Exercises for Analysis I, WWU Münster, Mathematisches Institut, WiSe 2015/16
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Hand in by Thursday, January 14, 2016 at 08:30 in the mail-box in the Hörsaal-gebäude.

Question 1

Justify why the function

$$\begin{aligned} f : \mathbb{R}^+ &\to \mathbb{R}^+ \\ x &\mapsto x^{p/q}, \end{aligned}$$

for $p \in \mathbb{Z}$, $q \in \mathbb{N}$, is differentiable and determine its derivative

(a) by means of the expression $f(x) = (x^{1/q})^p$,

(b) from the identity $(f(x))^q = x^p$.

Question 2

Show that the function

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R}, \\ x &\mapsto \begin{cases} x^2 \cos(1/x), & \text{for } x \neq 0, \\ 0, & \text{for } x = 0, \end{cases} \end{aligned}$$

is differentiable but not continuously differentiable, i. e. the derivative is not a continuous function. (During the lectures we are going to show that sin and cos are differentiable functions on $\mathbb{R}$ with $\sin' = \cos$ and $\cos' = -\sin$.)

Question 3

(a) Draw a sketch of the graphs of the following functions:

(i) $[-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$, $x \mapsto \sin x$,

(ii) $[0, \pi] \to \mathbb{R}$, $x \mapsto \cos x$,

(iii) $(-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, $x \mapsto \tan x := \frac{\sin x}{\cos x}$.

What is the image of these functions?

(b) Justify why the inverse functions

$$\arcsin := \sin^{-1}, \ \arccos := \cos^{-1}, \ \arctan := \tan^{-1}$$

exist over such images. Here, for instance, $\arcsin y$ is the angle x (in radians) such that $\sin x = y$. Remember that the measure of an angle in radians is, by definition, the length of the corresponding arc on the unit circle.

(c) Determine the derivative of the tangent function.

(d) Determine the derivative of the functions arcsin, arccos and arctan.

Question 4

Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be continuous. Show that f is injective if and only if f is strictly monotone.