\mathbb{RP}^d; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k = 0 \text{ or } (k = d \text{ odd}), \\ \mathbb{Z}/2, & 0 < k < d \text{ and } k \text{ odd}, \\ 0, & \text{otherwise.} \end{cases}$$

We compute

- $H_n(\mathbb{RP}^d; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2 \cong \begin{cases} \mathbb{Z}/2, & n = 0 \text{ or } 0 < n \leq d, n \text{ odd}, \\ 0, & \text{otherwise,} \end{cases}$
- $\text{Tor}_1^{\mathbb{Z}}(H_{n-1}(\mathbb{RP}^d; \mathbb{Z}), \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2, & 1 < n \leq d, n \text{ even}, \\ 0, & \text{otherwise,} \end{cases}$

so the universal coefficient theorem gives

$$H_n(\mathbb{RP}^d; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2, & 0 \leq n \leq d, \\ 0, & \text{otherwise.} \end{cases}$$

(In fact, for the standard CW structure on $\mathbb{RP}^d$, $C_*^{\text{CW}}(\mathbb{RP}^d; \mathbb{Z}/2)$ is given by

$$0 \longrightarrow \underset{d}{\mathbb{Z}/2} \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \cdots \xrightarrow{0} \underset{0}{\mathbb{Z}/2} \longrightarrow 0 \text{ .})$$

I.8. The Ext Functor

Definition I.8.1. A *cochain complex* consists of a family C^* of left or right R -modules and homomorphisms $\delta^* : C^* \rightarrow C^{*+1}$ with $\delta^{*+1} \circ \delta^* = 0$.

Observation. If M is a left R -module, then

$$\begin{aligned} \text{Hom}_R(-, M) : R\text{-chain} &\longrightarrow \mathbb{Z}\text{-cochain}, \\ (C_*, \partial_*) &\longmapsto (\text{Hom}_R(C_*, M), (-) \circ \partial_{*+1}), \\ (f_* : C_* \rightarrow D_*) &\longmapsto \left(\begin{array}{c} \text{Hom}_R(D_*, M) \longrightarrow \text{Hom}_R(C_*, M) \\ g_* \longmapsto g_* \circ f_* \end{array} \right) \end{aligned}$$

is a contravariant functor. For commutative R and an R -module M , this gives

$$\text{Hom}_R(-, M) : R\text{-chain} \longrightarrow R\text{-cochain}.$$

Let C^* be a cochain complex. Set

$$\begin{aligned} Z^n &:= \ker(\delta^n : C^n \rightarrow C^{n+1}) && \text{“cocycles”,} \\ B^n &:= \text{im}(\delta^{n-1} : C^{n-1} \rightarrow C^n) && \text{“coboundaries”.} \end{aligned}$$

Definition I.8.2. The n th cohomology of C^* is the R -module $H^n(C^*) := Z^n/B^n$.

Definition I.8.3. Let M, N be left R -modules and $P_* \rightarrow M$ a projective resolution. We define the abelian group

$$\text{Ext}_R^i(M, N) := H^i(\text{Hom}_R(P_*, N)).$$

Remark. As with Tor , one shows that $\text{Ext}_R^i(M, N)$ is independent of $P_* \rightarrow M$ and functorial in M and N . We have

$$\text{Ext}_R^i(M, N) = H^i(\text{Hom}_R(M, I^*))$$

for every injective resolution $N \rightarrow I^*$.

Proposition I.8.4. There is a natural isomorphism

$$\text{Ext}_R^0(M, N) \cong \text{Hom}_R(M, N).$$

Proof. Let $N \rightarrow I^*$ be an injective resolution. Consider

$$0 \longrightarrow \text{Hom}_R(M, N) \longrightarrow \text{Hom}_R(M, I^0) \longrightarrow \text{Hom}_R(M, I^1),$$

so

$$\begin{aligned} \text{Hom}_R(M, N) &= \ker(\text{Hom}_R(M, I^0) \rightarrow \text{Hom}_R(M, I^1)) \\ &= H^0(\text{Hom}_R(M, I^*)) \cong \text{Ext}_R^0(M, N). \end{aligned}$$

□

Theorem I.8.5. (Long exact Ext sequence) For a short exact sequence $0 \rightarrow M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow 0$ of left R -modules and a left R -module N , there are natural long exact sequences:

(i)

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Hom}_R(N, M_0) & \rightarrow & \text{Hom}_R(N, M_1) & \rightarrow & \text{Hom}_R(N, M_2) \\ & & & & \swarrow & & \\ & & \text{Ext}_R^1(N, M_0) & \rightarrow & \text{Ext}_R^1(N, M_1) & \rightarrow & \text{Ext}_R^1(N, M_2) \\ & & & & \swarrow & & \\ & & \text{Ext}_R^2(N, M_0) & \rightarrow & \dots & & \end{array}$$

(ii)

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Hom}_R(M_2, N) & \rightarrow & \text{Hom}_R(M_1, N) & \rightarrow & \text{Hom}_R(M_0, N) \\ & & & & \swarrow & & \\ & & \text{Ext}_R^1(M_2, N) & \rightarrow & \text{Ext}_R^1(M_1, N) & \rightarrow & \text{Ext}_R^1(M_0, N) \\ & & & & \swarrow & & \\ & & \text{Ext}_R^2(M_2, N) & \rightarrow & \dots & & \end{array}$$

Proof. Dual to [Theorem I.6.4](#) (“injective horseshoe lemma”).

□

Proposition I.8.6. P is projective $\iff \text{Ext}_R^i(P, N) = 0$ for $i \geq 1$ and every R -module N .

Proof. “ $\Rightarrow$ ”: Use $P \xrightarrow{\text{id}} P \rightarrow 0$. “ $\Leftarrow$ ”: By [I.8.5 \(i\)](#), $\text{Hom}_R(P, -)$ is exact, so P is projective by [I.4.3](#).

□

Proposition I.8.7.

- (i) $\text{Ext}_R^i\left(\bigoplus_{j \in I} M_j, N\right) \cong \prod_{j \in I} \text{Ext}_R^i(M_j, N).$
- (ii) $\text{Ext}_R^i\left(M, \prod_{j \in I} N_j\right) \cong \prod_{j \in I} \text{Ext}_R^i(M, N_j).$

Proof. (i) $P_*^j \rightarrow M_j$ gives $\bigoplus_{j \in I} P_*^j \rightarrow \bigoplus_{j \in I} M_j$,

$$\text{Hom}_R\left(\bigoplus_{j \in I} P_*^j, N\right) \cong \prod_{j \in I} \text{Hom}_R(P_*^j, N) \longrightarrow H^i(-).$$

□

Example. The abelian groups

$$H^i(G; M) := \text{Ext}_{\mathbb{Z}G}^i(\mathbb{Z}, M)$$

form the *group cohomology* of G with coefficients in M .

Theorem I.8.8. (Universal coefficient theorem for cochain complexes) Let R be a principal ideal domain, C_* a chain complex of projective R -modules with $C_n = 0$ for $n < 0$, and M an R -module. Then there is a natural short exact sequence

$$0 \longrightarrow \text{Ext}_R^1(H_{n-1}(C_*), M) \longrightarrow H^n(\text{Hom}_R(C_*, M)) \xrightarrow{\alpha_n} \text{Hom}_R(H_n(C_*), M) \longrightarrow 0$$

$\xleftarrow{s_n}$

$$[f] \longmapsto ([z] \mapsto f(z))$$

for all $n \geq 0$. The short exact sequence splits via s_n (natural in M , but in general not natural in C_*). □

Let N be a left $\mathbb{Z}G$ -module. The long exact Ext sequence associated to $0 \rightarrow I \rightarrow \mathbb{Z}G \rightarrow \mathbb{Z} \rightarrow 0$ gives

$$0 \longrightarrow \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}, N) \longrightarrow \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, N) \xrightarrow{\rho} \text{Hom}_{\mathbb{Z}G}(I, N)$$

$$\text{Ext}_{\mathbb{Z}G}^1(\mathbb{Z}, N) \longrightarrow 0.$$

Here $\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}, N) = N^G = \{n \in N : gn = n \text{ for all } g \in G\}$ (“invariants”). We have $\rho = 0$ if N is a trivial $\mathbb{Z}G$ -module, so

$$H^1(G; N) \cong \text{Hom}_{\mathbb{Z}G}(I, N).$$

Theorem I.8.9. For a trivial $\mathbb{Z}G$ -module N , we have

$$H^1(G; N) \cong \text{Hom}_{\mathbb{Z}}(G_{\text{ab}}, N).$$

Proof. Consider

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Group}}(G, N) &\xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}G}(I, N), \\ f &\mapsto (\varphi_f : g - 1 \mapsto f(g)), \\ (f_\varphi : g \mapsto \varphi(g - 1)) &\longleftarrow \varphi.\end{aligned}$$

Verifying that these maps are homomorphisms is an exercise. $\square$

Remark. The group $H^2(G; N) = \mathrm{Ext}_{\mathbb{Z}G}^2(\mathbb{Z}, N)$ classifies extensions

$$1 \longrightarrow N \longrightarrow E \xrightarrow{\varphi} G \longrightarrow 1,$$

such that the conjugation action of $G \cong E/N$ on the abelian normal subgroup $N \trianglelefteq E$ is given by the $\mathbb{Z}G$ -module structure of N :

$$ene^{-1} = \varphi(e) \cdot n \quad \text{for all } e \in E, n \in N,$$

up to isomorphism:

$$\begin{array}{ccccccc} & & & E & & & \\ & & \nearrow & \downarrow \cong & \searrow & & \\ 1 & \longrightarrow & N & & G & \longrightarrow & 1 \\ & & \searrow & \downarrow & \nearrow & & \\ & & & E' & & & \end{array}$$

Hence the name “Ext”.

If N is a trivial $\mathbb{Z}G$ -module, then $H^2(G; N)$ therefore classifies *central extensions* of G by N .

I.9. Künneth Theorems

Now let R be commutative.

Definition I.9.1. Let C_* and D_* be chain complexes over R . Their *tensor product* $C_* \otimes_R D_*$ is the chain complex

$$(C_* \otimes_R D_*)_n = \bigoplus_{p+q=n} C_p \otimes_R D_q$$

with

$$\partial^\otimes(x \otimes y) = \partial_p^{C_*}(x) \otimes y + (-1)^p x \otimes \partial_q^{D_*}(y), \quad x \otimes y \in C_p \otimes D_q.$$

Required:

$$\begin{aligned}\partial^\otimes \partial^\otimes(x \otimes y) &= \partial^\otimes(\partial_p(x) \otimes y + (-1)^p x \otimes \partial_q(y)) \\ &= (-1)^{p-1} \partial_p(x) \otimes \partial_q(y) + (-1)^p \partial_p(x) \otimes \partial_q(y) = 0.\end{aligned}$$

Definition I.9.2. The *Alexander–Whitney map* is

$$\begin{aligned}\mathrm{AW}_* : C_*^{\mathrm{sing}}(X \times Y; R) &\longrightarrow C_*^{\mathrm{sing}}(X; R) \otimes_R C_*^{\mathrm{sing}}(Y; R), \\ (\sigma \times \tau : \Delta_n \rightarrow X \times Y) &\longmapsto \sum_{i=0}^n \sigma|_{[v_0, \dots, v_i]} \otimes \tau|_{[v_i, \dots, v_n]}.\end{aligned}$$

Theorem I.9.3. (Eilenberg–Zilber) The chain map AW_* has a chain homotopy inverse EZ_* .

Corollary I.9.4.

$$H_*^{\text{sing}}(X \times Y; R) \cong H_*(C_*^{\text{sing}}(X; R) \otimes_R C_*^{\text{sing}}(Y; R)).$$

□

The proof of [Theorem I.9.3](#) is given in the next section. For now, recall (*Topology I*, IV.1.12): If X and Y are CW complexes and X or Y is locally compact, then $X \times Y$ is a CW complex with filtration

$$(X \times Y)^n = \bigcup_{p+q=n} X^p \times Y^q$$

and hence

$$\begin{aligned} C_n^{\text{CW}}(X \times Y; R) &\cong \bigoplus_{p+q=n} C_p^{\text{CW}}(X; R) \otimes_R C_q^{\text{CW}}(Y; R) \\ &= (C_*^{\text{CW}}(X; R) \otimes_R C_*^{\text{CW}}(Y; R))_n, \end{aligned}$$

so

$$H_n^{\text{CW}}(X \times Y; R) \cong H_n(C_*^{\text{CW}}(X; R) \otimes_R C_*^{\text{CW}}(Y; R)).$$

□

Theorem I.9.5. (Künneth theorem for chain complexes) Let C_* and D_* be chain complexes of projective modules over a principal ideal domain R , with $C_n = D_n = 0$ for $n < 0$. Then there is a natural short exact sequence

$$\begin{array}{ccccc} 0 & \longrightarrow & \bigoplus_{p+q=n} H_p(C_*) \otimes_R H_q(D_*) & \longrightarrow & H_n(C_* \otimes_R D_*) \\ & & & \searrow & \\ & & \bigoplus_{p+q=n-1} \text{Tor}_1^R(H_p(C_*), H_q(D_*)) & \longrightarrow & 0, \end{array}$$

which splits (not naturally).

Proof. First consider the special case $\partial_n^{C_*} = 0$ for all n . Then $\partial^\otimes(x \otimes y) = \pm x \otimes \partial(y)$, that is, up to sign, $C_* \otimes_R D_* \cong \bigoplus_{i \in \mathbb{Z}} C_i \otimes_R D_{*-i}$ and hence (by the universal coefficient theorem, since C_i is projective)

$$\begin{aligned} H_n(C_* \otimes_R D_*) &\cong \bigoplus_{i \in \mathbb{Z}} H_n(C_i \otimes_R D_{*-i}) \\ &\cong \bigoplus_{i \in \mathbb{Z}} C_i \otimes_R H_n(D_{*-i}) \\ &\cong \bigoplus_{i \in \mathbb{Z}} H_i(C_*) \otimes_R H_{n-i}(D_*) \end{aligned}$$

and

$$\text{Tor}_1^R(H_p(C_*), H_q(D_*)) \cong \text{Tor}_1^R(C_p, H_q(D_*)) \stackrel{C_p \text{ projective}}{=} 0,$$

so the theorem holds

if $\partial_*^{C_*} = 0$. In the general case, consider again

$$0 \longrightarrow Z_* \longrightarrow C_* \xrightarrow{\partial_*} B_{*-1} \longrightarrow 0,$$

where Z_* and B_* have zero differentials. Since R is a principal ideal domain and C_* is projective (hence free), Z_* and B_* are also projective. Thus

$$0 \rightarrow Z_* \otimes_R D_* \rightarrow C_* \otimes_R D_* \rightarrow B_{*-1} \otimes_R D_* \rightarrow 0$$

and we obtain

$$\begin{array}{ccc} \cdots \xrightarrow{\delta_{n+1}} \bigoplus_{p+q=n} Z_p \otimes_R H_q(D_*) & \longrightarrow & H_n(C_* \otimes_R D_*) \\ & \searrow & \\ & \bigoplus_{p+q=n} B_{p-1} \otimes_R H_q(D_*) & \xrightarrow{\delta_n} \cdots, \end{array}$$

where the connecting homomorphisms δ_* are again induced by the inclusions $i_* : B_* \subseteq Z_*$. Thus

$$0 \longrightarrow \operatorname{coker} \delta_{n+1} \longrightarrow H_n(C_* \otimes_R D_*) \longrightarrow \ker \delta_n \longrightarrow 0$$

and $\operatorname{coker} \delta_{n+1} = \bigoplus_{p+q=n} H_p(C_*) \otimes_R H_q(D_*)$, since

$$\operatorname{coker}(i_p \otimes \operatorname{id}_{H_q(D_*)}) = (\operatorname{coker} i_p) \otimes H_q(D_*),$$

because $(-) \otimes H_q(D_*)$ is right exact. Moreover, $\ker \delta_n \cong \bigoplus_{p+q=n-1} \operatorname{Tor}_1^R(H_p(C_*), H_q(D_*))$, by taking the direct sum of

$$0 \rightarrow \operatorname{Tor}_1^R(H_p(C_*), H_q(D_*)) \rightarrow B_p \otimes H_q(D_*) \xrightarrow{i_p \otimes \operatorname{id}_{H_q(D_*)}} Z_p \otimes H_q(D_*)$$

from the long exact Tor sequence (I.6.4). The discussion of splitting and naturality is similar to that for the universal coefficient theorem. $\square$

Theorem I.9.6. (Künneth theorem for spaces) Let R be a principal ideal domain, X and Y topological spaces, and $H_* = H_*^{\operatorname{sing}}(-; R)$. Then there is a natural short exact sequence

$$\begin{array}{ccc} 0 \longrightarrow \bigoplus_{p+q=n} H_p(X) \otimes_R H_q(Y) & \longrightarrow & H_n(X \times Y) \\ & \searrow & \\ & \bigoplus_{p+q=n-1} \operatorname{Tor}_1^R(H_p(X), H_q(Y)) & \longrightarrow 0, \end{array}$$

which splits (not naturally).

Proof. Theorems I.9.3 and I.9.5. $\square$

Corollary I.9.6. If $R = K$ is a field, then

$$\bigoplus_{p+q=n} H_p(X) \otimes_R H_q(Y) \cong H_n(X \times Y).$$

$\square$

For $K = \mathbb{Q}$, this gives

$$b_n(X \times Y) = \sum_{p+q=n} b_p(X) \cdot b_q(Y),$$

provided that X and Y have finite-dimensional homology and nonzero homology in only finitely many degrees.

