Hand in by Thursday, November 12, 2015 at 08:30 in the mail-box in the Hörsaal-gebäude.

Question 1

For $a, b \in \mathbb{R}$ with $a \leq b$ we define the sets

$$\begin{aligned} [a,\infty) &:= \{x \in \mathbb{R},\ a \leq x\}, & (a,\infty) &:= \{x \in \mathbb{R},\ a < x\}, \\ (-\infty,a] &:= \{x \in \mathbb{R},\ a \geq x\}, & (-\infty,a) &:= \{x \in \mathbb{R},\ a > x\}, \\ [a,b] &:= \{x \in \mathbb{R},\ a \leq x \leq b\}, & [a,b) &:= \{x \in \mathbb{R},\ a \leq x < b\}, \\ (a,b] &:= \{x \in \mathbb{R},\ a < x \leq b\}, & (a,b) &:= \{x \in \mathbb{R},\ a < x < b\}. \end{aligned}$$

Any of these sets is called a real *interval*. For $a = b$ the sets $[a, a)$, $(a, a]$ and (a, a) are empty, while $[a, a] = \{a\}$. Write the following sets as a union of real intervals and justify rigorously your answer:

(a) $\left\{x \in \mathbb{R} \setminus \{1, -1\} \;\middle|\; \dfrac{x-1}{x^2-1} > x\right\}$,

(b) $\left\{x \in \mathbb{R} \setminus \left\{-\dfrac{1}{2}, \dfrac{1}{2}\right\} \;\middle|\; \dfrac{1}{4} - \dfrac{x^2-1}{4x^2-1} < \varepsilon\right\}$ for some real number $\varepsilon > 0$.

Question 2

Deduce from the axioms of $\mathbb{R}$ the following statements, which were already mentioned during the lectures. Here a, b, a', b' are real numbers.

1. $a > b,\ a' > b' \Longrightarrow a + a' > b + b'$
2. $a > b > 0,\ a' > b' > 0 \Longrightarrow aa' > bb'$
3. $-(-a) = a$
4. $(a^{-1})^{-1} = a$ for $a \neq 0$
5. $-(a + b) = -a - b$
6. $(ab)^{-1} = a^{-1}b^{-1}$ for $ab \neq 0$
7. $-0 = 0$ and $1^{-1} = 1$
8. $a > 0 \Longrightarrow a^{-1} > 0$

Question 3

If $x, y \in \mathbb{R}^+$ we let

$$A(x,y) := \frac{x+y}{2}, \qquad G(x,y) := \sqrt{xy}, \qquad H(x,y) := \frac{2}{x^{-1}+y^{-1}}$$

define the so-called arithmetic, geometric and harmonic means of x and y.

(a) Using the order axioms show that $H(x, y) \leq G(x, y) \leq A(x, y)$.

(b) If one covers first a path of length s with speed x and then another path of length s with speed y, which of the three means above describes the average speed along the total path?

please turn over

Question 4

Deduce from the field axioms of $\mathbb{R}$ the following formulas for making computations with fractions, where $\frac{a}{b} := ab^{-1}$ for $a \in \mathbb{R}$, $b \in \mathbb{R}^*$. Specify at each step, which axiom you use.

(i) $\dfrac{a}{b} = \dfrac{c}{d} \iff ad = bc \quad (b, d \neq 0)$

(ii) $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd} \quad (b, d \neq 0)$

(iii) $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd} \quad (b, d \neq 0)$

(iv) $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{ad}{bc} \quad (b, c, d \neq 0)$

$*$ Bonus question

Let a_0 and b_0 be real numbers with $0 < b_0 \leq a_0$, and for all $n \in \mathbb{N}_0$ let us set

$$a_{n+1} := \frac{a_n + b_n}{2}, \quad b_{n+1} := \sqrt{a_n b_n}.$$

Show that for all $n \in \mathbb{N}_0$ the inequalities

$$b_n \leq b_{n+1} \leq a_{n+1} \leq a_n$$

hold and that

$$0 \leq a_{n+1} - b_{n+1} = \frac{(\sqrt{a_n} - \sqrt{b_n})^2}{2} = \frac{(a_n - b_n)^2}{2(\sqrt{a_n} + \sqrt{b_n})^2}.$$

$*$ Puzzle

Two old ladies leave their respective villages at sunrise and each of them walks in the direction of the village of the other. Both proceed with constant speed. At noon they meet and keep walking past each other without stopping. One lady reaches the village of the other at 16:00, the other lady reaches the village of the first at 21:00. At what time of that day did the sun rise?

Hint: The intellectual fun of this elementary question consists of finding a method of solution that can be immediately carried out in your mind.