Exercises for Analysis I, WWU Münster, Mathematisches Institut, WiSe 2015/16
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Hand in by Thursday, December 3, 2015 at 08:30 in the mail-box in the Hörsaal-gebäude.

Question 1

Let $(a_n)_{n\in\mathbb{N}}$ be some enumeration of $(0,1)\cap\mathbb{Q}$. Show that $\limsup_{n\to\infty} a_n = 1$ and $\liminf_{n\to\infty} a_n = 0$ by checking the corresponding definitions. Moreover, exhibit a concrete sub-sequence with limit 1 and a concrete sub-sequence with limit 0.

Question 2

Show the following characterization of the limit inferior:
We have $\liminf_{n\to\infty} a_n = b \in \mathbb{R}$ if and only if for all $\varepsilon > 0$ it holds:

i) $a_n > b - \varepsilon$ for all but finitely many $n \in \mathbb{N}$, and
ii) $a_n < b + \varepsilon$ for infinitely many $n \in \mathbb{N}$.

Hint: This statement is Part 2) from Theorem 10 in the lectures. Prove it following the argument used in Part 1) of the theorem and filling in the missing steps.

Question 3

Let $(a_n)_{n\in\mathbb{N}}$ be a bounded real sequence. Prove the following statements:

(a) $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$,

(b) $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ are accumulation points of the sequence,

(c) $\limsup_{n\to\infty} a_n = \sup\{c \in \mathbb{R} \mid c \text{ is an accumulation point of } (a_n)_{n\in\mathbb{N}}\}$.
(A corresponding statement holds for $\liminf_{n\to\infty} a_n$, but you do not need to prove it here.)

Question 4

Study the continuity of the following functions and prove your assertion:

(a) $f_1 : \mathbb{R} \to \mathbb{R}, \quad f_1(x) := \begin{cases} 1, & \text{if } x \leq 1, \\ 0, & \text{if } x > 1. \end{cases}$

(b) $f_2 : \mathbb{R}\setminus\{1\} \to \mathbb{R}, \quad f_2(x) := \begin{cases} 1, & \text{if } x \leq 1, \\ 0, & \text{if } x > 1. \end{cases}$

(c) $f_3 : \mathbb{R} \to \mathbb{R}, \quad f_3(x) := |x|$.

please turn over

* **Bonus question**

  A function $f : \mathbb{R} \to \mathbb{R}$ is called *monotone increasing*, if for all $x, y \in \mathbb{R}$ the implication $x \leq y \implies f(x) \leq f(y)$ holds true.

  (a) Let $f : \mathbb{R} \to \mathbb{R}$ be monotone increasing and $M \subset \mathbb{R}$ be bounded from above. Show that also $f(M)$ is bounded from above and that $\sup(f(M)) \leq f(\sup(M))$ holds.

  (b) Exhibit a monotone increasing function $g : \mathbb{R} \to \mathbb{R}$ and a set $M \subset \mathbb{R}$ bounded from above such that $\sup(g(M)) < g(\sup(M))$ holds.

  (c) Exhibit a function $h : \mathbb{R} \to \mathbb{R}$ and a set $M \subset \mathbb{R}$ bounded from above such that $h(M)$ is bounded from above, but $\sup(h(M)) > h(\sup(M))$ holds.

* **Puzzle**

  Let $D \subset \mathbb{R}$. We call $f : D \to D$ a *contraction*, if $|f(x) - f(y)| < |x - y|$ holds, for all $x, y \in D$ with $x \neq y$.

  Lipschitz-continuous functions $f : D \to D$ with Lipschitz constant $L < 1$ are contractions. Show with an example that the converse is not necessarily true.