Theorem I.9.6. (Künneth theorem for pairs) Let R be a principal ideal domain, (X, A) and (Y, B) closed cofibrations, and $H_* = H_*^{\text{sing}}(-; R)$. Then there is a natural short exact sequence

$$\begin{array}{ccc}
0 & \longrightarrow & \bigoplus_{p+q=n} H_p(X, A) \otimes_R H_q(Y, B) \\
& & \downarrow \\
& & H_n(X \times Y, X \times B \cup A \times Y) \\
& & \downarrow \\
& & \bigoplus_{p+q=n-1} \text{Tor}_1^R(H_p(X, A), H_q(Y, B)) \longrightarrow 0,
\end{array}$$

which splits (not naturally).

Proof. Show that

$$\begin{aligned}
& H_n(C_*^{\text{sing}}(X, A; R) \otimes_R C_*^{\text{sing}}(Y, B; R)) \\
& \cong H_n(C_*^{\text{sing}}(X \times Y, X \times B \cup A \times Y; R)).
\end{aligned}$$

using excision, among other tools. $\square$

I.10. Acyclic Models

Let $\mathcal{C}$ be a locally small category and R a commutative ring.

Definition I.10.1. An RC -module is a (covariant) functor $\mathcal{C} \rightarrow R\text{-Mod}$. Homomorphisms of RC -modules are natural transformations. This gives the category $RC\text{-Mod}$.

Examples.

- If $\mathcal{C} = (\bullet \rightrightarrows \text{id})$, then an RC -module M is an R -module, and

$$\text{Hom}_{RC}(M, N) = \text{Hom}_R(M(\bullet), N(\bullet)).$$

- If $\mathcal{C} = \underline{G}$ for a group G , then an RC -module is an R -module M with a homomorphism $G \rightarrow \text{Aut}_R M$, that is, an RG -module, and $\text{Hom}_{RC}(M, N) = \text{Hom}_{RG}(M(\bullet), N(\bullet))$:

$$\begin{array}{ccc}
M(\bullet) & \xrightarrow{g} & M(\bullet) \\
\eta \downarrow & & \downarrow \eta \\
N(\bullet) & \xrightarrow{g} & N(\bullet).
\end{array}$$

A sequence of RC -modules $0 \rightarrow M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow 0$ is called *exact* if $0 \rightarrow M_0(x) \rightarrow M_1(x) \rightarrow M_2(x) \rightarrow 0$ is exact for every $x \in \text{Ob}(\mathcal{C})$.

Definition I.10.2. An RC -module P is called *projective* if

$$\begin{array}{ccccc}
M & \longrightarrow & N & \longrightarrow & 0 \\
& \nwarrow \exists & \uparrow & & \\
& & P & &
\end{array}$$

For a projective RC -module P , each $P(x)$, for $x \in \text{Ob}(\mathcal{C})$, is a projective R -module (for an R -module N_0 , use the functor $y \mapsto \text{Hom}_R(R[\text{Hom}_{\mathcal{C}}(y, x)], N_0)$; it is right adjoint to evaluation at x and preserves epimorphisms). **Warning:** The converse does not hold!

Definition I.10.3. An RC -module F is called *free* if there are objects $(x_i)_{i \in I}$ in $\mathcal{C}$ such that F is naturally isomorphic to

$$\bigoplus_{i \in I} R[\text{Hom}_{\mathcal{C}}(x_i, \cdot)].$$

The objects x_i are called *models* of F .

Proposition I.10.4. Free RC -modules are projective.

Proof. Without loss of generality, $F = R[\text{Hom}_{\mathcal{C}}(x, \cdot)]$ for some $x \in \text{Ob}(\mathcal{C})$. We must show:

$$\begin{array}{ccccc} M & \xrightarrow{e} & N & \longrightarrow & 0 \\ & \nwarrow \bar{\eta} & \uparrow \eta & & \\ & & F & & \end{array}.$$

In particular, consider

$$\begin{array}{ccccc} M(x) & \xrightarrow{e(x)} & N(x) & \longrightarrow & 0 \\ & & \uparrow \eta(x) & & \\ & & R[\text{Hom}_{\mathcal{C}}(x, x)] & & \end{array}$$

Let $m \in M(x)$ be a preimage of $\eta(x)(\text{id}_x)$ under $e(x)$ and set $\bar{\eta}(x)(\text{id}_x) := m$. For $f \in \text{Hom}_{\mathcal{C}}(x, y)$, naturality of $\bar{\eta}$ then requires:

$$\bar{\eta}(y)(f) = \bar{\eta}(y)(f \circ \text{id}_x) = M(f)(\bar{\eta}(x)(\text{id}_x)) = M(f)(m).$$

By construction,

$$\begin{aligned} e(y)(\bar{\eta}(y)(f)) &= e(y)(M(f)(m)) = N(f)(e(x)(m)) \\ &= N(f)(\eta(x)(\text{id}_x)) = \eta(y)(F(f)(\text{id}_x)) \\ &= \eta(y)(f \circ \text{id}_x) = \eta(y)(f), \end{aligned}$$

that is, $e \circ \bar{\eta} = \eta$. □

Example. $C_n^{\text{sing}}(\cdot; R) : \text{Top} \longrightarrow R\text{-Mod}$ is a free $R\text{Top}$ -module with model Δ_n :

$$C_n^{\text{sing}}(\cdot; R) = R[\text{Hom}_{\text{Top}}(\Delta_n, \cdot)].$$

Thus $C_*^{\text{sing}}(\cdot; R)$ is a chain complex of projective $R\text{Top}$ -modules. The same proof as for [Theorem I.2.2](#) gives:

Theorem I.10.5. Let $P_{* \geq 0}$ be a chain complex of projective RC -modules. Let $Q_{* \geq 0}$ be a chain complex with $H_k(Q_*) = 0$ for $k > 0$ (objectwise). Then there is an isomorphism

$$[P_*, Q_*] \xrightarrow{\cong} \text{Hom}_{RC}(H_0 \circ P_*, H_0 \circ Q_*).$$

□

Theorem I.10.6. (Acyclic Models Theorem)

Let $F_{* \geq 0}$ be a chain complex of free RC -modules, and let M_* be the sets of models. Let G_* be an RC -chain complex with $H_k(G_*(x)) = 0$ for all $x \in M_k \cup M_{k+1}$ and $k \geq 1$. Then

$$[F_*, G_*] \xrightarrow{\cong} \text{Hom}_{RC}(H_0 \circ F_*, H_0 \circ G_*).$$

Proof. As in [Theorem I.2.2](#), with the natural transformations determined by their components at the models, as in [Proposition I.10.4](#). □

Proof of Theorem I.9.3. Set $\mathcal{C} = \text{Top} \times \text{Top}$,

$$F_* = C_*^{\text{sing}}(\cdot \times \cdot; R), \quad G_* = C_*^{\text{sing}}(\cdot; R) \otimes_R C_*^{\text{sing}}(\cdot; R).$$

We have

$$F_n = R[\text{Hom}_{\text{Top}}(\Delta_n, \cdot \times \cdot)] \cong R[\text{Hom}_{\mathcal{C}}(\Delta_n \times \Delta_n, \cdot \times \cdot)],$$

that is, F_n is a free $R\mathcal{C}$ -module with model $\Delta_n \times \Delta_n$. Similarly, we obtain

$$\begin{aligned} G_n &= \bigoplus_{n=p+q} C_p^{\text{sing}}(\cdot; R) \otimes_R C_q^{\text{sing}}(\cdot; R) \\ &= \bigoplus_{n=p+q} R[\text{Hom}_{\text{Top}}(\Delta_p, \cdot)] \otimes_R R[\text{Hom}_{\text{Top}}(\Delta_q, \cdot)] \\ &\cong \bigoplus_{n=p+q} R[\text{Hom}_{\text{Top}}(\Delta_p, \cdot) \times \text{Hom}_{\text{Top}}(\Delta_q, \cdot)] \\ &\cong \bigoplus_{n=p+q} R[\text{Hom}_{\mathcal{C}}(\Delta_p \times \Delta_q, \cdot \times \cdot)], \end{aligned}$$

that is, G_n is a free $R\mathcal{C}$ -module with models $\Delta_p \times \Delta_q$ for $p + q = n$. For all p, q , we have $\Delta_p \times \Delta_q \simeq \bullet$, hence

$$H_k(F_*(\Delta_p \times \Delta_q)) = H_k(G_*(\Delta_p \times \Delta_q)) = 0$$

for all p, q and $k \geq 1$. Consider

$$\begin{array}{ccc} H_0(G_*(X \times Y)) & \xrightarrow{\Psi(X \times Y)} & H_0(F_*(X \times Y)) \\ \parallel & & \parallel \\ H_0(C_*^{\text{sing}}(X; R) \otimes C_*^{\text{sing}}(Y; R)) & \longrightarrow & H_0^{\text{sing}}(X \times Y; R) \end{array}$$

$$[x \otimes y] \longmapsto [(x, y)].$$

The natural transformation Ψ is well-defined because $\partial_1^{\otimes} = \partial_1^X \otimes \text{id} + \text{id} \otimes \partial_1^Y$, which implies that Ψ takes boundaries to boundaries. By Theorem I.10.6, Ψ determines, up to chain homotopy, the chain map $\text{EZ}_*: G_* \rightarrow F_*$ with $H_0(\text{EZ}_*) = \Psi$. Since

$$H_0(\text{EZ}_*) \circ H_0(\text{AW}_*) = \text{id} \quad \text{and} \quad H_0(\text{AW}_*) \circ H_0(\text{EZ}_*) = \text{id},$$

we obtain $\text{EZ}_* \circ \text{AW}_* \simeq \text{id}_*$ and $\text{AW}_* \circ \text{EZ}_* \simeq \text{id}_*$. □

II. Cohomology

II.1. Eilenberg–Steenrod Cohomology Theories

Let R be a commutative ring.

Definition II.1.1. A *cohomology theory* with values in $R\text{-Mod}$ consists of contravariant functors

$$H^n: \mathbf{Top}^{(2)} \longrightarrow R\text{-Mod} \quad \text{for } n \in \mathbb{Z}$$

and natural transformations

$$\delta^n: H^n \circ \partial \longrightarrow H^{n+1} \quad \text{for } n \in \mathbb{Z}$$

with $\partial: \mathbf{Top}^{(2)} \rightarrow \mathbf{Top}^{(2)}$, $(X, A) \mapsto (A, \emptyset)$, such that homotopy invariance, the long exact sequence, and excision hold as in *Topology I*, I.5.1, with the arrows reversed.

Definition II.1.2. The *singular cochain complex* of (X, A) with coefficients in R is

$$C_{\text{sing}}^*(X, A; R) := \text{Hom}_R(C_{\text{sing}}^*(X, A; \mathbb{Z}), R).$$

The n th *singular cohomology* is

$$H_{\text{sing}}^n(X, A; R) := H^n(C_{\text{sing}}^*(X, A; R)).$$

Observation.

$$\begin{aligned} C_{\text{sing}}^*(X, A; R) &\cong \text{Hom}_R(C_{\text{sing}}^*(X, A; \mathbb{Z}), R) \\ &\cong \text{Hom}_R(R \otimes_{\mathbb{Z}} C_{\text{sing}}^*(X, A; \mathbb{Z}), R) \\ &\cong \text{Hom}_{\mathbb{Z}}(C_{\text{sing}}^*(X, A; \mathbb{Z}), R). \end{aligned}$$

If $A = \emptyset$, then

$$\begin{aligned} \dots &\cong \text{Hom}_{\mathbb{Z}}(F_{\mathbb{Z}}(\text{Hom}_{\mathbf{Top}}(\Delta^*, X)), R) \\ &\cong \text{Hom}_{\mathbf{Set}}(\text{Hom}_{\mathbf{Top}}(\Delta^*, X), R). \end{aligned}$$

Theorem II.1.3. Singular cohomology is a cohomology theory.

Proof.

1. Homotopy invariance: $f \simeq g \Rightarrow C_{\text{sing}}^*(f) \simeq_{P_*} C_{\text{sing}}^*(g)$, and $(\cdot) \circ P_*$ is a (co)chain homotopy from $(\cdot) \circ C_{\text{sing}}^*(f)$ to $(\cdot) \circ C_{\text{sing}}^*(g)$.
2. Long exact sequence: The short exact sequence

$$0 \longrightarrow C_{\text{sing}}^*(A; R) \longrightarrow C_{\text{sing}}^*(X; R) \longrightarrow C_{\text{sing}}^*(X, A; R) \longrightarrow 0$$


splits degreeewise because $\{\sigma: \Delta^* \rightarrow X : \text{im } \sigma \cap (X \setminus A) \neq \emptyset\}$ is a basis of $C_*^{\text{sing}}(X, A; R)$. It follows that

$$0 \longrightarrow \text{Hom}_R(C_*^{\text{sing}}(X, A; R), R) \longrightarrow \text{Hom}_R(C_*^{\text{sing}}(X; R), R) \longrightarrow \text{Hom}_R(C_*^{\text{sing}}(A; R), R) \longrightarrow 0$$

is exact. (The dual of) Theorem *Topology I*, II.2.2, gives the required natural long exact sequence.

3. In the proof of Theorem *Topology I*, II.4.2, for $A \subseteq Y \subseteq X$ with $\overline{A} \subseteq \overset{\circ}{Y}$, we showed that the inclusion induces a chain homotopy equivalence

$$C_*^{\text{sing}}(X \setminus A, Y \setminus A; R) \simeq C_*^{\text{sing}}(X, Y; R).$$

As in 1., dualizing gives

$$C_{\text{sing}}^*(X, Y; R) \simeq C_{\text{sing}}^*(X \setminus A, Y \setminus A; R).$$

□

Remark. As before, for an R -module M ,

$$H_{\text{sing}}^*(\cdot; M) := H^*(\text{Hom}_R(C_*^{\text{sing}}(\cdot; R), M))$$

is also a cohomology theory with values in $R\text{-Mod}$.

Theorem II.1.4. $(H_{\text{sing}}^*, \delta^*)$ satisfies the *dimension axiom*

$$H_{\text{sing}}^n(\bullet; M) = 0 \quad \text{for } n \neq 0$$

and the *additivity axiom* (multiplicativity axiom!)

$$H_{\text{sing}}^*\left(\coprod_{i \in I} X_i; M\right) \cong \prod_{i \in I} H_{\text{sing}}^*(X_i; M).$$

Proof. Dimension:

$$C_{\text{sing}}^*(\bullet; M) = 0 \longrightarrow M_0 \xrightarrow{0} M_1 \xrightarrow{\text{id}} M_2 \xrightarrow{0} M_3 \xrightarrow{\text{id}} \dots$$

Additivity: as in [Proposition I.8.7](#).

□

Theorem II.1.5. (Universal Coefficient Theorem for Singular Cohomology)

Let R be a principal ideal domain. Then there is a natural short exact sequence

$$0 \longrightarrow \text{Ext}_R^1(H_{n-1}(X, A; R), R) \longrightarrow H^n(X, A; R) \longrightarrow \text{Hom}_R(H_n(X, A; R), R) \longrightarrow 0,$$

which splits (not naturally).

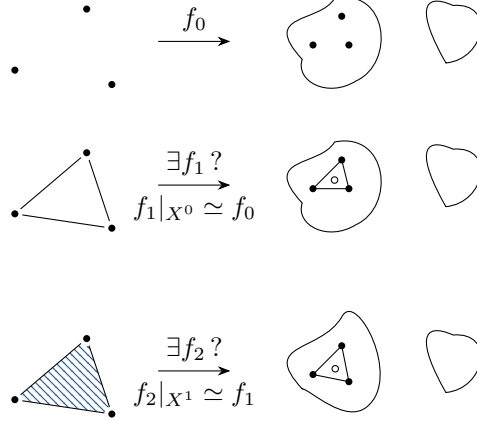
Proof. Apply [Theorem I.8.8](#) to $C_*^{\text{sing}}(X, A; R)$.

□

$\Rightarrow$ Cohomology is determined by homology.

II.2. Motivation

1. Cohomology is often relevant in homotopy-theoretic contexts, for example, obstruction theory:



For codomain Y and $n \geq 3$:

$$a: \sigma_n \mapsto [f_{n-1}|_{\partial\sigma_n}] \in \pi_{n-1}(Y), \quad \delta a(\sigma_{n+1}) = a(\partial\sigma_{n+1}) = 0$$

$\rightsquigarrow [a] \in H^n(X; \pi_{n-1}(Y))$: “extension obstruction.”¹

Another example: characteristic classes. Let G be a topological group, and let $b_G: \mathbf{Top} \rightarrow \mathbf{Set}$,

$$b_G(X) = \{\text{principal } G\text{-bundles over } X\} / \cong.$$

A *characteristic class* is a natural transformation $b_G \rightarrow H^*$. For $f: X \rightarrow Y$, we obtain

$$\begin{array}{ccc} b_G(Y) & \xrightarrow{f^*} & b_G(X) \\ c \downarrow & & \downarrow c \\ H^*(Y) & \xrightarrow{f^*} & H^*(X), \end{array} \quad \text{that is, } c(f^*P) = f^*c(P).$$

We have $b_G \cong [\cdot, BG]$. “Yoneda lemma”:

$$\text{Characteristic classes with respect to } H^* = \text{Nat}(b_G, H^*) = H^*(BG).$$

2. Relation to global analysis.

Let M be a smooth manifold of dimension n , and let Ω^*M be the cochain complex of differential forms on M . Consider

$$\begin{aligned} \Omega^*M & \xrightarrow{I_*} \text{Hom}_{\mathbb{Z}}(C_*^{\text{smooth}}(M), \mathbb{R}) \xrightarrow[\text{(nontrivial)}]{\simeq} C_{\text{sing}}^*(M; \mathbb{R}) \\ \omega & \longmapsto (\sigma \mapsto \int \sigma^* \omega). \end{aligned}$$

¹This is a schematic description, not a fully specified general theorem. In general, local coefficients are required. Constant coefficients require the relevant π_1 -action to be trivial.

