Hand in by Thursday, October 29, 2015 at 08:30 in the corresponding mail-box in the Hörsaalgebäude (numbers of the mail-boxes of the exercise groups on the web-page of the course).

Important notes:

- Some of the concepts and notations on this exercise sheet will be clarified for the first time in the next lecture.
- On the majority of exercise sheets you will find bonus questions. They are marked by an $*$. With these questions you can collect extra points.

Question 1

Let M, N, M', N' be sets. Show through an example that the set

$$(M \times N) \setminus (M' \times N')$$

is in general different from the set

$$(M \setminus M') \times (N \setminus N').$$

Show, on the other hand, that $(M \times N) \setminus (M' \times N')$ can always be described as the union of two sets of the form $A \times B$.

Here $M \times N$ denotes the **Cartesian product** of the sets M and N, i. e.

$$M \times N = \{(x, y) \mid x \in M, y \in N\}.$$

Hint: Draw a schematic picture (e. g. with subsets of $\mathbb{R}$), which illustrates the assignment.

Question 2

Consider sets A, B, C and maps $f : A \to B$, $g : B \to C$. The map f is called **injective**, if for all $x, x' \in A$ the following statement holds: if $x \neq x'$, then also $f(x) \neq f(x')$. The map f is called **surjective**, if for every $y \in B$, there exists an $x \in A$ with $f(x) = y$. Analogue definitions apply to g. Show that:

(a) If f and g are injective, so is also the composition $g \circ f : A \to C$.

(b) If $g \circ f$ is injective, so is also f.

(c) If $g \circ f$ is injective and f is surjective, then g is injective.

(d) Show through an example that the condition "f is surjective" in (c) cannot be dropped.

please turn over

Question 3

Let M, N be sets and $f : M \to N$ be a map. Moreover, let A and B be subsets of M, and let C and D be subsets of N. Prove or disprove (through a counterexample) the following statements:

(a) $f(A \cup B) = f(A) \cup f(B)$

(b) $f(A \cap B) = f(A) \cap f(B)$

(c) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$

(d) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

Here, for instance, $f^{-1}(C)$ denotes the pre-image (also called inverse image) of C under the map f, namely the set

$$f^{-1}(C) = \{x \in M \mid f(x) \in C\}.$$

∗ (e) For the false statements indicate which inclusion symbol must replace the equality symbol, in order to get a true statement. Then give a proof of these corrected statements.

Question 4

Formulate the following propositions using the quantifiers $\forall$ and $\exists$. Then write down the formal negation of such propositions. Translate these negated proposition back to "colloquial language". Here $I \subset \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is a function.

(a) For every $x_0 \in I$ and every $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x \in I$ with $|x - x_0| < \delta$, we have that $|f(x) - f(x_0)| < \varepsilon$.

(b) For every $\varepsilon > 0$, there is a $\delta > 0$ such that for every $x_0 \in I$ and every $x \in I$ with $|x - x_0| < \delta$, we have that $|f(x) - f(x_0)| < \varepsilon$.

Remark: Here we are dealing with the definition of **continuity** and **uniform continuity** respectively, which we will get to know in detail later on.

∗ **Bonus question**

Here is a list of five propositions, which refer to each other. Which of these propositions are true, which are false?

(i) Exactly one proposition from this list is false.

(ii) Exactly two propositions from this list are false.

(iii) Exactly three propositions from this list are false.

(iv) Exactly four propositions from this list are false.

(v) Exactly five propositions from this list are false.