Exercises for Analysis I, WWU Münster, Mathematisches Institut, WiSe 2015/16
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Hand in by Thursday, December 17, 2015 at 08:30 in the mail-box in the Hörsaal-gebäude.

Question 1

Determine which of the following subsets $K_1, \dots, K_4$ of $\mathbb{R}$ are compact.

(a) $K_1 := \{\frac{1}{n} \mid n \in \mathbb{N}\}$

(b) $K_2 := f([0,1])$ under the map $f : \mathbb{R} \setminus \{-3\} \to \mathbb{R}$, $f(x) := \dfrac{x^2 - 6x + 7}{x + 3}$

(c) $K_3 := \{\frac{1}{x} \mid x \in (0,1]\}$

(d) $K_4 := \bigcap_{n \in \mathbb{N}} [-\frac{1}{n}, 1]$

Question 2

Give (possibly distinct) examples of a continuous map $f : \mathbb{R} \to \mathbb{R}$ such that

(a) the pre-image of a connected set is not connected,
(b) the image of an open set is not open,
(c) the image of a closed set is not closed,
(d) the pre-image of a compact set is not compact.

Question 3

(a) Let $d \in \mathbb{N}$ be fixed and for all $n \in \mathbb{N}$ let $K_n \subset \mathbb{R}^d$ be a compact non-empty set such that $K_{n+1} \subset K_n$ for all $n \in \mathbb{N}$. Show that $\bigcap_{n \in \mathbb{N}} K_n \neq \emptyset$ holds.

(b) Construct for some $d \in \mathbb{N}$ and for all $n \in \mathbb{N}$ a closed non-empty set $A_n \subset \mathbb{R}^d$ in such a way that $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$, but $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$ holds.

Question 4

Prove Lemma 4': Let $n \in \mathbb{N}$ and $D \subset \mathbb{R}^n$ be given. Then the following statement holds:

$$M \subset D \text{ open in } D \Longleftrightarrow \forall\, x \in M\ \exists\, r > 0 : B_r(x) \cap D \subset M.$$

please turn over

* **Bonus question**

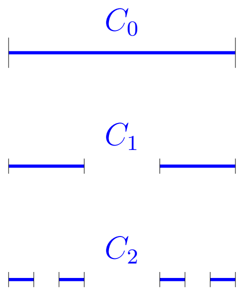
Let us define $C_0 := [0, 1]$, and

$$C_n = \bigcup_{k=1}^{2^n} I_{n,k}\ ,$$

where the $I_{n,k}$ are 2^n pairwise disjoint closed intervals of length 3^{-n} such that C_{n+1} is the union of all disjoint closed intervals $I_{n+1,2k-1}$, $I_{n+1,2k}$ obtained by deleting the open middle third from $I_{n,k}$ of C_n. Namely,

$$C_{n+1} := \bigcup_{k=1}^{2^n} (I_{n+1,2k-1} \cup I_{n+1,2k}) = \bigcup_{k=1}^{2^{n+1}} I_{n+1,k}.$$

Sketch:



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In this way we construct recursively a sequence $C_0, C_1, \ldots$ of subsets of $\mathbb{R}$, and we define $C := \bigcap_{n\in\mathbb{N}} C_n$. This set is called *Cantor Set*.

Show that:

(a) $C = \partial C$,

(b) C is uncountable
(for instance by giving a bijection between C and the set of all 0-1-sequences),

(c) every point of C is an accumulation point of C,

(d) the total length of C_n goes to 0 as $n \to \infty$.