Then Stokes' theorem gives

$$\begin{aligned} I^{*+1}(d\omega)(\sigma: \Delta^{*+1} \rightarrow M) &= \int_{\Delta^{*+1}} \sigma^* d\omega = \int_{\Delta^{*+1}} d(\sigma^* \omega) \\ &= \int_{\partial \Delta^{*+1}} i^* \sigma^* \omega = \int_{\partial \Delta^{*+1}} (\sigma \circ i)^* \omega = \delta^*(I^*(\omega))(\sigma), \end{aligned}$$

with $i: \partial \Delta^{*+1} \hookrightarrow \Delta^{*+1}$; that is, I_* is a cochain map.

Theorem (de Rham). $H^n(I_*): H_{\text{dR}}^n(M) \xrightarrow{\cong} H_{\text{sing}}^n(M; \mathbb{R})$. □

3. Product structure. Set

$$C_{\text{CW}}^*(X) = H_{\text{sing}}^*(X^*, X^{*-1}) = \text{Hom}_R(C_*^{\text{CW}} X, R).$$

The map

$$C_{\text{CW}}^*(X) \otimes C_{\text{CW}}^*(X) \longrightarrow C_{\text{CW}}^*(X \times X), \quad e_p^* \otimes e_q^* \longmapsto (e_p \times e_q)^*,$$

induces

$$H_{\text{CW}}^p(X) \times H_{\text{CW}}^q(X) \longrightarrow H_{\text{CW}}^{p+q}(X \times X).$$

Using $\Delta: X \rightarrow X \times X$, $x \mapsto (x, x)$, we obtain the product

$$\cup: H_{\text{CW}}^p(X) \times H_{\text{CW}}^q(X) \longrightarrow H_{\text{CW}}^{p+q}(X \times X) \xrightarrow{\Delta^*} H_{\text{CW}}^{p+q}(X).$$

“Cup product” $\rightsquigarrow H_{\text{CW}}^*(X) := \bigoplus_{n \geq 0} H_{\text{CW}}^n(X)$ is a *graded ring*. (Caution: Δ is not cellular! Choose a cellular approximation of the diagonal: $\Delta' \simeq \Delta$.)

II.3. Multiplicative Structures and Graded Algebras

Definition II.3.1. Let (H^*, δ^*) be a cohomology theory with coefficients in R . A *multiplicative structure* assigns to each space X an element $1_X \in H^0(X)$ and to subspaces $A, B \subseteq X$ a family of R -bilinear maps

$$\cup: H^p(X, A) \times H^q(X, B) \longrightarrow H^{p+q}(X, A \cup B)$$

such that

1. $\cup$ is natural with respect to $f: (X; A, B) \rightarrow (Y; A', B')$;
2. $\cup$ is graded-commutative: for $u \in H^p(X, A)$ and $v \in H^q(X, B)$,

$$u \cup v = (-1)^{p \cdot q} v \cup u;$$

3. $\cup$ is associative: $u \cup (v \cup w) = (u \cup v) \cup w$;
4. for $u \in H^p(X, A)$, we have $1_X \cup u = u \cup 1_X = u$;
5. if $j: A \hookrightarrow X$ is a subspace inclusion, $u \in H^p(A)$, and $v \in H^q(X)$, then

$$\delta^p(u) \cup v = \delta^{p+q}(u \cup H^q(j)(v)).$$

Remarks. We still speak of a multiplicative structure if it is defined only for certain triads $(X; A, B)$ (for example, when $(A \cup B; A, B)$ is excisive).

Definition II.3.2. An R -algebra is an R -module A with an R -bilinear map $m: A \times A \rightarrow A$ (equivalently, an R -homomorphism $A \otimes_R A \rightarrow A$) such that $(A, +, m)$ is a ring. A *graded R -algebra* is an R -algebra A with a decomposition $A = \bigoplus_{p \in \mathbb{Z}} A^p$ such that $m(A^p \times A^q) \subseteq A^{p+q}$. Elements of A^p are called *homogeneous (of degree p)*. A graded R -algebra $A = \bigoplus_p A^p$ is called *graded-commutative*

if $m(u, v) = (-1)^{pq}m(v, u)$ for $u \in A^p$ and $v \in A^q$. For $a \in A^p$, we write $|a| = \deg a = p$.

Examples.

1. If H^* is a cohomology theory with a multiplicative structure, then

$$H^*(X) = \bigoplus_{p \in \mathbb{Z}} H^p(X)$$

is a graded-commutative graded R -algebra.

2. The polynomial ring $Q = R[x_1, \dots, x_n]$ is an R -algebra. Assign fixed degrees $\deg x_i \in \mathbb{Z}_{\geq 0}$ to the generators (= variables). Then $Q = \bigoplus_{p \in \mathbb{Z}} Q^p$, with

$$Q^p = \left\langle x_1^{i_1} \cdots x_n^{i_n} : \sum_{j=1}^n i_j \deg x_j = p \right\rangle_R$$

is a graded R -algebra. It is graded-commutative if $\deg x_i \in 2\mathbb{Z}_{\geq 0}$ for all $i = 1, \dots, n$ and/or if $\text{char } R = 2$. (It is always commutative.)

3. The *exterior algebra* $\Lambda_R[x_1, \dots, x_n]$ is the free R -module $R[x_{i_1} \cdots x_{i_k} : i_1 < \cdots < i_k]$ with fixed odd degrees $\deg x_i \in \mathbb{Z}_{\geq 0}$ and

$$\Lambda_R[x_1, \dots, x_n] = \bigoplus_p \Lambda_R^p[x_1, \dots, x_n],$$

where

$$\Lambda_R^p[x_1, \dots, x_n] = R \left[x_{i_1} \cdots x_{i_k} : i_1 < \cdots < i_k, \sum_j \deg x_{i_j} = p \right].$$

Multiplication is determined by the rules $x_i x_j = -x_j x_i$ and $x_i x_i = 0$ and is graded-commutative.

Definition II.3.3. If $A^* = \bigoplus_p A^p$ and $B^* = \bigoplus_p B^p$ are graded R -algebras, then their *tensor product* is the graded R -algebra

$$A^* \otimes_R B^* := \bigoplus_{n \in \mathbb{Z}} \bigoplus_{p+q=n} A^p \otimes_R B^q$$

with $(a \otimes b)(a' \otimes b') = (-1)^{|a'| \cdot |b|} a a' \otimes b b'$ for $a \in A^p$, $a' \in A^{p'}$, $b \in B^q$, $b' \in B^{q'}$. If A^* and B^* are graded-commutative, then so is $A^* \otimes_R B^*$.

Example.

$$\Lambda_R[x_1, \dots, x_n] \otimes \Lambda_R[y_1, \dots, y_m] = \Lambda_R[x_1, \dots, x_n, y_1, \dots, y_m].$$

II.4. Multiplicative Structures on Singular Cohomology

Set $C_* := C_*^{\text{sing}}(\cdot; \mathbb{Z})$ and consider the natural cochain map

$$\begin{aligned} \text{Hom}_{\mathbb{Z}}(C_* X, R) \otimes_{\mathbb{Z}} \text{Hom}_{\mathbb{Z}}(C_* Y, R) &\longrightarrow \text{Hom}_{\mathbb{Z}}(C_* X \otimes_{\mathbb{Z}} C_* Y, R \otimes_{\mathbb{Z}} R), \\ f \otimes g &\longmapsto (x \otimes y \longmapsto f(x) \otimes g(y)). \end{aligned}$$

Composing with $\text{Hom}_{\mathbb{Z}}(\text{AW}_*, R \otimes_{\mathbb{Z}} R \xrightarrow{\cdot} R)$ gives an R -balanced map, hence

$$\text{Hom}_{\mathbb{Z}}(C_*X, R) \otimes_R \text{Hom}_{\mathbb{Z}}(C_*Y, R) \longrightarrow \text{Hom}_{\mathbb{Z}}(C_*(X \times Y), R).$$

Passing to cohomology gives

$$\begin{aligned} H^p(\text{Hom}_{\mathbb{Z}}(C_*X, R)) \otimes_R H^q(\text{Hom}_{\mathbb{Z}}(C_*Y, R)) &\xrightarrow{\alpha} \\ H^{p+q}(\text{Hom}_{\mathbb{Z}}(C_*X, R) \otimes_R \text{Hom}_{\mathbb{Z}}(C_*Y, R)) &\longrightarrow H^{p+q}(X \times Y; R) \end{aligned}$$

with $\alpha([f] \otimes [g]) = [f \otimes g]$, so altogether we obtain

$$\times : H^p(X; R) \otimes_R H^q(Y; R) \longrightarrow H^{p+q}(X \times Y; R),$$

the (*cohomological*) *cross product*.

Definition II.4.1. The (*singular*) *cup product*

$$\cup : H^p(X; R) \otimes_R H^q(X; R) \longrightarrow H^{p+q}(X; R)$$

is the composite

$$H^p(X; R) \otimes_R H^q(X; R) \xrightarrow{\times} H^{p+q}(X \times X; R) \xrightarrow{\Delta^*} H^{p+q}(X; R)$$

with $\Delta : X \rightarrow X \times X$, $x \mapsto (x, x)$.

Remark. As we saw at the end of II.2, for a cell complex X , the *cellular cohomology*

$$H_{\text{CW}}^*(X; R) := H^*(\text{Hom}_R(C_*^{\text{CW}}(X; R), R))$$

also has a *cellular cup product*.

The definition of AW_* gives the following explicit description of the singular cup product of a p -cocycle φ and a q -cocycle ψ :

$$\varphi \cup \psi = \left(\sigma \mapsto \varphi(\sigma|_{[v_0, \dots, v_p]}) \cdot \psi(\sigma|_{[v_p, \dots, v_{p+q}]} \right).$$

Let

$$\begin{aligned} \varphi &\in \text{Hom}_{\mathbb{Z}}(C_p(X, A), R) = \{\chi \in \text{Hom}_{\mathbb{Z}}(C_pX, R) : \chi|_{C_pA} = 0\}, \\ \psi &\in \text{Hom}_{\mathbb{Z}}(C_q(X, B), R) = \{\chi \in \text{Hom}_{\mathbb{Z}}(C_qX, R) : \chi|_{C_qB} = 0\}. \end{aligned}$$

It follows that

$$\varphi \cup \psi \in \text{Hom}_{\mathbb{Z}}(C_*X / (C_*A + C_*B), R).$$

If $(A \cup B; A, B)$ is excisive, then

$$H^*(\text{Hom}_{\mathbb{Z}}(C_*X / (C_*A + C_*B), R)) \cong H^*(\text{Hom}_{\mathbb{Z}}(C_*(X, A \cup B), R))$$

and we obtain

$$\cup : H^p(X, A; R) \otimes_R H^q(X, B; R) \longrightarrow H^{p+q}(X, A \cup B; R).$$

Theorem II.4.2. The cup product is a multiplicative structure on singular cohomology.

Proof. For simplicity, consider only the case $A = B = \emptyset$. Let $1_X \in H^0(X)$ be represented by the constant cocycle $C_0(X) \rightarrow R$, $\sigma^0 \mapsto 1$. Then 4. is clear. 1., 3., and 5. follow by straightforward calculations.

2. (Graded commutativity.) Consider the natural chain map (involution)

$$\begin{aligned}\rho_*: C_*X &\longrightarrow C_*X, \\ \sigma^n &\longmapsto (-1)^{\frac{n(n+1)}{2}} \cdot \sigma|_{[v_n, \dots, v_0]}.\end{aligned}$$

The natural transformations $\rho_*: C_*(\cdot) \rightarrow C_*(\cdot)$ form a chain map of $\mathbb{Z}\text{Top}$ -chain complexes with $\rho_0 = \text{id}_{C_0(\cdot)}$. By [Theorem I.10.6](#), $\rho_* \simeq \text{id}_*$. Moreover,

$$\rho^* \varphi \cup \rho^* \psi = (-1)^{|\varphi| \cdot |\psi|} \cdot \rho^*(\psi \cup \varphi),$$

because

$$\begin{aligned}(\rho^* \varphi \cup \rho^* \psi)(\sigma^{p+q}) &= (-1)^{\frac{p(p+1)}{2}} \varphi(\sigma|_{[v_p, \dots, v_0]}) \cdot (-1)^{\frac{q(q+1)}{2}} \psi(\sigma|_{[v_{p+q}, \dots, v_p]}) \\ &= (-1)^{\frac{p^2+q^2+p+q+(p+q)(p+q+1)}{2}} (\rho^*(\psi \cup \varphi))(\sigma^{p+q})\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}(p^2 + q^2 + p + q + (p+q)(p+q+1)) &= \frac{1}{2}(2p^2 + 2q^2 + 2p + 2q + 2pq) \\ &= p^2 + p + q^2 + q + pq \equiv pq \pmod{2}.\end{aligned}$$

□

Proposition II.4.3. For $\varphi \in H^p(X; R)$ and $\psi \in H^q(Y; R)$,

$$\varphi \times \psi = \text{pr}_X^* \varphi \cup \text{pr}_Y^* \psi.$$

Remark. This also holds for $\varphi \in H^p(X, A; R)$ and $\psi \in H^q(Y, B; R)$.

Proof.

$$\begin{array}{ccccc} & H^*(X; R) \otimes_R H^*(Y; R) & & & \\ \text{pr}_X^* \otimes \text{pr}_Y^* \swarrow & & \searrow \times & & \\ H^*(X \times Y; R) & & & & H^*(X \times Y; R) \\ \otimes_R H^*(X \times Y; R) \swarrow \times & & & & \searrow (\text{pr}_X \times \text{pr}_Y)^* \\ & H^*((X \times Y) \times (X \times Y); R) & & & \\ \cup \swarrow & \downarrow \Delta^* & & & \searrow \text{id} \\ & H^*(X \times Y; R) & & & \end{array}$$

The upper square commutes by naturality of $\times$, the left triangle by the definition of $\cup$, and the right triangle because $(\text{pr}_X \times \text{pr}_Y) \circ \Delta = \text{id}$. □

Proposition II.4.4. The cross product

$$H^*(X; R) \otimes_R H^*(Y; R) \xrightarrow{\times} H^*(X \times Y; R)$$

is a homomorphism of graded R -algebras.

Proof. We must show:

$$\times((a \otimes b) \cdot (a' \otimes b')) = \times(a \otimes b) \cup \times(a' \otimes b').$$

$$\text{Left-hand side} = (-1)^{|b||a'|}(a \cup a') \times (b \cup b').$$

$$\text{Right-hand side} = (a \times b) \cup (a' \times b')$$

$$\begin{aligned} &\stackrel{\text{II.4.3}}{=} (\text{pr}_X^* a \cup \text{pr}_Y^* b) \cup (\text{pr}_X^* a' \cup \text{pr}_Y^* b') \\ &= (-1)^{|b||a'|}(\text{pr}_X^* a \cup \text{pr}_X^* a') \cup (\text{pr}_Y^* b \cup \text{pr}_Y^* b') \\ &= (-1)^{|b||a'|} \text{pr}_X^*(a \cup a') \cup \text{pr}_Y^*(b \cup b') \\ &\stackrel{\text{II.4.3}}{=} (-1)^{|b||a'|}(a \cup a') \times (b \cup b'). \end{aligned}$$

□

Theorem II.4.5. (Cohomological Künneth theorem for spaces) Let R be a principal ideal domain, and let (X, A) and (Y, B) be closed cofibrations such that $H_p(X, A)$ is finitely generated for every p . Then, for every $n \geq 0$, there is a natural short exact sequence

$$\begin{aligned} 0 &\longrightarrow \bigoplus_{p+q=n} H^p(X, A; R) \otimes_R H^q(Y, B; R) \\ &\xrightarrow{\times} H^n(X \times Y, X \times B \cup A \times Y; R) \\ &\longrightarrow \bigoplus_{p+q=n+1} \text{Tor}_1^R(H^p(X, A; R), H^q(Y, B; R)) \longrightarrow 0, \end{aligned}$$

which splits (not naturally).

Proof. The general case is proved as in homology ([Theorem I.9.6](#)). If R is a field, the cohomological theorem follows from the homological one:

$$\begin{array}{ccc} \bigoplus_{p+q=n} H^p(X; R) \otimes_R H^q(Y; R) & \xrightarrow{\times} & H^n(X \times Y; R) \\ \downarrow \cong \text{universal coefficient theorem} & & \downarrow \\ \bigoplus_{p+q=n} \text{Hom}_R(H_p(X; R), R) \otimes_R \text{Hom}_R(H_q(Y; R), R) & & \\ \downarrow \cong \alpha & \text{universal coefficient theorem} & \cong \\ \bigoplus_{p+q=n} \text{Hom}_R(H_p(X; R) \otimes_R H_q(Y; R), R) & & \\ \downarrow \cong & & \downarrow \\ \text{Hom}_R\left(\bigoplus_{p+q=n} H_p(X; R) \otimes_R H_q(Y; R), R\right) & \xleftarrow[\text{Künneth}]{\cong} & \text{Hom}_R(H^n(X \times Y; R), R) \end{array}$$

where $\alpha(\varphi \otimes \psi)(x \otimes y) = \varphi(x) \cdot \psi(y)$. Since $\dim_R H_p(X; R) < \infty$, α is an isomorphism. □

II.5. Computing Cup Products

Let $H^* = H_{\text{sing}}^*(\cdot; \mathbb{Z})$.

