Hand in by Thursday, January 21, 2016 at 08:30 in the mail-box in the Hörsaal-gebäude.

Question 1

Let $\tau : \mathbb{N}_0 \to \mathbb{N}_0$ be a *bounded reordering*, i. e. τ is bijective and there exists $d \in \mathbb{N}_0$ with the property that

$$|\tau(n) - n| \leq d \text{ for all } n \in \mathbb{N}_0.$$

Without using the Reordering Theorem prove that: A series $\sum_{n=0}^{\infty} a_n$, with $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}_0$, is absolutely convergent if and only if the series $\sum_{n=0}^{\infty} a_{\tau(n)}$ is absolutely convergent.

Question 2 (Root test)

Let $\sum_{n=0}^{\infty} a_n$ be a series with $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}_0$.

(a) Suppose that there exists $\theta \in \mathbb{R}$ with $0 < \theta < 1$ and $n_0 \in \mathbb{N}$ such that

$$\sqrt[n]{|a_n|} \leq \theta \text{ for all } n \geq n_0.$$

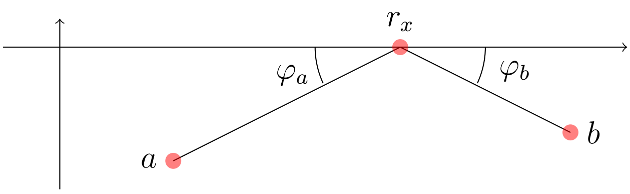
Show that the series is absolutely convergent.

(b) Suppose that $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$. Show that the series is absolutely convergent.

Question 3

The points $a, b \in \mathbb{R}^2$ are connected by a ray of light, which is reflected by a mirror parallel to the x-axis at a point $r_x = (x, 0) \in \mathbb{R}^2$. Let $L(x)$ be the length of the whole path covered by the ray, i.e. $L(x) = |r_x - a| + |r_x - b|$, and let φ_a and φ_b be the angles (with values in $[0, \frac{\pi}{2}]$), that the ray forms with the mirror in the direction of a and of b, respectively.

Sketch:



The points a and b can have different y-coordinate, as the picture on the left shows.

(a) Show that: $L'(x) = 0 \;\Leftrightarrow\; \varphi_a = \varphi_b$.

(b) Does the function L attain a maximum or a minimum at the point x with $L'(x) = 0$?

Question 4

Determine the following limit values with the help of l'Hospital Rule:

$$\text{(a)} \lim_{x\to 0} \frac{1 - \cos\frac{x}{2}}{1 - \cos x} \qquad \text{(b)} \lim_{x\to 0} \frac{\sin x - x\cos x}{x \sin x}$$

Hint: Use that l'Hospital Rule can be applied multiple times. However, each time you apply it, do not forget to check that the hypotheses of the rule are satisfied (in relation to this see the bonus question on the back).

please turn over

* Bonus question

What problems arise in determining the following limits with the aid of l'Hospital Rule? How can you circumvent such problems?

$$\text{(i)} \lim_{x\to 0+} \frac{\sin x + \cos x}{x} \qquad \text{(ii)} \lim_{x\to 0} \frac{x^2\cos(1/x)}{\sin x}$$

(a) Show that the limit in (i) does not exist, even if an inaccurate application of l'Hospital Rule would yield a quotient which tends to 1 as $x \to 0$. Which of the hypotheses of the theorem about l'Hospital Rule is not satisfied here?

(b) Show that trying to use l'Hospital Rule to compute the limit in (ii) leads nowhere. Prove, using a different argument, that the desired limit exists and it is equal to 0.

(c) Formulate l'Hospital Rule for a limit with $x \to \infty$ and a quotient whose numerator and denominator tend to ∞.

(d) For $f(x) = x + \sin x \cos x$, $g(x) = f(x)e^{\sin x}$ the limit $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ does not exist, even if

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{2\cos x}{x + \sin x \cos x + 2\cos x} e^{-\sin x} = 0.$$

Why cannot we apply l'Hospital Rule here?