Exercises for Analysis I, WWU Münster, Mathematisches Institut, WiSe 2015/16
P. Albers, K. Halupczok **Sheet Nr. 2**

Hand in by Thursday, November 5, 2015 at 08:30 in the corresponding mail-box in the Hörsaalgebäude (numbers of the mail-boxes of the exercise groups on the web-page of the course).

Question 1

If $n, k \in \mathbb{N}$ and $1 \leq k \leq n$, let $M(n,k)$ be the number of different ways to choose k numbers in $\{1, \dots, n\}$ in such a way that any two of the k numbers are not contiguous. (Thus, for instance, the choice $\{1, 3, 4, 8\}$ is forbidden, as 3 and 4 are contiguous.)

(a) Show that for all $n, k \in \mathbb{N}$ with $2 \leq k \leq n$ the following equality holds:

$$M(n+1, k) = M(n, k) + M(n-1, k-1).$$

Hint: divide the subsets of $\{1, \dots, n\}$ of k elements into two groups: those which contain the element $n+1$ and those which do not contain it.

(b) Prove, making use of mathematical induction, that $M(n,k) = \binom{n-k+1}{k}$ for all $n, k \in \mathbb{N}$ with $1 \leq k \leq n$.

Question 2

Show, using mathematical induction, that for all $n \in \mathbb{N}$ there holds:

(a) $\quad (x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ for all $x \in \mathbb{R}$, $\qquad$ (b) $\quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

Question 3

Let $x \in \mathbb{R} \setminus \{1\}$ and let $\ell, n \in \mathbb{N}$. Derive a formula for $\sum_{k=\ell}^{n} x^k$ in three different ways, namely

(a) by reduction to $x^\ell \cdot \sum_{k=\ell}^{n} x^{k-\ell}$, $\quad$ (b) by reduction to $\sum_{k=0}^{n-\ell} x^{k+\ell}$,

(c) by using $\sum_{k=\ell}^{n} x^k = \sum_{k=0}^{n} x^k - \sum_{k=0}^{\ell-1} x^k$.

Question 4

We want to prove that all natural numbers are equal, e. g. $3 = 7$. To this purpose let us define for $a, b \in \mathbb{N}$ the maximum $\max(a,b)$ as the largest of the two numbers a, b. For $a = b$ we have $\max(a,b) := a = b$. Let $\mathcal{A}_n$ be the proposition: "If $a, b \in \mathbb{N}$ with $\max(a,b) = n$, then $a = b$.".

(i) **Base case:** $\mathcal{A}_1$ is true, since $a = b = 1$ implies $\max(a,b) = 1$.

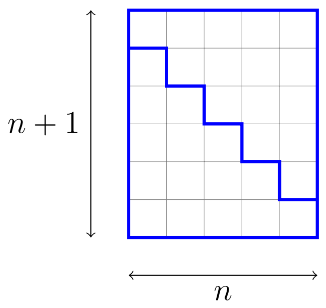
(ii) **Inductive step:** Let us suppose that $\mathcal{A}_n$ is true for a particular $n \in \mathbb{N}$. Let $a, b \in \mathbb{N}$ with $\max(a,b) = n+1$. Let us set $\alpha = a-1$, $\beta = b-1$. Then we have $\max(\alpha, \beta) = n$ and, therefore, $\alpha = \beta$, as $\mathcal{A}_n$ holds true. This implies $a = b$ and, as a consequence, also $\mathcal{A}_{n+1}$ holds true.

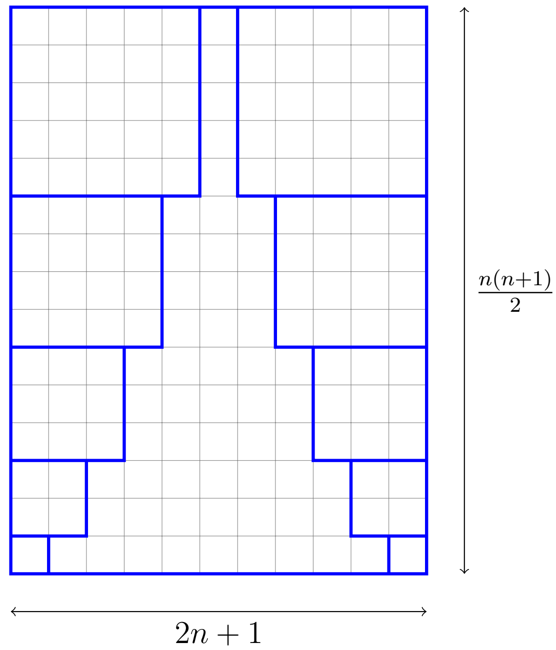
Let now $a, b \in \mathbb{N}$ be arbitrary, $r := \max(a,b)$. Since $\mathcal{A}_n$ is true for all $n \in \mathbb{N}$, in particular $\mathcal{A}_r$ is true and, therefore, $a = b$. Where is the fallacy in the argument?

please turn over

∗ **Bonus question**

The induction method in Question 2 (b) has clearly the disadvantage that one has to know in advance the formula for the sum, in order to prove it. Derive the formula $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ of the "small Gauß" and the formula in Question 2 (b) using the following pictures.





∗ **Puzzle**

Let a regular polygon with n sides be given. How many are the triangles, whose vertices are also vertices of the polygon, but whose sides are not sides of the polygon? For example, for $n = 7$ there are seven such triangles.