Observation. If

$$H^k(X) \cong \begin{cases} \mathbb{Z} & k = 0, 1, \\ 0 & \text{otherwise,} \end{cases}$$

then $H^*(X) \cong \Lambda_{\mathbb{Z}}[\alpha]$ with $|\alpha| = 1$, since $1_X \cup \alpha = \alpha \cup 1_X = \alpha$ and $\alpha \cup \alpha \in H^2(X) = 0$.

Example. By the universal coefficient theorem (II.1.5),

$$H^*(S^1) \cong \text{Hom}_{\mathbb{Z}}(H_*(S^1), \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 0, 1, \\ 0 & \text{otherwise,} \end{cases}$$

since $\text{Ext}_{\mathbb{Z}}^1(H_n(S^1), \mathbb{Z}) = 0$ for all $n \geq 1$, because $H_n(S^1)$ is always projective. Thus

$$H^*(S^1) \cong \Lambda[\alpha] \quad \text{with } |\alpha| = 1.$$

Theorem II.5.1. We have $H^*(T^n) \cong \Lambda[\alpha_1, \dots, \alpha_n]$, $|\alpha_i| = 1$.

Proof. Induction on n . Base case: $\checkmark$. The induction hypothesis gives

$$\begin{aligned} H^*(T^n) &\cong H^*(T^{n-1} \times S^1) \xleftarrow[\cong (\text{II.4.5})]{\times} H^*(T^{n-1}) \otimes_{\mathbb{Z}} H^*(S^1) \\ &\cong \Lambda[\alpha_1, \dots, \alpha_{n-1}] \otimes \Lambda[\alpha_n] \cong \Lambda[\alpha_1, \dots, \alpha_n]. \end{aligned}$$

□

Corollary II.5.2. If $T^2 = U_1 \cup U_2$ with U_1, U_2 open, then U_1 or U_2 is not contractible.

Proof. Otherwise, $\alpha_1 \cup \alpha_2 = 0$ by Problem 2 on Sheet 8.

□

As we already know,

$$b_k(T^n) = \text{rank}_{\mathbb{Z}} H_k(T^n) = \text{rank}_{\mathbb{Z}} H^k(T^n) = \text{rank}_{\mathbb{Z}} \Lambda^k[\alpha_1, \dots, \alpha_n] = \binom{n}{k}.$$

Example. Construct a space X with

$$H^*(X) \cong \Lambda[\alpha_1, \alpha_2, \alpha_3] / (\alpha_1 \cdot \alpha_2), \quad |\alpha_i| = 1.$$

In particular, we require

$$\begin{aligned} H^0(X) &\cong \mathbb{Z} \cdot 1_X, & H^1(X) &\cong \mathbb{Z}[\{\alpha_1, \alpha_2, \alpha_3\}], \\ H^2(X) &\cong \mathbb{Z}[\{\alpha_1\alpha_3, \alpha_2\alpha_3\}], & H^3(X) &\cong 0. \end{aligned}$$

Consider T^3 with its standard cell structure. For a subcomplex $i: X \hookrightarrow T^3$, the map

$$H^*(i): H^*(T^3) \longrightarrow H^*(X)$$

is surjective, since $C_*^{\text{CW}}(X) \xrightarrow{i_*} C_*^{\text{CW}}(T^3)$ is split injective, so

$$\text{Hom}_{\mathbb{Z}}(C_*^{\text{CW}}(T^3), \mathbb{Z}) \xrightarrow{i^*} \text{Hom}_{\mathbb{Z}}(C_*^{\text{CW}}(X), \mathbb{Z})$$

is split surjective, and $C_*^{\text{CW}}(T^3)$, and hence also $\text{Hom}_{\mathbb{Z}}(C_*^{\text{CW}}(T^3), \mathbb{Z})$, has trivial differentials. Thus

$$H^*(X) \cong H^*(T^3) / \ker H^*(i).$$

The proof of [Theorem II.5.1](#) shows that under $H^*(T^3) \cong H_{\text{CW}}^*(T^3)$ (use the universal coefficient theorem and the five lemma), the ring generators α_i correspond to the dual 1-cells e_i^* , and the cup product $\alpha_i \cdot \alpha_j$ corresponds to the cell product $(e_i \times e_j)^*$. Thus the subcomplex $X \subseteq T^3$ obtained by removing the cells $e_1 \times e_2$ and $e_1 \times e_2 \times e_3$ has the desired property.

Warning. Such *realization problems* are difficult in general!

Let X and Y be path-connected CW complexes. Consider $X \vee Y$ with $p_X: X \vee Y \rightarrow X$ and $p_Y: X \vee Y \rightarrow Y$. We have

$$H^k(X \vee Y) \cong \begin{cases} \mathbb{Z} & k = 0, \\ H^k(X) \oplus H^k(Y) & k \geq 1. \end{cases}$$

Proposition II.5.3. Let $\alpha \in H^k(X)$ and $\beta \in H^\ell(Y)$, $k, \ell \geq 1$. Then

$$p_X^* \alpha \cup p_Y^* \beta = 0.$$

Proof. Let $i_X: X \rightarrow X \vee Y$ and $i_Y: Y \rightarrow X \vee Y$ be the inclusions. Then

$$\begin{aligned} i_X^*(p_X^* \alpha \cup p_Y^* \beta) &= (p_X \circ i_X)^* \alpha \cup (p_Y \circ i_X)^* \beta \\ &= \alpha \cup 0 = 0, \end{aligned}$$

since

$$\begin{array}{ccc} & \{*\} & \\ X & \xrightarrow{p_Y \circ i_X} & Y \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} & H^*(\{*\}) & \\ H^*(X) & \xleftarrow{(p_Y \circ i_X)^*} & H^*(Y) \end{array},$$

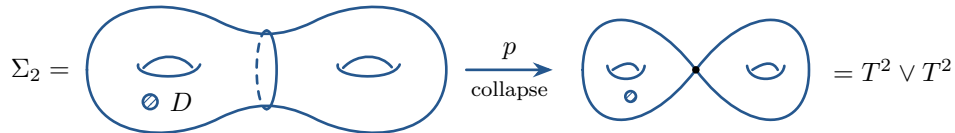
and $H^*(\{*\}) = 0$ for $* \geq 1$. Similarly, $i_Y^*(p_X^* \alpha \cup p_Y^* \beta) = 0 \cup \beta = 0$. Since

$$i_X^* \oplus i_Y^*: H^k(X \vee Y) \xrightarrow{\cong} H^k(X) \oplus H^k(Y),$$

the claim follows. $\square$

Example. The spaces $X = T^2$ and $Y = S^1 \vee S^1 \vee S^2$ have isomorphic homology and cohomology modules, but different cohomology rings.

Example.



Clearly, $H_0(p)$ and $H_1(p)$ are isomorphisms.

$$H_2(p): H_2(\Sigma_2) \cong \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \cong H_2(T^2 \vee T^2)$$

is the diagonal map $x \mapsto x \oplus x$: if $p_1, p_2: T^2 \vee T^2 \rightarrow T^2$ are the two projections, then

$$\begin{array}{ccc}
H_2(\Sigma_2) & \xrightarrow{(p_1 \circ p)_*} & H_2(T^2) \\
\downarrow \cong \text{long exact sequence } (*) & & \downarrow \cong \text{long exact sequence (similarly)} \\
H_2(\Sigma_2, \Sigma_2 \setminus D) & \longrightarrow & H_2(T^2, T^2 \setminus D) \\
\downarrow \cong & & \downarrow \cong \\
\tilde{H}_2(\Sigma_2/(\Sigma_2 \setminus D)) & \xrightarrow{\cong} & \tilde{H}_2(T^2/(T^2 \setminus D)).
\end{array}$$

$$(*) \quad H_2(\Sigma_2) \xrightarrow{\cong} H_2(\Sigma_2, \Sigma_2 \setminus D) \xrightarrow{0} H_1(\Sigma_2 \setminus D) \xrightarrow{\cong} H_1(\Sigma_2) \longrightarrow 0,$$

since $0 \rightarrow \ker f \rightarrow \mathbb{Z}^n \xrightarrow{f} \mathbb{Z}^n \rightarrow 0$ splits; $\ker f$ is free, so $\ker f = 0$. The universal coefficient theorem gives

$$\begin{array}{ccc}
H^2(T^2 \vee T^2) & \xrightarrow{p^*} & H^2(\Sigma_2) \\
\wr \parallel & & \wr \parallel \\
\text{Hom}_{\mathbb{Z}}(H_2(T^2 \vee T^2), \mathbb{Z}) & \xrightarrow{H_2(p)^*} & \text{Hom}_{\mathbb{Z}}(H_2(\Sigma_2), \mathbb{Z}) \\
\wr \parallel & & \wr \parallel \\
\mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{+} & \mathbb{Z}.
\end{array}$$

Consider

$$\begin{array}{ccccc}
H^1(\Sigma_2) \otimes_{\mathbb{Z}} H^1(\Sigma_2) & \xrightarrow{\cup} & H^2(\Sigma_2) & & \\
\uparrow p^* \otimes p^* \cong & & \uparrow p^* + & & \\
H^1(T^2 \vee T^2) \otimes_{\mathbb{Z}} H^1(T^2 \vee T^2) & \xrightarrow{\cup} & H^2(T^2 \vee T^2) & & \\
\uparrow \text{pr}_i^* \otimes \text{pr}_i^* & & \uparrow \text{pr}_i^* & & \\
H^1(T^2) \otimes_{\mathbb{Z}} H^1(T^2) & \xrightarrow{\cup} & H^2(T^2) & &
\end{array}$$

Writing (by the universal coefficient theorem)

$$\begin{aligned}
H^0(\Sigma_2) &\cong \mathbb{Z} \cdot 1_{\Sigma_2}, \\
H^1(\Sigma_2) &\cong \mathbb{Z} \cdot \alpha_1 \oplus \mathbb{Z} \alpha_2 \oplus \mathbb{Z} \beta_1 \oplus \mathbb{Z} \beta_2, \\
H^2(\Sigma_2) &\cong \mathbb{Z} \gamma,
\end{aligned}$$

we obtain

$$\alpha_1 \cup \alpha_2 = \gamma, \quad \beta_1 \cup \beta_2 = \gamma, \quad \alpha_i \cup \beta_j = 0, \quad \alpha_i^2 = \beta_i^2 = 0.$$

Remark. The generalization to Σ_g is clear.

Theorem II.5.4. $H^*(\mathbb{RP}^n; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[\alpha]/(\alpha^{n+1})$, $|\alpha| = 1$.

Proof. Write $H^*(\cdot) = H^*(\cdot; \mathbb{Z}/2)$, $P^n = \mathbb{RP}^n$, $\otimes = \otimes_{\mathbb{Z}/2}$. By *Topology I*, IV.3.3.3,

$$C_*^{\text{CW}}(P^n; \mathbb{Z}/2) = \cdots \longrightarrow 0 \longrightarrow \mathbb{Z}/2 \xrightarrow[n]{0} \mathbb{Z}/2 \xrightarrow[n-1]{0} \cdots \xrightarrow[0]{0} \mathbb{Z}/2 \longrightarrow 0,$$

hence

$$C_{\text{CW}}^*(P^n; \mathbb{Z}/2) = \cdots \longleftarrow 0 \longleftarrow \underset{n}{\mathbb{Z}/2} \overset{0}{\longleftarrow} \underset{n-1}{\mathbb{Z}/2} \overset{0}{\longleftarrow} \cdots \overset{0}{\longleftarrow} \underset{0}{\mathbb{Z}/2} \longleftarrow 0,$$

that is,

$$H^k(P^n) \cong \begin{cases} \mathbb{Z}/2 & 0 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the k -skeleton satisfies $(P^n)^k = P^k$, so $H^k(P^{n-1} \hookrightarrow P^n)$ is an isomorphism for $k \leq n-1$. It remains to show that for $\langle \alpha_k \rangle \cong H^k(P^n)$, we have $\alpha_1 \cup \alpha_{n-1} = \alpha_n$. Set $\ell = n - k$. More generally, we show that $\alpha_k \cup \alpha_\ell = \alpha_n$. We have embeddings

$$\begin{aligned} P^k &\subseteq P^n, & [x_0 : \cdots : x_k] &\longmapsto [x_0 : \cdots : x_k : 0 : \cdots : 0], \\ P^\ell &\subseteq P^n, & [x_0 : \cdots : x_\ell] &\longmapsto [0 : \cdots : 0 : x_0 : \cdots : x_\ell], \end{aligned}$$

that is,

$$P^k \cap P^\ell = \{[0 : \cdots : 0 : 1 : 0 : \cdots : 0]_n\}.$$

Consider the affine chart

$$\begin{aligned} \varphi: U = \{[x_0 : \cdots : x_n] \mid x_k \neq 0\} &\xrightarrow{\cong} \mathbb{R}^n, \\ [x_0 : \cdots : x_n] &\longmapsto \left(\frac{x_0}{x_k}, \dots, \frac{x_{k-1}}{x_k}, \frac{x_{k+1}}{x_k}, \dots, \frac{x_n}{x_k} \right). \end{aligned}$$

Let $p := [0 : \cdots : 0 : 1 : 0 : \cdots : 0] = \varphi^{-1}(0)$. We obtain the diagram

$$\begin{array}{ccc} H^k(P^n) \otimes H^\ell(P^n) & \xrightarrow{\cup} & H^n(P^n) \\ \uparrow & & \uparrow \\ H^k(P^n, P^n \setminus P^\ell) \otimes H^\ell(P^n, P^n \setminus P^k) & \xrightarrow{\cup} & H^n(P^n, P^n \setminus \{p\}) \\ \downarrow (\varphi^{-1})^* & & \downarrow (\varphi^{-1})^* \\ H^k(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^\ell) \otimes H^\ell(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^k) & \xrightarrow{\cup} & H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \\ \uparrow \text{pr}_1^* \otimes \text{pr}_2^* \cong & \nearrow \times & \text{II.4.3 (relative)} \\ H^k(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\}) \otimes H^\ell(\mathbb{R}^\ell, \mathbb{R}^\ell \setminus \{0\}) & & \end{array}$$

Here $\text{pr}_1: \mathbb{R}^k \times \mathbb{R}^\ell \rightarrow \mathbb{R}^k$ and $\text{pr}_2: \mathbb{R}^k \times \mathbb{R}^\ell \rightarrow \mathbb{R}^\ell$ are the projections. Moreover,

$$\begin{array}{ccc} H^k(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\}) \otimes H^\ell(\mathbb{R}^\ell, \mathbb{R}^\ell \setminus \{0\}) & \xrightarrow{\times} & H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \\ \downarrow \cong & & \downarrow \cong \\ H^k(I^k, \partial I^k) \otimes H^\ell(I^\ell, \partial I^\ell) & \xrightarrow{\times} & H^n(I^n, \partial I^n) \\ \downarrow \cong & & \downarrow \cong \\ H^k(T^k, T^k \setminus D^k) \otimes H^\ell(T^\ell, T^\ell \setminus D^\ell) & \xrightarrow{\times} & H^n(T^n, T^n \setminus D^n) \\ \downarrow \cong \text{long exact sequence} & & \downarrow \cong \text{long exact sequence} \\ H^k(T^k) \otimes H^\ell(T^\ell) & \xrightarrow[\cong \text{ (II.4.5)}]{\times} & H^n(T^n) \end{array}$$

To show that the highlighted arrows are isomorphisms, consider

$$\begin{array}{ccccccc}
H^k(P^n) & \xleftarrow{\cong} & H^k(P^n, P^{k-1}) & \xleftarrow{(*)} & H^k(P^n, P^n \setminus P^\ell) & \longrightarrow & H^k(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^\ell) \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \cong \\
H^k(P^k) & \xleftarrow{\cong} & H^k(P^k, P^{k-1}) & \xleftarrow{\cong} & H^k(P^k, P^k \setminus \{p\}) & \xrightarrow[\text{excision}]{\cong} & H^k(\mathbb{R}^k, \mathbb{R}^k \setminus \{0\}) \\
& & \text{deformation retract} & & & &
\end{array}$$

The arrow $(*)$ (and hence all the arrows) is an isomorphism, since P^{k-1} is a deformation retract of $P^n \setminus P^\ell$:

$$[x_0 : \cdots : x_{k-1} : (1-t) \cdot x_k : \cdots : (1-t) \cdot x_n].$$

□

Theorem II.5.5.

- $H^*(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$, $|\alpha| = 2$.
- $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[\alpha]$, $|\alpha| = 1$.
- $H^*(\mathbb{CP}^\infty; \mathbb{Z}) \cong \mathbb{Z}[\alpha]$, $|\alpha| = 2$.

Proof. The argument for $H^*(\mathbb{CP}^n; \mathbb{Z})$ is similar to the one above.

$$H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) = \bigoplus_{k \geq 0} H^k(\mathbb{RP}^\infty; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[\alpha]$$

is clear as an isomorphism of graded $\mathbb{Z}/2$ -vector spaces. For $\alpha, \beta \in H^*(\mathbb{RP}^\infty; \mathbb{Z}/2)$, we have

$$\alpha, \beta, \alpha \cup \beta \in \bigoplus_{k=0}^N H^k(\mathbb{RP}^\infty; \mathbb{Z}/2) \cong \bigoplus_{k=0}^N H^k(\mathbb{RP}^N; \mathbb{Z}/2)$$

for $N \gg 0$. The argument for $H^*(\mathbb{CP}^\infty; \mathbb{Z})$ is the same.

