**Exercises for Analysis I, WWU Münster, Mathematisches Institut, WiSe 2015/16**
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Hand in by Thursday, November 26, 2015 at 08:30 in the mail-box in the Hörsaal-gebäude.

## Question 1

Determine the limit value and the accumulation points of the real sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$, $(d_n)_{n\in\mathbb{N}}$, whenever these values exist. If the limit value exist, determine for every $\varepsilon > 0$ an explicit $N_0 \in \mathbb{N}$ such that the elements of the sequence after the $N_0$-th differ at most by $\varepsilon$ from the limit value. Determine also the supremum and the infimum of the images of the sequences in $\mathbb{R}$.

$$a_n := \left(1 - \frac{1}{n^2}\right)^n, \qquad b_n := \begin{cases} \frac{1}{n}, & \text{if } n \text{ odd}, \\ 1, & \text{if } n \text{ even}, \end{cases} \qquad c_n := (-1)^n \cdot \sqrt[n]{n} + \frac{1}{n},$$

$$d_n := \begin{cases} \frac{1}{n^2}, & \text{if there exists } k \in \mathbb{N} \text{ with } n = 3k, \\ 1 - \frac{1}{n}, & \text{if there exists } k \in \mathbb{N}_0 \text{ with } n = 3k+1, \\ 2 + \frac{n+1}{n}, & \text{if there exists } k \in \mathbb{N}_0 \text{ with } n = 3k+2. \end{cases}$$

## Question 2

Prove the following fact: A real sequence $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if and only if

$$\forall\, \varepsilon > 0\ \exists\, m \in \mathbb{N}\ \forall\, k \in \mathbb{N}_0 :\ |a_{m+k} - a_m| < \varepsilon.$$

## Question 3

(a) Using the Cauchy criterion, show that the sequence $(a_n)_{n\in\mathbb{N}}$ defined recursively by

$$a_1 := 1,\ a_{n+1} := \frac{2 + a_n}{1 + a_n} \text{ for all } n \in \mathbb{N}$$

is convergent.

Hint: Observe that for all $n \in \mathbb{N}$, there holds $1 \leq a_n \leq 2$.

(b) Determine the limit value $c \in \mathbb{R}$ of the sequence.

Hint: 1.) In general it holds: if $\lim_{n\to\infty} a_n = c$, then also $\lim_{n\to\infty} a_{n+1} = c$,

2.) use/prove that $(a_n)_{n\in\mathbb{N}}$ and $\left(\frac{2+a_n}{1+a_n}\right)_{n\in\mathbb{N}}$ have the same limit value.

∗ (c) Determine with the ideas of this question the limit value of

$$1 + \frac{1}{1 + \frac{1}{1+\dots}} = \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}.$$

## Question 4

Show with the aid of the Cauchy criterion

(a) the divergence of the harmonic series $(a_n)_{n\in\mathbb{N}}$ with $a_n := \sum_{k=1}^{n} \frac{1}{k}$ for $n \in \mathbb{N}$,

(b) the convergence of the series $(b_n)_{n\in\mathbb{N}}$ with $b_n := \sum_{k=2}^{n} \frac{1}{k(k-1)}$ for $n \in \mathbb{N}$, (Hint: Write $b_n - b_m$ as the difference of two partial sums of the harmonic series.)

(c) the convergence of the sequence $(c_n)_{n\in\mathbb{N}}$ with $c_n := \sum_{k=1}^{n} \frac{1}{k^2}$ for $n \in \mathbb{N}$.

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* **Bonus question**

Show that the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ from the Bonus question in Sheet 3 converge towards a common limit value $M = M(a_0, b_0)$.

Furthermore, show that the estimate

$$|a_{n+1} - M| \leq C|a_n - M|^2 \qquad \text{(and likewise } |b_{n+1} - M| \leq C|b_n - M|^2\text{)}$$

holds for some real number $C > 0$ and all $n \in \mathbb{N}$.

*Observation:*

(1) The meaning of the estimate is that at every step the number of decimal digits, that the elements of the sequences have in common with the limit value, approximately doubles.

(2) For the starting values $a_0 := 1$, $b_0 := \frac{1}{\sqrt{2}}$ let us define $c_n := \sqrt{a_n^2 - b_n^2}$ and

$$\pi_n := \frac{2a_{n+1}^2}{1 - \sum_{k=0}^{n} 2^k c_k^2}$$

for all $n \in \mathbb{N}_0$. It can be shown that

$$0 < \pi - \pi_{n+1} \leq \frac{(\pi - \pi_n)^2}{2^{n+1}\pi^2}$$

for all $n \in \mathbb{N}$. This method computes numerically $\pi$ in one of the fastest ways known at the present.

* **Puzzle**

Consider the harmonic series and erase from it all the summands whose denominator contains the digit 9 in its decimal representation. Show that the partial sums obtained in this way converge.

Hint: First show that the number of the summands whose denominator lies between$10^{m-1} - 1$ and $10^m - 1$ is equal to $9^m - 9^{m-1}$. Thereby, prove that the limit of the series is bounded from above.