Hand in by Thursday, December 10, 2015 at 08:30 in the mail-box in the Hörsaal-gebäude.

Question 1

(a) Construct a function $f : D \to \mathbb{C}$, $D \subset \mathbb{C}$, and a convergent sequence $(x_n)_{n\in\mathbb{N}} \subset D$ with limit $x_0 \in D$ in such a way that $\lim\limits_{n\to\infty} f(x_n) = f(x_0)$, but f is not continuous at x_0. Why this does not contradict the continuity criterion given in terms of sequences, which was proven in the lectures?

(b) Determine whether the functions $f, g : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0, \end{cases} \qquad g(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

are continuous at $x_0 = 0$.

Question 2

Reminder: A function $f : D \to \mathbb{C}$, $D \subset \mathbb{C}$, is called *continuous* exactly when the following statement holds:

$$\forall\, x_0 \in D\ \forall\, \varepsilon > 0\ \exists\, \delta = \delta(x_0, \varepsilon) > 0\ \forall\, x \in D : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

If the δ in the statement does *not* depend on x_0, we call the function *uniformly continuous.*

Show that: The function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $f(x) := \sqrt{x}$, is uniformly continuous,
but the function $f : \mathbb{R}_{>0} \to \mathbb{R}$, $f(x) := \frac{1}{x}$ it is not.

Question 3

Let $M \subset \mathbb{R}^n$ be a non-empty set. Show that:

(a) The set $\mathring{M} = M \setminus \partial M$ is open.

(b) For every open set U with $U \subset M$, there holds $U \subset M \setminus \partial M = \mathring{M}$.

(c) It holds $\mathring{M} = \bigcup \{U \subset \mathbb{R}^n \mid U \subset M \text{ and } U \text{ open}\}$.

Remark (not to be proven): Analogously we have:
(a) $\overline{M} = M \cup \partial M$ is closed,
(b) A closed and $M \subset A \Rightarrow \overline{M} \subset A$,
(c) $\overline{M} = \bigcap \{A \subset \mathbb{R}^n \mid M \subset A \text{ and } A \text{ closed}\}$.

Question 4

Find the boundary ∂M of the following subsets M of $\mathbb{R}^2$ and determine which M are open and which are closed.

(a) $M_1 = B_1((0,0)) \setminus \{(0,0)\}$

(b) $M_2 = [1,2] \times [3,4)$

(c) $M_3 = \{x \in [1,2] \times [3,4] \mid x \in \mathbb{Q}^2\}$

(d) $M_4 = [1,2) \times \{0\}$

please turn over

∗ **Nikolaus Question**

Nikolaus hangs spheres ($\circ$) and tinsels ($-$) to a subset $T \subset \mathbb{R}^2$ so that the sets

$$T, \quad \mathring{T}, \quad \overline{T}, \quad \mathring{\overline{T}}, \quad \overline{\mathring{T}}, \quad \overline{\mathring{\overline{T}}}, \quad \mathring{\overline{\mathring{T}}},$$

are all distinct. Exhibit explicitly one such a set $T \subset \mathbb{R}^2$.