□

We also compute $H^*(\mathbb{RP}^\infty; \mathbb{Z})$ by applying the covariant functor

$$H^*(\mathbb{RP}^\infty; \cdot) : \text{Rings} \longrightarrow \mathbb{Z}\text{-Alg}$$

to $\mathbb{Z} \rightarrow \mathbb{Z}/2$. At the level of cochain complexes:

$$\begin{array}{ccccccc}
\cdots & \xleftarrow{\cdot 2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{\cdot 2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{\quad} & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xleftarrow{0} & \mathbb{Z}/2 & \xleftarrow{0} & \mathbb{Z}/2 & \xleftarrow{0} & \mathbb{Z}/2 & \xleftarrow{0} & \mathbb{Z}/2 & \xleftarrow{\quad} & 0
\end{array}$$

It follows that $H^*(\mathbb{RP}^\infty; \mathbb{Z}) \rightarrow H^*(\mathbb{RP}^\infty; \mathbb{Z}/2)$ is injective in positive degrees, with image $\bigoplus_{k \geq 0} H^{2k}(\mathbb{RP}^\infty; \mathbb{Z}/2)$, that is,

$$H^*(\mathbb{RP}^\infty; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(2\alpha) \quad \text{with } |\alpha| = 2.$$

Since $H^*(\mathbb{RP}^\infty; \mathbb{Z}) \xrightarrow{\cong} H^*(\mathbb{RP}^{2n}; \mathbb{Z})$ for $* \leq 2n$, we obtain

$$H^*(\mathbb{RP}^{2n}; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(2\alpha, \alpha^{n+1}), \quad |\alpha| = 2.$$

For $\mathbb{RP}^{2n+1}$, the cellular cochain complex ends in

$$\begin{array}{ccccccc}
 0 & \longleftarrow & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{\cdot 2} & \dots \xleftarrow{0} \mathbb{Z} \longleftarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longleftarrow & \mathbb{Z}/2 & \xleftarrow{0} & \mathbb{Z}/2 & \xleftarrow{0} & \dots \xleftarrow{0} \mathbb{Z}/2 \longleftarrow 0 \\
 & & 2n+1 & & & &
 \end{array}$$

Therefore,

$$H^*(\mathbb{RP}^{2n+1}; \mathbb{Z}) \cong \mathbb{Z}[\alpha, \beta] / (2\alpha, \alpha^{n+1}, \alpha\beta, \beta^2), \quad |\alpha| = 2, \quad |\beta| = 2n + 1.$$

Observation.

$$H^*(\mathbb{RP}^{2n+1}; \mathbb{Z}) \cong H^*(\mathbb{RP}^{2n} \vee S^{2n+1}; \mathbb{Z})$$

as $\mathbb{Z}$ -algebras, but for $n \geq 1$, we have $\mathbb{RP}^{2n+1} \not\cong \mathbb{RP}^{2n} \vee S^{2n+1}$, for example because

$$H^*(\mathbb{RP}^{2n+1}; \mathbb{Z}/2) \not\cong H^*(\mathbb{RP}^{2n} \vee S^{2n+1}; \mathbb{Z}/2),$$

since for $\langle \alpha \rangle \cong H^1(\dots; \mathbb{Z}/2)$, we have $\alpha^{2n+1} \neq 0$ on the left and $= 0$ on the right.

II.6. Applications of the Cup Product

Recall. The *Hopf map* is

$$\begin{array}{ccc}
 S^3 & \xrightarrow{\eta} & S^2 \\
 \downarrow & & \wr \\
 \mathbb{C} \times \mathbb{C} & \longrightarrow & \mathbb{C} \cup \{\infty\}
 \end{array} \quad (\text{stereographic projection}).$$

$$(z_1, z_2) \longmapsto z_1/z_2$$

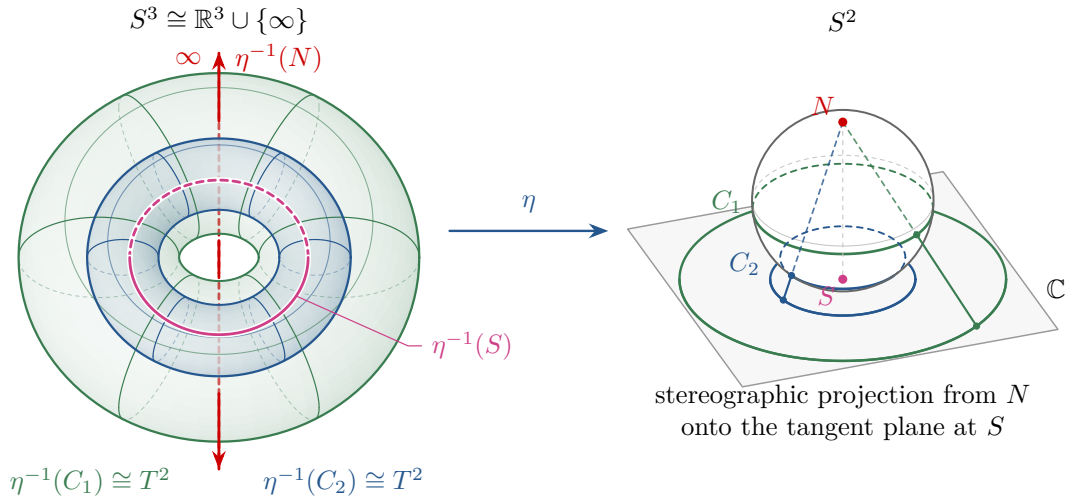
Use polar coordinates to visualize it:

$$\eta(r_1 e^{i\varphi_1}, r_2 e^{i\varphi_2}) = \frac{r_1}{r_2} e^{i(\varphi_1 - \varphi_2)},$$

that is,

$$\eta^{-1}(r \cdot S^1) = \{(r_1 e^{i\varphi_1}, r_2 e^{i\varphi_2}) \mid r_1^2 + r_2^2 = 1, r_1/r_2 = r\},$$

so r_1, r_2 are fixed, and φ_1, φ_2 are arbitrary. Hence the preimages of circles under η are tori.



Note that the fibers of η are circles ($\varphi_1 - \varphi_2$ is fixed). Since η^{-1} (north pole) and η^{-1} (south pole) are linked, so are any other two fibers! (Proof: Move the base points continuously to the north and south poles.)

Theorem II.6.1. The Hopf map is not nullhomotopic.

Lemma II.6.2. If $f: X \rightarrow Y$ is nullhomotopic, then $j: Y \rightarrow C_f$ admits a retraction.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow i & & \downarrow j \\ CX & \xrightarrow{\bar{f}} & C_f \end{array} \quad \begin{array}{c} \lrcorner \\ \nearrow r \end{array}$$

Proof. Let $h: X \times I \rightarrow Y$ satisfy $h_0 = f$, with h_1 constant at y_0 . Define

$$r: C_f \longrightarrow Y, \quad z \longmapsto \begin{cases} z, & z \in j(Y), \\ h(x, t), & z = \bar{f}(x, t) \in \bar{f}(CX). \end{cases}$$

r is continuous and $r \circ j = \text{id}_Y$. □

Proof of Theorem II.6.1. We have

$$\begin{array}{ccc} S^3 & \xrightarrow{\eta} & \mathbb{CP}^1 \cong S^2 \\ \downarrow & & \downarrow j \\ C(S^3) \cong D^4 & \longrightarrow & \mathbb{CP}^2 \cong C_\eta. \end{array} \quad \lrcorner$$

It remains to show that $\mathbb{CP}^1 \xrightarrow{j} \mathbb{CP}^2$ does not admit a retraction. Suppose there were a retraction $r: \mathbb{CP}^2 \rightarrow \mathbb{CP}^1$ with $r \circ j = \text{id}_{\mathbb{CP}^1}$. Then

$$\begin{array}{ccc} H^*(\mathbb{CP}^1; \mathbb{Z}) & \xrightarrow{r^*} & H^*(\mathbb{CP}^2; \mathbb{Z}) \\ \wr \parallel & & \wr \parallel \\ \mathbb{Z}[\alpha]/(\alpha^2) & & \mathbb{Z}[\beta]/(\beta^3) \end{array}$$

with $|\alpha| = |\beta| = 2$ is injective, so $r^*(\alpha) = k \cdot \beta$ for some $k \in \mathbb{Z} \setminus \{0\}$, since r^* preserves degrees. Thus

$$0 = r^*(0) = r^*(\alpha^2) = (r^*(\alpha))^2 = k^2 \cdot \beta^2 \neq 0,$$

a contradiction. □

Similarly, the Hopf maps

$$\nu: S^7 \longrightarrow S^4 \quad \text{and} \quad \sigma: S^{15} \longrightarrow S^8$$

arise from the quaternion and octonion algebras:

The *quaternion algebra* is

$$\mathbb{H} = \langle 1, i, j, k \rangle_{\mathbb{R}} \quad \text{with} \quad \underbrace{i^2 = j^2 = k^2 = ijk = -1}_{\text{Broome Bridge, Dublin}}.$$

It follows that $ij = -ijk^2 = k$, $jk = -i^2jk = i$, $ki = k(k(-j)) = j$; $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. *Quaternionic conjugation* is given by

$$a + bi + cj + dk \mapsto a - bi - cj - dk.$$

The *octonion algebra* (also called the *Cayley numbers*) is $\mathbb{O} = \mathbb{H} \times \mathbb{H}$ with multiplication (not associative!)

$$\begin{aligned} \mathbb{O} \times \mathbb{O} &\longrightarrow \mathbb{O}, \\ (v_1, v_2) \cdot (w_1, w_2) &= (v_1 \cdot w_1 - \bar{w}_2 \cdot v_2, v_2 \cdot \bar{w}_1 + w_2 \cdot v_1). \end{aligned}$$

The algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are *division algebras* in the following sense.

Definition II.6.3. A (finite-dimensional real) *division algebra* is a pair $(\mathbb{R}^n, \cdot)$ with $n \geq 1$, where $\cdot : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies:

- 1.) $a \cdot (b + c) = a \cdot b + a \cdot c$, $(b + c) \cdot a = b \cdot a + c \cdot a$, $a, b, c \in \mathbb{R}^n$.
- 2.) $r \cdot (a \cdot b) = (r \cdot a) \cdot b = a \cdot (r \cdot b)$, $a, b \in \mathbb{R}^n$, $r \in \mathbb{R}$.
- 3.) There is an $e \in \mathbb{R}^n$ such that $a \cdot e = e \cdot a = a$, $a \in \mathbb{R}^n$.
- 4.) $x \mapsto a \cdot x$, $x \mapsto x \cdot a$ define bijections $\mathbb{R}^n \rightarrow \mathbb{R}^n$ for all $a \in \mathbb{R}^n \setminus \{0\}$.

(Conditions 1 and 2 mean that $\cdot$ is $\mathbb{R}$ -bilinear.)

Definition II.6.4. The *quaternionic* and *octonionic Hopf maps* are

$$\begin{array}{ccc} S^7 & \xrightarrow{\nu} & S^4 \\ \downarrow & \wr & \\ \mathbb{H} \times \mathbb{H} & & \mathbb{H} \cup \{\infty\} \end{array} \quad \text{and} \quad \begin{array}{ccc} S^{15} & \xrightarrow{\sigma} & S^8 \\ \downarrow & \wr & \\ \mathbb{O} \times \mathbb{O} & & \mathbb{O} \cup \{\infty\} \end{array}$$

defined by $(z_1, z_2) \mapsto z_1 \cdot z_2^{-1}$.

(Note that $\mathbb{H}$ and $\mathbb{O}$ are normed, with $\|x\|^2 = x \cdot \bar{x} = \bar{x} \cdot x$, so for $x^{-1} := \bar{x}/\|x\|^2$, we have $x \cdot x^{-1} = x^{-1} \cdot x = 1$. Alternatively, for all $a, b \in \mathbb{O}$, $\langle a, b \rangle \subseteq \mathbb{O}$ is associative, and $a^{-1} \cdot a = 1$ implies $a \cdot a^{-1} = a \cdot a^{-1} \cdot a \cdot a^{-1}$. Hence, by condition 1,

$$a \cdot a^{-1} \cdot (1 - a \cdot a^{-1}) = 0,$$

so condition 4 implies that $a \cdot a^{-1} = 1$.)

As before, it follows that ν and σ are not nullhomotopic:

$$\begin{array}{ccc} S^7 & \xrightarrow{\nu} & \mathbb{HP}^1 \cong S^4 \\ \downarrow & \lrcorner & \downarrow \\ D^8 & \longrightarrow & \mathbb{HP}^2 \cong C_\nu \end{array} \quad \begin{array}{ccc} S^{15} & \xrightarrow{\sigma} & \mathbb{OP}^1 \cong S^8 \\ \downarrow & \lrcorner & \downarrow \\ D^{16} & \longrightarrow & \mathbb{OP}^2 \cong C_\sigma, \end{array}$$

$$\begin{aligned} H^*(\mathbb{HP}^2; \mathbb{Z}) &\cong \mathbb{Z}[\alpha]/(\alpha^3), & |\alpha| &= 4, \\ H^*(\mathbb{OP}^2; \mathbb{Z}) &\cong \mathbb{Z}[\alpha]/(\alpha^3), & |\alpha| &= 8. \end{aligned}$$

Corollary II.6.5. The homotopy groups $\pi_3(S^2; *)$, $\pi_7(S^4; *)$, and $\pi_{15}(S^8; *)$ are nontrivial.

Remark.

$$\begin{aligned} \langle \Sigma \eta \rangle &= \pi^1 = \operatorname{colim}_{k \rightarrow \infty} \pi_{k+1}(S^k) \cong \mathbb{Z}/2, \\ \langle \Sigma \nu \rangle &= \pi^3 = \operatorname{colim}_{k \rightarrow \infty} \pi_{k+3}(S^k) \cong \mathbb{Z}/24, \\ \langle \Sigma \sigma \rangle &= \pi^7 = \operatorname{colim}_{k \rightarrow \infty} \pi_{k+7}(S^k) \cong \mathbb{Z}/240 \quad (\text{difficult}). \end{aligned}$$

Warning. We cannot define $\mathbb{OP}^n$ for $n > 2$, since proving that $(z_1, \dots, z_n) \sim \lambda(z_1, \dots, z_n)$ for $\lambda \in \mathbb{O}^*$ defines an equivalence relation requires associativity.

Theorem II.6.6. (Hopf) If $(\mathbb{R}^n, \cdot)$ is a division algebra, then n is a power of 2.

Proof. The claim is clear for $n = 1$; now let $n > 1$. Define

$$g: S^{n-1} \times S^{n-1} \longrightarrow S^{n-1}, \quad (x, y) \longmapsto \frac{x \cdot y}{\|x \cdot y\|}.$$

We have $g(-x, y) = g(x, -y) = -g(x, y)$, so we obtain $\bar{g}: \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} \rightarrow \mathbb{RP}^{n-1}$ and hence

$$\begin{array}{ccc} H^*(\mathbb{RP}^{n-1}; \mathbb{Z}/2) & \xrightarrow{\bar{g}^*} & H^*(\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}; \mathbb{Z}/2) \\ \wr & & \wr \\ (\mathbb{Z}/2)[\gamma]/(\gamma^n) & & (\mathbb{Z}/2)[\alpha, \beta]/(\alpha^n, \beta^n) \end{array}$$

with $\alpha, \beta \in H^1(\mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}; \mathbb{Z}/2)$ given by

$$\alpha = \text{pr}_1^*(\gamma), \quad \beta = \text{pr}_2^*(\gamma)$$

for $\text{pr}_i: \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} \rightarrow \mathbb{RP}^{n-1}$. Write $\bar{g}^*(\gamma) = k\alpha + \ell\beta$. Let

$$j_i: \mathbb{RP}^{n-1} \longrightarrow \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1}, \quad x \longmapsto (x, e), (e, x) \quad (e \in (\mathbb{R}^n, \cdot)).$$

Then $\bar{g} \circ j_i = \text{id}_{\mathbb{RP}^{n-1}}$, so

$$\begin{aligned} \gamma &= j_i^* \circ \bar{g}^*(\gamma) = k j_i^*(\alpha) + \ell j_i^*(\beta) \\ &= k j_i^*(\text{pr}_1^*(\gamma)) + \ell j_i^*(\text{pr}_2^*(\gamma)) \\ &= \begin{cases} k\gamma & i = 1, \\ \ell\gamma & i = 2, \end{cases} \end{aligned}$$

that is, $k = \ell = 1$, so $\bar{g}^*(\gamma) = \alpha + \beta$. It follows that

$$\begin{aligned} 0 &= \bar{g}^*(0) = \bar{g}^*(\gamma^n) = (\bar{g}^*(\gamma))^n = (\alpha + \beta)^n \\ &= \sum_{k=0}^n \binom{n}{k} \alpha^k \beta^{n-k} = \sum_{k=1}^{n-1} \binom{n}{k} \alpha^k \beta^{n-k}. \end{aligned}$$

Thus $\binom{n}{k} \equiv 0 \pmod{2}$ for $k = 1, \dots, n-1$, or equivalently, $(1+x)^n = 1+x^n$ in $(\mathbb{Z}/2)[x]$. Write $n = n_1 + \dots + n_r$, where each n_i is a power of 2 and $n_1 < \dots < n_r$. Then

$$(1+x)^n = (1+x)^{n_1} \cdots (1+x)^{n_r} = (1+x^{n_1}) \cdots (1+x^{n_r}).$$

Considering the binary expansions of the n_i , we see that $(1+x^{n_1}) \cdots (1+x^{n_r})$ has exactly 2^r terms. Thus $(1+x)^n = 1+x^n$ implies $r = 1$. $\square$

III. Poincaré Duality

(R commutative, $H_* = H_*^{\text{sing}}$, $H^* = H_{\text{sing}}^*$.)

