Exercises for Analysis I, WWU Münster, Mathematisches Institut, WiSe 2015/16
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Hand in by Thursday, January 28, 2016 at 08:30 in the mail-box in the Hörsaal-gebäude.

Question 1

Let $a > 0$ be given. Prove the following properties of the generalized exponential function with base a:

(a) $a^x : \mathbb{R} \to \mathbb{R}^+$ is continuous.

(b) $a^{x+y} = a^x a^y$ for all $x, y \in \mathbb{R}$.

(c) $a^{\frac{p}{q}} = \sqrt[q]{a^p}$ for all $q \in \mathbb{N}$, $q \geq 2$, $p \in \mathbb{Z}$.

(d) $(a^x)^y = a^{xy}$ for all $x, y \in \mathbb{R}$.

(e) $a^x b^x = (ab)^x$ for all $x \in \mathbb{R}$, $b > 0$.

(f) $\frac{1}{a^x} = a^{-x}$ for all $x \in \mathbb{R}$.

Show, moreover, that the function a^x is differentiable and compute its derivative.

Question 2

Let $\log_a : \mathbb{R}^+ \to \mathbb{R}$ be the logarithm to base a, i. e. the inverse function of $a^x : \mathbb{R} \to \mathbb{R}^+$, $a > 0$.

(a) Prove that there holds $\log_a(x) = \dfrac{\log x}{\log a}$, for all $x > 0$.

(b) Determine the derivative of $\log_a(x)$ in two ways. First, by applying the theorem about the derivative of the inverse function. Second, by differentiating the equation in (a).

Question 3

Show that the identity $\mathrm{id} : \mathbb{R} \to \mathbb{R},\ x \mapsto x$ is integrable on $[0, 1]$ and there holds $\int_0^1 x dx = \frac{1}{2}$.
Hint: Prove that for all $\varepsilon > 0$ there exists a partition of the form

$$Z_n = (0 = x_0, \frac{1}{n}, \frac{2}{n}, \ldots, 1 = x_n)$$

with $O(Z_n, \mathrm{id}) - U(Z_n, \mathrm{id}) < \varepsilon$.

Remark: The map id is integrable on all intervals $[a, b]$ and there holds $\int_a^b x dx = \frac{b^2 - a^2}{2}$.

Question 4

Use the Intermediate Value Theorem for continuous functions to show the following *Mean Value Theorem of integral calculus*: If $f : [a, b] \to \mathbb{R}$ is continuous, then there exists $c \in [a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b - a).$$