Hand in by Thursday, November 19, 2015 at 08:30 in the mail-box in the Hörsaal-gebäude.

### Question 1

For which $x \in \mathbb{R}$ the following inequalities are true?

(a) $|x^2 - 1| < 2x$,

(b) $\dfrac{|x-3|}{1-|x-1|} < 1$ for $x \neq 0, 2$.

Write your answer as a union of real intervals and justify it rigorously.

### Question 2

Do the following subsets of $\mathbb{R}$ possess a supremum, an infimum, a maximum or a minimum, respectively? Justify carefully your answer and compute explicitly these quantities in the cases where they exist.

(a) $\left\{ \dfrac{|x|}{1+|x|} \;\middle|\; x \in \mathbb{R} \right\}$

(b) $\left\{ \dfrac{x}{1+x} \;\middle|\; x > -1 \right\}$

(c) $\left\{ x + \dfrac{1}{x} \;\middle|\; \dfrac{1}{2} < x \leq 2 \right\}$

(d) $\left\{ \dfrac{1}{(-3)^m} + \dfrac{n}{2n-1} \;\middle|\; m \in \mathbb{N}_0,\ n \in \mathbb{N} \right\}$

### Question 3

Let $A, B \subset \mathbb{R}$ be two non-empty subsets of $\mathbb{R}$ bounded from above. Show that the set

$$A + B := \{x + y \mid x \in A,\ y \in B\}$$

is also bounded from above and that the following relation holds

$$\sup(A+B) = \sup(A) + \sup(B).$$

### Question 4

(a) Let $\ell, k \in \mathbb{N}_0$ and $A_0, \ldots, A_\ell, B_0, \ldots, B_k \in \mathbb{R}$. Let also $a_n := A_\ell n^\ell + A_{\ell-1} n^{\ell-1} + \cdots + A_0$ and $b_n := B_k n^k + B_{k-1} n^{k-1} + \cdots + B_0$ for all $n \in \mathbb{N}$. Let us assume in addition that $A_\ell, B_k \neq 0$.

Show that the sequence $\left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}}$ converge if and only if $k \geq \ell$ and that in this case

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \begin{cases} \frac{A_\ell}{B_\ell}, & \text{falls } k = \ell, \\ 0, & \text{falls } k > \ell. \end{cases}$$

(b) Let us denote by $i$ the imaginary unit in $\mathbb{C}$. Determine whether the two sequences

$$i^n + \frac{1}{n} \quad \text{und} \quad \sqrt{n+1} - \sqrt{n}.$$

converge or not and compute the value of the limit, if it exists.

please turn over

∗ **Bonus question**

Construct an explicit bijection between the open real interval $(0, 1)$ and $\mathbb{R}$.

Note: In your construction you are not allowed to use any function that was not discussed in the lectures, such as for example the function arctan.

∗ **Puzzle**

(a) The "Hilbert" Hotel has a countably infinite set of rooms. At the moment of your arrival at the hotel the rooms are unfortunately all already occupied. Nevertheless the hotel receptionist finds a room for you by cleverly redistributing the guests in such a way that no guest has to leave the hotel or has to share the room with someone else. How is that possible?

(b) Now also a couch of Canto(u)rs arrives with a countably infinite set of passengers. Can these guests be accommodated, as well?