The simplest case: For a closed d -dimensional manifold M , we have

$$H^{d-k}(M; \mathbb{Z}/2) \cong H_k(M; \mathbb{Z}/2).$$

(For example, $M = T^d$: $\dim_{\mathbb{Z}/2} H^{d-k}(M; \mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H_k(M; \mathbb{Z}/2)$
 $\iff \binom{d}{d-k} = \binom{d}{k}$.) “Local property $\Rightarrow$ global consequence.”

III.1. The Cap Product

Let X be a topological space and R a commutative ring. We define

$$\begin{aligned} \cap: C_{k+\ell}^{\text{sing}}(X; R) \otimes_R C_{\text{sing}}^{\ell}(X; R) &\longrightarrow C_k^{\text{sing}}(X; R), \\ \sigma \otimes \varphi &\longmapsto \varphi(\sigma|_{[v_0, \dots, v_{\ell}]}) \cdot \sigma|_{[v_{\ell}, \dots, v_{k+\ell}]}. \end{aligned}$$

Lemma III.1.1. We have $\partial\sigma \cap \varphi = \sigma \cap \delta\varphi + (-1)^{\ell} \partial(\sigma \cap \varphi)$.

Proof.

$$\begin{aligned} \partial\sigma \cap \varphi &= \sum_{r=0}^{k+\ell} (-1)^r \sigma|_{[v_0, \dots, \widehat{v}_r, \dots, v_{k+\ell}]} \cap \varphi \\ &= \sum_{r=0}^{\ell+1} (-1)^r \varphi(\sigma|_{[v_0, \dots, \widehat{v}_r, \dots, v_{\ell+1}]}) \cdot \sigma|_{[v_{\ell+1}, \dots, v_{k+\ell}]} \\ &\quad - (-1)^{\ell+1} \varphi(\sigma|_{[v_0, \dots, v_{\ell}]}) \cdot \sigma|_{[v_{\ell+1}, \dots, v_{k+\ell}]} \\ &\quad + \sum_{r=\ell+1}^{\ell+k} (-1)^r \varphi(\sigma|_{[v_0, \dots, v_{\ell}]}) \cdot \sigma|_{[v_{\ell}, \dots, \widehat{v}_r, \dots, v_{k+\ell}]} \\ &= \sigma \cap \delta\varphi + \varphi(\sigma|_{[v_0, \dots, v_{\ell}]}) \sum_{r=\ell}^{\ell+k} (-1)^r \sigma|_{[v_{\ell}, \dots, \widehat{v}_r, \dots, v_{k+\ell}]} \\ &= \sigma \cap \delta\varphi + (-1)^{\ell} \partial(\sigma \cap \varphi). \end{aligned}$$

□

It follows that

$$\begin{aligned} \text{cycle} \cap \text{cocycle} &= \text{cycle}, \\ \text{boundary} \cap \text{cocycle} &= \text{boundary}, \\ \text{cycle} \cap \text{coboundary} &= \text{boundary}. \end{aligned}$$

Definition III.1.2. The induced homomorphism

$$\cap: H_{k+\ell}(X; R) \otimes_R H^\ell(X; R) \longrightarrow H_k(X; R)$$

is called the *cap product*. It has the relative versions

$$\begin{aligned} \cap: H_{k+\ell}(X, A; R) \otimes_R H^\ell(X; R) &\longrightarrow H_k(X, A; R), \\ \cap: H_{k+\ell}(X, A; R) \otimes_R H^\ell(X, A; R) &\longrightarrow H_k(X; R), \\ \cap: H_{k+\ell}(X, A \cup B; R) \otimes_R H^\ell(X, A; R) &\longrightarrow H_k(X, B; R), \end{aligned}$$

the last of these for an excisive triad $(A \cup B; A, B)$. Extending distributively, we obtain

$$H_*(X, A; R) \otimes H^*(X; R) \longrightarrow H_*(X, A; R)$$

where $H_* = \bigoplus_{p \geq 0} H_p$, $H^* = \bigoplus_{p \geq 0} H^p$.

Proposition III.1.3. The cap product makes $H_*(X, A; R)$ a right $H^*(X; R)$ -module.

Functoriality: Given $f: (X, A) \rightarrow (Y, B)$, $H_*(X, A) \rightarrow H_*(Y, B)$ is an $H^*(Y)$ -module homomorphism, where, for $\alpha \in H^*(Y)$ and $x \in H_*(X, A)$, we define $x \cdot \alpha := x \cap f^* \alpha$.

Thus, for $x \in H_*(X, A)$ and $\alpha \in H^*(Y)$, we have

$$f_*(x \cap f^* \alpha) = f_* x \cap \alpha.$$

Proof. Distributivity and $x \cap 1_X = x$ are clear.

We must show: For $\sigma \in C_{k+\ell}^{\text{sing}}(X, A)$, $\alpha \in C_{\text{sing}}^p(X)$, $\beta \in C_{\text{sing}}^q(X)$, $p + q = \ell$, we have $(\sigma \cap \alpha) \cap \beta = \sigma \cap (\alpha \cup \beta)$.

$$\begin{aligned} \sigma \cap \alpha &= \alpha(\sigma|_{[v_0, \dots, v_p]}) \cdot \sigma|_{[v_p, \dots, v_{k+\ell}]}, \\ (\sigma \cap \alpha) \cap \beta &= \alpha(\sigma|_{[v_0, \dots, v_p]}) \cdot \beta(\sigma|_{[v_p, \dots, v_\ell]}) \cdot \sigma|_{[v_\ell, \dots, v_{k+\ell}]}, \\ \sigma \cap (\alpha \cup \beta) &= (\alpha \cup \beta)(\sigma|_{[v_0, \dots, v_\ell]}) \cdot \sigma|_{[v_\ell, \dots, v_{k+\ell}]}. \end{aligned}$$

We show functoriality at the chain level: For $\sigma \in C_{k+\ell}^{\text{sing}}(X, A)$, $\alpha \in C_{\text{sing}}^\ell(Y)$, we have

$$\begin{aligned} f_*(\sigma \cap f^* \alpha) &= f_* \left(\alpha(f_* \sigma|_{[v_0, \dots, v_\ell]}) \cdot \sigma|_{[v_\ell, \dots, v_{k+\ell}]} \right) \\ &= \alpha(f_* \sigma|_{[v_0, \dots, v_\ell]}) \cdot f_* \sigma|_{[v_\ell, \dots, v_{k+\ell}]} \\ &= f_* \sigma \cap \alpha. \end{aligned}$$

□

By the tensor–Hom adjunction, the natural homomorphism

$$H^k(X, A; R) \longrightarrow \text{Hom}_R(H_k(X, A; R), R)$$

yields the homomorphism

$$H^k(X, A; R) \otimes_R H_k(X, A; R) \longrightarrow R.$$

Definition III.1.4. The associated bilinear form

$$\langle \cdot, \cdot \rangle: H^k(X, A; R) \times H_k(X, A; R) \longrightarrow R$$

is called the *Kronecker pairing*.

Proposition III.1.5. For $\alpha \in H^k(X, A)$, $\beta \in H^\ell(X, A)$, $x \in H_{k+\ell}(X, A)$, we have

$$\langle \alpha \cup \beta, x \rangle = \langle \beta, x \cap \alpha \rangle.$$

Proof. Check at the chain level (homework!). □

III.2. Orientations

Let M be a d -dimensional manifold.

Lemma III.2.1. For $x \in M$, we have

$$H_k(M, M \setminus \{x\}; R) \cong \begin{cases} R & k = d, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Choose a neighborhood $U \subseteq M$ of x with $U \cong \mathbb{R}^d$. Then

$$\begin{aligned} H_k(M, M \setminus \{x\}; R) &\cong H_k(U, U \setminus \{x\}) && \text{(excision)} \\ &\cong H_k(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}) \cong H_k(D^d, S^{d-1}). \end{aligned}$$

See *Topology I*, III.1.10. □

Definition III.2.2. An R -orientation of M is a family of generators $\mu_x \in H_d(M, M \setminus \{x\}; R)$, $x \in M$, such that $(\mu_x)_{x \in M}$ satisfies the following compatibility condition: For each $x \in M$, there is a neighborhood $U \subseteq M$ of x and $\mu_U \in H_d(M, M \setminus U; R)$ such that, for all $y \in U$, $i_* \mu_U = \mu_y$, where $i: (M, M \setminus U) \rightarrow (M, M \setminus \{y\})$.

The μ_x are called *local R -orientations*; M is called *R -orientable* if M admits an R -orientation, and M is called *R -oriented* if an R -orientation has been chosen. When R is omitted from the notation, $R = \mathbb{Z}$ is understood.

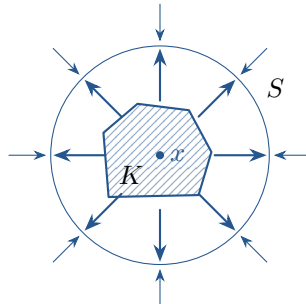
Proposition III.2.3. Every manifold is $\mathbb{Z}/2$ -orientable.

Proof. This follows from $\text{Aut}(\mathbb{Z}/2) = 1$. □

Proposition III.2.4. Let $K \subseteq M$ be compact. Then

- (i) $H_i(M, M \setminus K; R) = 0$ for $i > d$.
- (ii) If $u \in H_d(M, M \setminus K; R)$ is trivial in $H_d(M, M \setminus \{x\}; R)$ for all $x \in K$, then $u = 0$.
- (iii) If $(\mu_x)_{x \in M}$ is an R -orientation of M , then there is a unique element $\mu_K \in H_d(M, M \setminus K; R)$ that restricts to μ_x for all $x \in K$.

Proof. Step 1. $M = \mathbb{R}^d$, $K \subseteq \mathbb{R}^d$ convex.



Choose a sphere $S \subset M \setminus K$ and $x \in K$. Since K is convex, $S \subset M \setminus K$ and $S \subset M \setminus \{x\}$ are deformation retracts. The five lemma gives

$$H_k(M, M \setminus K) \cong H_k(M, M \setminus \{x\}).$$

Thus (i), (ii), and (iii) follow from [Lemma III.2.1](#).

Step 2. If the proposition holds for K_1 , K_2 , and $K_1 \cap K_2$, then it also holds for $K_1 \cup K_2$. Using *Topology I*, II.4.1 and the five lemma, we obtain a short exact sequence of chain complexes

$$0 \rightarrow C_*^{\text{sing}}(M | K_1 \cup K_2) \rightarrow C_*^{\text{sing}}(M | K_1) \oplus C_*^{\text{sing}}(M | K_2) \rightarrow Q_* \rightarrow 0,$$

where

$$Q_* := \frac{C_*^{\text{sing}}(M)}{C_*^{\text{sing}}(M \setminus K_1) + C_*^{\text{sing}}(M \setminus K_2)}.$$

Subdivision shows that the natural projection $Q_* \rightarrow C_*^{\text{sing}}(M | K_1 \cap K_2)$ is a chain homotopy equivalence. The short exact sequence therefore induces the following long exact Mayer–Vietoris sequence:

$$\begin{aligned} H_{k+1}(M | K_1 \cap K_2) \rightarrow H_k(M | K_1 \cup K_2) \rightarrow H_k(M | K_1) \oplus H_k(M | K_2) \rightarrow H_k(M | K_1 \cap K_2), \\ u_1 \oplus u_2 \longmapsto u_1 - u_2. \end{aligned}$$

Here $H_{k+1}(M | K_1 \cap K_2) = 0$ for $k > d - 1$ and $H_k(M | K_1) = H_k(M | K_2) = 0$ for $k > d$. $\Rightarrow$ (i).

For $k = d$, it follows that the homomorphism

$$\begin{aligned} H_d(M | K_1 \cup K_2) &\rightarrow H_d(M | K_1) \oplus H_d(M | K_2), \\ u &\mapsto (u_1, u_2) \end{aligned}$$

is injective. If u is trivial in $H_d(M | \{x\})$ for all $x \in K_1 \cup K_2$, then u_i is trivial in $H_d(M | \{x\})$ for all $x \in K_i$ ($i = 1, 2$). The hypothesis gives $u_i = 0$, and injectivity therefore gives $u = 0$. $\Rightarrow$ (ii).

By uniqueness, $\mu_{K_1 \cap K_2}$ is the image of μ_{K_i} under $H_d(M | K_i) \rightarrow H_d(M | K_1 \cap K_2)$. Thus $\mu_{K_1} \oplus \mu_{K_2}$ lies in the kernel in the Mayer–Vietoris sequence and hence has a preimage $\mu_{K_1 \cup K_2} \in H_d(M | K_1 \cup K_2)$, which is unique because $H_{d+1}(M | K_1 \cap K_2) = 0$. $\Rightarrow$ (iii).

Step 3. The proposition holds for $M = \mathbb{R}^d$ and $K = \bigcup_{i=1}^n K_i$ with each K_i convex. Induction on n : $n = 1$ is Step 1.

Write $K = \bigcup_{i=1}^n K_i = K_n \cup \bigcup_{i=1}^{n-1} K_i$. We have

$$K_n \cap \bigcup_{i=1}^{n-1} K_i = \bigcup_{i=1}^{n-1} \underbrace{K_n \cap K_i}_{\text{convex}}.$$

Thus the assertion follows from the induction hypothesis and Step 2.

Step 4. The proposition holds for $M = \mathbb{R}^d$ and compact $K \subseteq \mathbb{R}^d$.

For $[z] \in H_k(\mathbb{R}^d | K)$, let $K_0 \subseteq \mathbb{R}^d \setminus K$ be the union of the simplices of the boundary ∂z . Since K_0, K are compact, hence closed, and $\mathbb{R}^d$ is normal, there is $\delta > 0$ with $d(K_0, K) > \delta$. Since K is compact, there is a union of closed balls $W = \bigcup_{i=1}^n \overline{B}_\delta(x_i) \supseteq K$ with $x_i \in K$, so $W \cap K_0 = \emptyset$ and hence

$$[z] \in \text{im}(H_k(\mathbb{R}^d | W) \rightarrow H_k(\mathbb{R}^d | K)).$$

But Step 3 gives $H_i(\mathbb{R}^d | W) = 0$ for $i > d$. $\Rightarrow$ (i).

For (ii), in view of Step 3, it remains to show: If $[z]$ vanishes in $H_d(\mathbb{R}^d \mid \{x\})$ for all $x \in K$, then $[z]$ also vanishes in $H_d(\mathbb{R}^d \mid \{x\})$ for all $x \in W$. So let $y \in \overline{B}_\delta(x_i) =: B$. Then we have

$$\begin{array}{ccccc} H_d(\mathbb{R}^d \mid W) & \longrightarrow & H_d(\mathbb{R}^d \mid B) & \xrightarrow{\cong} & H_d(\mathbb{R}^d \mid \{x_i\}) \\ \alpha \mapsto & & \alpha_B \mapsto & & \alpha_{x_i} \end{array}$$

and $\alpha_{x_i} = 0$ implies $\alpha_B = 0$ and hence also $\alpha_y = 0$.

For (iii), choose $R \gg 0$ such that $B := B_R(0) \supseteq K$. Then, for each $x \in B$, the map $H_d(\mathbb{R}^d \mid B) \xrightarrow{\cong} H_d(\mathbb{R}^d \mid \{x\})$ is an isomorphism. For each generator $\alpha \in H_d(\mathbb{R}^d \mid B)$, let

$$P_\alpha := \{x \in B : \alpha \text{ maps to } \mu_x\}.$$

Thus $\bigcup_{\alpha \text{ a generator}} P_\alpha = B$ is a disjoint union. But local compatibility of $(\mu_x)_{x \in \mathbb{R}^d}$ implies that each $P_\alpha \subseteq B$ is open: For $x \in P_\alpha$, there is an open ball U with $x \in U \subseteq B$ and $\mu_U \in H_d(\mathbb{R}^d \mid U)$ such that $\mu_U \mapsto \mu_y$ for all $y \in U$. Since $\alpha \mapsto \mu_x$, it follows that $\alpha \mapsto \mu_U \mapsto \mu_y$ for all $y \in U$, so $U \subseteq P_\alpha$. Since B is connected, there is therefore a unique generator $\alpha \in H_d(\mathbb{R}^d \mid B)$ with $\alpha \mapsto \mu_x$ for all $x \in B \supseteq K$.

Step 5. The proposition holds if $K \subseteq U \subseteq M$ with $U \cong \mathbb{R}^d$. By excision, $H_k(M \mid K) \cong H_k(U \mid K) \rightsquigarrow$ Step 4.

Step 6. The general case. For each $x \in K$, choose a neighborhood $U_x \subseteq M$ with $\varphi_x: U_x \xrightarrow{\cong} \mathbb{R}^d$ and $\varphi_x(x) = 0$. Since K is compact, there are $x_1, \dots, x_n \in K$ with $\bigcup_{i=1}^n \varphi_{x_i}^{-1}(D^d) \supseteq K$. By Step 5, the proposition holds for $\varphi_{x_i}^{-1}(D^d) \cap K$ and all intersections of these sets. By Step 2, the proposition follows for K . $\square$

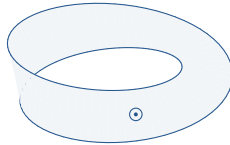
Remark. For each $x \in M$, we have

$$\begin{array}{ccccc} x & \in & B & \subseteq & U & \subseteq & M \\ \downarrow & & \wr & & \wr & & \\ 0 & \in & D^d & \subseteq & \mathbb{R}^d & & \end{array}$$

and hence

$$\begin{array}{ccc} H_d(M \mid B) & \longrightarrow & H_d(M \mid \{x\}) \\ \wr & & \wr \\ H_d(U \mid B) & & H_d(U \mid \{x\}) \\ \wr & & \wr \\ H_d(\mathbb{R}^d \mid D^d) & \cong & H_d(\mathbb{R}^d \mid \{0\}), \end{array}$$

that is, local orientations can be chosen compatibly in neighborhoods of points, but generally not globally:



If M is a smooth manifold, the two generators of $H_d(M \mid \{x\}; \mathbb{Z}) \cong \mathbb{Z}$ correspond to the two “orientations” of the $\mathbb{R}$ -vector space $T_x M$ ($\{\text{bases}\}/\sim$ with $B \sim B' \iff \det M_{BB'} > 0$).

Theorem III.2.5. Let M be a *closed* (that is, compact and without boundary) connected d -manifold. Then

1.) If M is R -orientable, then

$$H_d(M; R) \xrightarrow{\cong} H_d(M \mid \{x\}; R), \quad \mu \mapsto \mu_x$$

is an isomorphism for every $x \in M$.

2.) If M is not R -orientable, then $\mu \mapsto \mu_x$ is injective with image isomorphic to $\{r \in R : 2r = 0\}$.

3.) $H_i(M; R) = 0$ for $i > d$.

Proof. For 1.) Surjectivity follows from [Proposition III.2.4 \(iii\)](#) with $K = M$. We have

$$\begin{aligned} \ker(H_d(M; R) \longrightarrow H_d(M \mid \{x\}; R)) \\ = \ker(H_d(M; R) \longrightarrow H_d(M \mid \{x\}; R) \xrightarrow{\cong} H_d(M \mid U; R)) \end{aligned}$$

with U as in the preceding remark. Thus, if $\mu \in H_d(M; R)$ satisfies $\mu \mapsto \mu_x = 0$, then $\mu \mapsto \mu_y = 0$ for all $y \in U$. Since M is connected, $\mu \mapsto \mu_y = 0$ for all $y \in M$. By [Proposition III.2.4 \(ii\)](#), $\mu = 0$.

For 2.) Injectivity holds as just shown. For the second assertion, use the two-sheeted “orientation covering” (for example, $S^2 \rightarrow \mathbb{RP}^2$, $T^2 \rightarrow K$) [[Hatcher](#), pp. 234–236].

For 3.) [Proposition III.2.4 \(i\)](#) with $K = M$. □

Definition III.2.6. Let M be as in [III.2.5](#) and R -orientable. Then every generator $[M] \in H_d(M; R)$ is called a *fundamental class* of M .

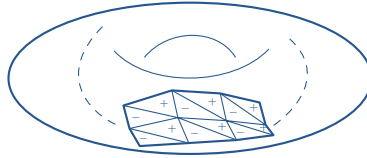
Observation. By [Theorem III.2.5 \(i\)](#), orientations and fundamental classes are in one-to-one correspondence, and there are $|R^*|$ distinct fundamental classes.

- If M is triangulated and $[M] = [\sum_{i=1}^k a_i \cdot \sigma_i]$ is a simplicial fundamental class under the isomorphism of *Topology I*, [III.2.3](#), then for $x \in \sigma_i(\Delta^d)$ we have

$$\begin{array}{ccc} H_d(M; R) & \xrightarrow{\cong} & H_d(M \mid \{x\}; R) \cong R \\ [M] & \longmapsto & a_i[\sigma_i] \leftarrow \text{generator} \end{array}$$

so $a_i \in R^*$. Moreover, $\bigcup_{i=1}^k \sigma_i(\Delta^d) = M$, since $[M] \mapsto 0 \in H_d(M \mid \{x\})$ if $x \notin \bigcup_{i=1}^k \sigma_i(\Delta^d)$.

$R = \mathbb{Z}$, $M = T^2$, $[T^2]$:



III.3. The Degree of Maps Between Manifolds

Definition III.3.1. Let M, N be connected, oriented, closed, d -dimensional manifolds and $f: M \rightarrow N$. Then the *degree* of f is the unique integer $\deg f \in \mathbb{Z}$ such that

$$H_d(f; \mathbb{Z})([M]) = (\deg f) \cdot [N].$$

Clearly, for $M = N = S^d$, this definition agrees with Definition III.4.1 in *Topology I*.

Observations. If f is not surjective, then $\deg f = 0$.

- $\deg(f \circ g) = (\deg f) \cdot (\deg g)$.
- $\deg(f: M \rightarrow N) = -\deg(f: M^- \rightarrow N)$, where M^- carries the opposite orientation.

Remark. For smooth maps between smooth manifolds, $\deg f$ agrees with the invariant from differential topology [Differential Topology seminar, *Topology I*, III.4]. In particular, the formula

$$\deg f = \sum_{x \in f^{-1}(y)} \operatorname{sgn} \det D_x f$$

holds for every regular value $y \in N$.

III.4. The Duality Theorem

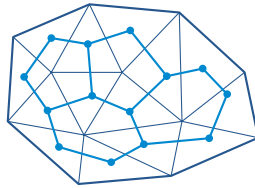
Theorem III.4.1. (Poincaré duality). Let M be a connected, closed, R -oriented, d -dimensional manifold. Then

$$P = [M] \cap (\cdot): H^{d-k}(M; R) \xrightarrow{\cong} H_k(M; R)$$

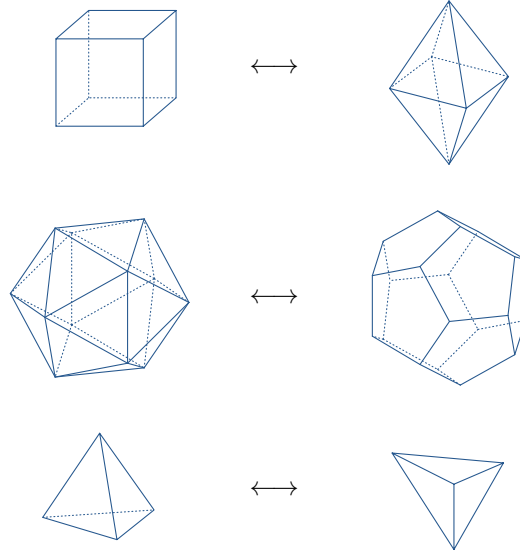
is an isomorphism for every k .

Idea [Poincaré]: Every cell structure on M yields a *dual cell structure*:

$$\{k\text{-cells}\} \longleftrightarrow \{(d-k)\text{-cells}\}.$$



- Examples with $M = S^2$:



$\rightsquigarrow$ We obtain a *nondegenerate* “intersection pairing”

$$C_*(M) \otimes C_{d-*}(M) \longrightarrow R.$$

$\rightsquigarrow$ By adjunction, this gives an isomorphism $C_{d-*}(M) \xrightarrow{\cong} C^*(M)$. $\rightsquigarrow H_{d-k}(M) \cong H^k(M)$. $\square$

Preliminaries on colimits of R -modules. Let $(I, \leq)$ be a *directed set*, that is, “ $\leq$ ” is a partial order on I , and for $x, y \in I$ there is always $z \in I$ with $x \leq z, y \leq z$. Setting

$$\text{Hom}_I(x, y) = \begin{cases} * & x \leq y, \\ \emptyset & \text{otherwise,} \end{cases}$$

makes I a small category.

A diagram $F: I \rightarrow \mathcal{C}$ is called a *directed system* in $\mathcal{C}$, and $\text{colim } F$ (if it exists) is called the *directed limit* or *direct limit* of F (notation: $\text{colim } F = \varinjlim_{i \in I} F(i)$). Now let $\mathcal{C} = R\text{-Mod}$. We observe:

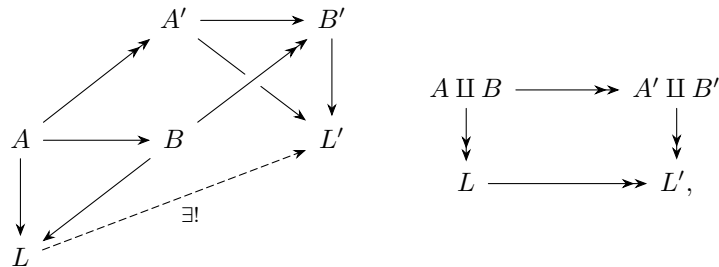
- Direct limits are exact: If $F_1 \rightarrow F_2 \rightarrow F_3$ are natural transformations such that

$$0 \longrightarrow F_1(i) \longrightarrow F_2(i) \longrightarrow F_3(i) \longrightarrow 0$$

is exact for all $i \in I$, then

$$0 \longrightarrow \varinjlim_i F_1(i) \longrightarrow \varinjlim_i F_2(i) \longrightarrow \varinjlim_i F_3(i) \longrightarrow 0$$

is exact. (Colimits are generally only right exact,



but are also left exact over directed systems.)

- If $\mathcal{C}$ is the category of submodules of an R -module N and $F(i \leq j): F(i) \subseteq F(j)$, then

$$\varinjlim_i F(i) = \bigcup_{i \in I} F(i) \quad (\text{by the universal property}).$$

- If $J \subseteq I$ is *cofinal* (that is, $\forall i \in I \exists j \in J: i \leq j$), then

$$\varinjlim_{j \in J} F(j) \cong \varinjlim_{i \in I} F(i).$$

Definition III.4.2. Let X be a topological space, and let $C_c^*(X; R) \subseteq C_{\text{sing}}^*(X; R)$ be the subcomplex of all $\alpha: C_*^{\text{sing}}(X; R) \rightarrow R$ for which there is a compact set $K_\alpha \subseteq X$ such that $\alpha(\sigma) = 0$ for all $\sigma: \Delta^* \rightarrow X$ with $\sigma(\Delta^*) \cap K_\alpha = \emptyset$. We call

$$H_c^*(X; R) := H^*(C_c^*(X; R))$$

the *singular cohomology with compact support* of X .

Note. If X is compact, then $H_c^*(X; R) = H^*(X; R)$. □

The cocone $C^*(X, X \setminus K; R) \rightarrow C_c^*(X; R)$, with $K \subseteq X$ compact, induces an isomorphism

$$\begin{aligned} & \varinjlim_{\substack{K \subseteq X \\ K \text{ compact}}} C^*(X, X \setminus K; R) \xrightarrow{\cong} C_c^*(X; R). \\ & = \bigcup_{\substack{K \subseteq X \\ K \text{ compact}}} C^*(X, X \setminus K; R) \end{aligned}$$

Thus we obtain

$$H^* \left(\varinjlim_{\substack{K \subseteq X \\ K \text{ compact}}} C^*(X, X \setminus K; R) \right) \xrightarrow{\cong} H_c^*(X; R).$$

Proposition III.4.3. $H^*: R\text{-cochain} \rightarrow R\text{-Mod}$ preserves direct limits.

Proof.

$$\begin{aligned} 0 & \longrightarrow \varinjlim Z^n \longrightarrow \varinjlim C^n \longrightarrow \varinjlim C^{n+1} \\ 0 & \longrightarrow \varinjlim Z^{n-1} \longrightarrow \varinjlim C^{n-1} \longrightarrow \varinjlim B^n \longrightarrow 0 \\ 0 & \longrightarrow \varinjlim B^n \longrightarrow \varinjlim Z^n \longrightarrow \varinjlim H^n \longrightarrow 0 \end{aligned}$$

Thus

$$\begin{aligned} \varinjlim Z^n &= Z^n(\varinjlim C^*), \\ \varinjlim B^n &= \varinjlim C^{n-1} / \varinjlim Z^{n-1} = \varinjlim C^{n-1} / Z^{n-1}(\varinjlim C^*) = B^n(\varinjlim C^*), \\ \varinjlim H^n(C^*) &= Z^n(\varinjlim C^*) / B^n(\varinjlim C^*) = H^n(\varinjlim C^*). \end{aligned}$$

□

We therefore have an isomorphism

$$H_c^*(X; R) \cong \varinjlim_{K \subseteq X} H^*(X | K; R).$$

Now let M be a d -dimensional manifold with an R -orientation $(\mu_x)_{x \in M}$. By [Proposition III.2.4](#) (iii), for every compact $K \subseteq M$ there is a unique $\mu_K \in H_d(M | K; R)$ with $\mu_K \mapsto \mu_x$ for all $x \in K$. Uniqueness implies $\mu_L \mapsto \mu_K$ for $K \subseteq L$, so

$$\begin{array}{ccc} H^{d-k}(M | K; R) & & \\ \downarrow i^* & \searrow \mu_K \cap (-) & \\ H^{d-k}(M | L; R) & \xrightarrow{\mu_L \cap (-)} & H_k(M; R) \end{array}$$

$$(i_*(\mu_L \cap i^*(\alpha))) = i_*\mu_L \cap \alpha = \mu_K \cap \alpha$$

is a cocone over the directed system $H^{d-k}(M | K; R)$, with $K \subseteq M$ compact. We obtain

$$P_k(M): H_c^{d-k}(M; R) \cong \varinjlim_{K \subseteq M} H^{d-k}(M | K; R) \longrightarrow H_k(M; R).$$

Theorem III.4.4. (Poincaré duality, noncompact version) Let M be a d -dimensional R -oriented manifold. Then

$$P_k(M): H_c^{d-k}(M; R) \longrightarrow H_k(M; R)$$

is an isomorphism for every k .

The observation above shows that [III.4.4](#) $\Rightarrow$ [III.4.1](#).

Now let $M = U \cup V$ with U, V open. For any compact $K \subseteq U$, $L \subseteq V$, dualizing the short exact sequence of chain complexes in Step 2 of the proof of [Proposition III.2.4](#),

$$\begin{aligned} 0 \longrightarrow C_*^{\text{sing}}(M | K_1 \cup K_2) &\longrightarrow C_*^{\text{sing}}(M | K_1) \oplus C_*^{\text{sing}}(M | K_2) \\ &\longrightarrow Q_* \longrightarrow 0, \end{aligned}$$

$$Q_* := C_*^{\text{sing}}(M) / (C_*^{\text{sing}}(M \setminus K_1) + C_*^{\text{sing}}(M \setminus K_2)),$$

where subdivision gives a chain homotopy equivalence between Q_* and $C_*^{\text{sing}}(M | K_1 \cap K_2)$, yields the Mayer–Vietoris sequence

$$\begin{aligned} \cdots \longrightarrow H^k(M | K \cap L) &\longrightarrow H^k(M | K) \oplus H^k(M | L) \\ &\longrightarrow H^k(M | K \cup L) \longrightarrow H^{k+1}(M | K \cap L) \longrightarrow \cdots \end{aligned}$$

By excision, we obtain

$$\begin{aligned} \cdots \longrightarrow H^k(U \cap V | K \cap L) &\longrightarrow H^k(U | K) \oplus H^k(V | L) \\ &\longrightarrow H^k(M | K \cup L) \longrightarrow H^{k+1}(U \cap V | K \cap L) \longrightarrow \cdots \end{aligned}$$

Choose increasing compact exhaustions $K_n \subseteq U$, $L_n \subseteq V$ with $K_n \subseteq \text{int}_U K_{n+1}$, $L_n \subseteq \text{int}_V L_{n+1}$ and $\bigcup_{n=1}^{\infty} K_n = U$, $\bigcup_{n=1}^{\infty} L_n = V$. Then $\bigcup_{n=1}^{\infty} (K_n \cup L_n) = M$. Applying $\varinjlim_n$ gives

$$\cdots \longrightarrow H_c^k(U \cap V) \longrightarrow H_c^k(U) \oplus H_c^k(V) \longrightarrow H_c^k(M) \longrightarrow H_c^{k+1}(U \cap V) \longrightarrow \cdots$$

Proposition III.4.5. The diagram

$$\begin{array}{ccccccc}
\cdots \rightarrow H_c^k(U \cap V) & \longrightarrow & H_c^k(U) \oplus H_c^k(V) & \longrightarrow & H_c^k(M) & \longrightarrow & H_c^{k+1}(U \cap V) \rightarrow \cdots \\
\downarrow P(U \cap V) & & \downarrow P(U) - P(V) & & \downarrow P(M) & & \downarrow P(U \cap V) \\
\cdots \rightarrow H_{d-k}(U \cap V) & \longrightarrow & H_{d-k}(U) \oplus H_{d-k}(V) & \longrightarrow & H_{d-k}(M) & \longrightarrow & H_{d-k-1}(U \cap V) \rightarrow \cdots
\end{array}$$

commutes up to sign.

Proof. For the first two squares, this follows from the cocone description of the maps involved. For the third square, we must use the construction of the connecting homomorphism in the snake lemma [Hatcher, Lemma 3.36, p. 246]. $\square$

Proof (of Theorem III.4.4). Two induction steps:

- I) If $M = U \cup V$ with $U, V \subseteq M$ open and $P(U)$, $P(V)$, and $P(U \cap V)$ are isomorphisms, then so is $P(M)$: this follows immediately from Proposition III.4.5 and the five lemma.
- II) For open subsets $U_1 \subseteq U_2 \subseteq \cdots \subseteq M$ with $\bigcup_{i=1}^{\infty} U_i = M$, if $P(U_i)$ is an isomorphism for every i , then $P(M)$ is also an isomorphism: by excision,

$$H_c^*(U_i) \cong \varinjlim_{K \subseteq U_i} H^*(U_i | K) \cong \varinjlim_{K \subseteq U_i} H^*(M | K)$$

and hence

$$\varinjlim_{i \rightarrow \infty} H_c^*(U_i) \cong \varinjlim_{i \rightarrow \infty} \varinjlim_{K \subseteq U_i} H^*(M | K) \cong H_c^*(M).$$

On the other hand,

$$\varinjlim_{i \rightarrow \infty} H_*(U_i) = H_*(M),$$

since every singular chain has compact support and is therefore contained in some U_i . By assumption,

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
H_c^*(U_i) & \xrightarrow[\cong]{P(U_i)} & H_*(U_i) \\
\downarrow & & \downarrow \\
H_c^*(U_{i+1}) & \xrightarrow[\cong]{P(U_{i+1})} & H_*(U_{i+1}) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

are isomorphic directed systems, and hence

$$P(M): H_c^*(M) \cong \varinjlim_{i \rightarrow \infty} H_c^*(U_i) \xrightarrow{\cong} \varinjlim_{i \rightarrow \infty} H_*(U_i) \cong H_*(M).$$

We now turn to the proof itself:

Step 1. The case $M = \mathbb{R}^d$. We have

$$H_c^*(\mathbb{R}^d) \cong \varinjlim_{K \subseteq \mathbb{R}^d} H^*(\mathbb{R}^d | K) \cong \varinjlim_{n \rightarrow \infty} H^*(\mathbb{R}^d | B_n(0)),$$

since the family $\{B_n(0)\}_{n=1}^\infty$ of closed balls is cofinal in $\{K \subseteq \mathbb{R}^d \text{ compact}\}$. The system $H^*(\mathbb{R}^d \mid B_n(0))$ is clearly constant, so

$$H_c^*(\mathbb{R}^d) \cong H^*(\mathbb{R}^d \mid B_1(0)) \cong H^*(D^d, S^{d-1}) \cong \begin{cases} R & * = d, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly,

$$H_{d-*}(\mathbb{R}^d) \cong H_{d-*}(\{\cdot\}) \cong \begin{cases} R & * = d, \\ 0 & \text{otherwise.} \end{cases}$$

We therefore only need to show that

$$P_0(\mathbb{R}^d): H_c^d(\mathbb{R}^d) \longrightarrow H_0(\mathbb{R}^d)$$

is an isomorphism. As we saw above, it suffices to show that

$$\mu_{B_1(0)} \cap (-): H^d(\mathbb{R}^d \mid B_1(0)) \longrightarrow H_0(\mathbb{R}^d)$$

is an isomorphism. Since $H_{d-1}(\mathbb{R}^d \mid B_1(0))$ is projective, the universal coefficient theorem implies that

$$\begin{aligned} H^d(\mathbb{R}^d \mid B_1(0)) &\xrightarrow{\cong} R, \\ \alpha &\longmapsto \langle \alpha, \mu_{B_1(0)} \rangle \end{aligned}$$

is an isomorphism. Likewise,

$$\begin{aligned} H_0(\mathbb{R}^d) &\xrightarrow{\cong} R, \\ \nu &\longmapsto \langle 1_{H^0(\mathbb{R}^d)}, \nu \rangle \end{aligned}$$

is an isomorphism (a generator maps to a generator). Thus $\alpha \mapsto \mu_{B_1(0)} \cap \alpha$ is an isomorphism if and only if

$$\alpha \longmapsto \langle 1_{H^0(\mathbb{R}^d)}, \mu_{B_1(0)} \cap \alpha \rangle$$

is an isomorphism. The latter holds because

$$\begin{aligned} \langle 1_{H^0(\mathbb{R}^d)}, \mu_{B_1(0)} \cap \alpha \rangle &= \langle \alpha \cup 1_{H^0(\mathbb{R}^d)}, \mu_{B_1(0)} \rangle \\ &= \langle \alpha, \mu_{B_1(0)} \rangle. \end{aligned}$$

Step 2. The case of an open subset $M \subseteq \mathbb{R}^d$. We first show, by induction on r , that for every finite union $V = V_1 \cup \cdots \cup V_r$ of open, bounded, convex sets $V_i \subseteq \mathbb{R}^d$, the map $P_*(V)$ is an isomorphism. Base case: For $V = V_1$, we have $V \cong \mathbb{R}^d$, so $P_*(V)$ is an isomorphism by Step 1. Induction step: By the induction hypothesis, $P(V_1 \cup \cdots \cup V_{r-1})$ and $P_*((V_1 \cap V_r) \cup \cdots \cup (V_{r-1} \cap V_r))$ are isomorphisms. Moreover, $P_*(V_r)$ is an isomorphism by Step 1. Thus $P_*(V)$ is an isomorphism by I.

Now $\mathbb{R}^d \supseteq M = \bigcup_{i=0}^\infty U_i$ is a countable union of open balls. Setting $V_j := \bigcup_{i < j} U_i$, we see from the preceding argument that $P_*(V_j)$ is an isomorphism for every j . By II, $P_*(M)$ is therefore also an isomorphism.

Step 3. The general case. We first show, by induction on r , that for every finite union $V = V_1 \cup \cdots \cup V_r$ of open sets $V_i \subseteq M$ with $V_i \cong U_i \subseteq \mathbb{R}^d$ open, $P_*(V)$ is an isomorphism: proceed exactly as above (simply replace “Step 1” with “Step 2”).

Now M is second countable, so $M = \bigcup_{i=1}^\infty U_i$ with $\mathbb{R}^d \cong U_i \subseteq M$ open. Setting $V_j := \bigcup_{i < j} U_i$, we see from the preceding argument that $P_*(V_j)$ is an isomorphism for every j . By II, $P_*(M)$ is therefore also an isomorphism. $\square$

III.5. Consequences of Poincaré Duality

Theorem III.5.1. Let M be a closed odd-dimensional manifold. Then $\chi(M) = 0$.

Proof. Let K be a field. As in *Topology I*, Theorem IV.2.8, we see that for a finite chain complex C_* of finite-dimensional K -vector spaces,

$$\sum_k (-1)^k \dim_K C_k = \sum_k (-1)^k \dim_K H_k(C_*).$$

Since M is $\mathbb{Z}/2$ -orientable, Poincaré duality with $d = \dim M$ shows that

$$\begin{aligned} \dim_{\mathbb{Z}/2} H_{d-k}(M; \mathbb{Z}/2) &\stackrel{\text{universal coefficient theorem}}{=} \dim_{\mathbb{Z}/2} H^{d-k}(M; \mathbb{Z}/2) \\ &\stackrel{\text{Poincaré duality}}{=} \dim_{\mathbb{Z}/2} H_k(M; \mathbb{Z}/2). \end{aligned}$$

For every cell structure on M (if one exists), we therefore have

$$\begin{aligned} \chi(M) &= \sum_{k=0}^d (-1)^k |\{k\text{-cells}\}| = \sum_{k=0}^d (-1)^k \dim_{\mathbb{Z}/2} C_k^{\text{CW}}(M; \mathbb{Z}/2) \\ &= \sum_{k=0}^d (-1)^k \dim_{\mathbb{Z}/2} H_k^{\text{CW}}(M; \mathbb{Z}/2) = 0. \end{aligned}$$

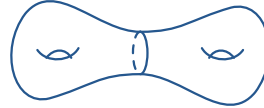
□

Definition III.5.2. A d -dimensional *manifold with boundary* is a second-countable Hausdorff space M such that every $x \in M$ has a neighborhood $U \subseteq M$ with $U \cong \mathbb{R}^d$ or $U \cong \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_d \geq 0\}$. The points that have no neighborhood $U \cong \mathbb{R}^d$ form the *boundary* $\partial M \subseteq M$. Clearly, ∂M is a $(d-1)$ -dimensional manifold without boundary.



Definition III.5.3. Let M be a manifold with boundary ∂M . The *double* of M is the manifold without boundary defined by

$$\begin{array}{ccc} \partial M & \xrightarrow{\quad} & M \\ \downarrow & \lrcorner & \downarrow \\ M & \xrightarrow{\quad} & DM \end{array}$$



Theorem III.5.4. If M is a compact odd-dimensional manifold with boundary, then $\chi(M) = \frac{\chi(\partial M)}{2}$.

Proof. By additivity of the Euler characteristic (*Topology I*, IV.2.10),

$$2\chi(M) - \chi(\partial M) = \chi(DM) \stackrel{\text{III.5.1}}{=} 0.$$

□

Corollary III.5.4. If a manifold M is a boundary (that is, $M = \partial W$ for a compact manifold W), then $\chi(M)$ is even.

Proof. If $\dim M$ is odd, then $\chi(M) = 0 \in 2\mathbb{Z}$. If $\dim M$ is even, the claim follows because $\chi(W) \in \mathbb{Z}$ by definition. $\square$

- Thus the surfaces N_{2t-1} , which are nonorientable over $\mathbb{Z}$ and have odd genus, are not boundaries, since $\chi(N_{2t-1}) = 2 - (2t - 1) = 2(1 - t) + 1$.
- In particular, $\mathbb{RP}^2$, and more generally $\mathbb{RP}^{2n}$, is not a boundary, since $\chi(\mathbb{RP}^{2n}) = 1$.
- Since $\chi(\mathbb{CP}^{2n}) = 2n + 1$, the space $\mathbb{CP}^{2n}$ is a $\mathbb{Z}$ -oriented manifold that is not a boundary.
- N_{2t} , $\mathbb{RP}^{2n+1}$, and $\mathbb{CP}^{2n+1}$ are boundaries! (“null-bordant” $\rightsquigarrow$ bordism)

Theorem III.5.5. Let M be closed and R -oriented. Then the R -bilinear map

$$\begin{aligned} H^{d-k}(M; R) \times H^k(M; R) &\longrightarrow R, \\ (\alpha, \beta) &\longmapsto \langle \alpha \cup \beta, [M] \rangle \end{aligned}$$

is nondegenerate (that is, the induced maps

$$\begin{aligned} \varphi: H^k(M; R) &\longrightarrow \text{Hom}_R(H^{d-k}(M; R), R), \\ \psi: H^{d-k}(M; R) &\longrightarrow \text{Hom}_R(H^k(M; R), R) \end{aligned}$$

are isomorphisms) if R is a field, or if $R = \mathbb{Z}$ and we replace $H^*(M; R)$ by $H^*(M; R)/\text{torsion}$.

Proof. For $\alpha \in H^{d-k}(M; R)$, the composition

$$\begin{aligned} F: H^k(M; R) &\xrightarrow{h} \text{Hom}(H_k(M; R), R) \\ &\xrightarrow{P_k(M)^*} \text{Hom}(H^{d-k}(M; R), R) \xrightarrow{\text{ev}_\alpha} R \end{aligned}$$

is given by $F(\beta) = \langle \beta, [M] \cap \alpha \rangle = \langle \alpha \cup \beta, [M] \rangle$. Since h (by the assumptions and the universal coefficient theorem) and $P_k(M)^*$ (by Poincaré duality) are isomorphisms, φ is also an isomorphism. Since

$$\begin{aligned} \langle \alpha \cup \beta, [M] \rangle &= (-1)^{k(d-k)} \langle \beta \cup \alpha, [M] \rangle \\ &= (-1)^{k(d-k)} \langle \alpha, [M] \cap \beta \rangle, \end{aligned}$$

ψ is also an isomorphism. $\square$

Corollary III.5.6. Let M be connected, closed, and oriented, and let $\alpha \in H^{d-k}(M; \mathbb{Z})$ be such that its image $\bar{\alpha} \in H^{d-k}(M; \mathbb{Z})/\text{torsion}$ is nonzero and primitive in this quotient. Then there is a $\beta \in H^k(M; \mathbb{Z})$ such that $\alpha \cup \beta$ generates $H^d(M; \mathbb{Z}) \cong \mathbb{Z}$.

Proof. By assumption, $\langle \bar{\alpha} \rangle \cong \mathbb{Z}$ is a direct summand of $H^{d-k}(M; \mathbb{Z})/\text{torsion}$, so there is a nontrivial $\mathbb{Z}$ -linear form $\delta: H^{d-k}(M; \mathbb{Z})/\text{torsion} \rightarrow \mathbb{Z}$ with $\bar{\alpha} \mapsto 1$. By III.5.5, every representative β of $\varphi^{-1}(\delta)$ satisfies $\langle \alpha \cup \beta, [M] \rangle = 1$. Thus $\alpha \cup \beta \in H^d(M; \mathbb{Z})$ is a generator. $\square$

Definition III.5.7. For a closed oriented $4k$ -dimensional manifold M , the symmetric $\mathbb{Z}$ -bilinear form

$$\begin{aligned} (H^{2k}(M; \mathbb{Z})/\text{torsion}) \times (H^{2k}(M; \mathbb{Z})/\text{torsion}) &\longrightarrow \mathbb{Z}, \\ (\alpha, \beta) &\longmapsto \langle \alpha \cup \beta, [M] \rangle \end{aligned}$$

is called the *intersection form* of M .

Definition III.5.8. For M as in III.5.7, we define the *signature* $\sigma(M) \in \mathbb{Z}$ to be the signature of its intersection form $(\#\{\text{positive eigenvalues}\} - \#\{\text{negative eigenvalues}\})$ after applying $(-) \otimes_{\mathbb{Z}} \mathbb{R}$.

Example. $\sigma(\mathbb{CP}^{2n}) = 1$.

Remark.

$$\sigma: \text{“bordism ring of } 4k\text{-manifolds”} \longrightarrow \mathbb{Z}$$

is a ring homomorphism!

Poincaré duality often simplifies the computation of a manifold’s cohomology ring:

Example. $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$ is an H^* -isomorphism for $* \leq 2n - 2$. By the induction hypothesis, $H^*(\mathbb{CP}^{n-1}) \cong \mathbb{Z}[\alpha]/(\alpha^n)$, $|\alpha| = 2$. By III.5.6, there is an $r \in \mathbb{Z}$ such that $\alpha \cup r\alpha^{n-1}$ generates $H^{2n}(\mathbb{CP}^n)$. In particular, $r = \pm 1$, so $\alpha^n \in H^{2n}(\mathbb{CP}^n)$ is also a generator, and

$$H^*(\mathbb{CP}^n) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1}).$$

Similarly:

$$H^*(\mathbb{HP}^n) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1}), \quad |\alpha| = 4.$$

$$H^*(\mathbb{RP}^n; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[\alpha]/(\alpha^{n+1}).$$

Definition. A connected, closed d -dimensional manifold is called a *homology sphere* if $H_*(M) \cong H_*(S^d)$.

Proposition III.5.9. A connected closed 3-manifold with $H_1(M) = 0$ is a homology sphere.

Proof. Since $(\pi_1 M)_{\text{ab}} = 1$, there is no $H \leq \pi_1 M$ with $[\pi_1 M : H] = 2$. Thus M has no connected two-sheeted covering, so it is orientable. It follows that

$$\begin{aligned} H_2(M) &\xrightarrow{\text{Poincaré duality}} \cong H^1(M) \xrightarrow{\text{universal coefficient theorem}} \cong \text{Hom}(H_1(M), \mathbb{Z}) \cong 0, \\ H_3(M) &\xrightarrow{\text{Poincaré duality}} \cong H^0(M) \xrightarrow{\text{universal coefficient theorem}} \cong \text{Hom}(H_0(M), \mathbb{Z}) \cong \mathbb{Z}. \end{aligned}$$

□

Poincaré: There is a 3-dimensional homology sphere M with $M \not\cong S^3$. (The quotient of a solid dodecahedron obtained by identifying opposite faces with a 36° rotation. $\pi_1 M \cong I^*$, the binary icosahedral group, $(\pi_1 M)_{\text{ab}} = 1$.)

Theorem III.5.10. (Poincaré conjecture) A simply connected d -dimensional homology sphere is homeomorphic to S^d .

Proof.

- $d = 2$: clear.
- $d \geq 5$: [S. Smale, 1961], Fields Medal 1966.
- $d = 4$: [M. Freedman, 1982], Fields Medal 1986.
- $d = 3$: [G. Perelman, 2003], Fields Medal 2006 (declined). The only solved *Millennium Prize Problem*.

□

III.6. Where Do We Go from Here?

- Homotopy theory, in particular:
- Classification of fiber bundles ($b_G(X) \cong [X, BG]$).
- Characteristic classes (natural transformations $c: b_G \rightarrow H^*$, Yoneda: $\text{Nat}(b_G, H^*) \cong H^*(BG)$).
- Cohomology operations (natural transformations $h: H^* \rightarrow H^{*+k}$, for example: $\cup: H^n \rightarrow H^{2n}$, Steenrod algebra).
- K-theory ($\text{Vect}(X)$ with $\oplus, \sim$ Grothendieck $\sim K^0(X)$).
- Spectral sequences $F \hookrightarrow E \rightarrow B$, $H^p(B, H^q(F)) \Rightarrow H^{p+q}(E)$.
- Spectra and the homotopy-theoretic construction of cohomology.

$$H^n(X; G) \xrightarrow{\cong} [X, K(G, n)]$$

$$f^*c \longleftarrow \quad \quad \quad \dashv f$$

$$\begin{aligned} H^n(K(G, n); G) &\cong \text{Hom}(H_n(K(G, n)), G) \\ &\cong \text{Hom}(\pi_n(K(G, n)), G) \\ &\cong \text{Hom}(G, G), \\ c &\longmapsto \text{id}_G. \end{aligned}$$

$$\Omega K(G, n) \simeq K(G, n-1) \quad \longrightarrow \quad \text{Brown representability theorem:}$$

(reduced cellular) cohomology theories are represented by “ Ω -spectra.”