Exercises for Analysis I, WWU Münster, Mathematisches Institut, WiSe 2015/16
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WITHOUT HANDING IN, Discussion with tutors by arrangement on January 11/12/13.

Question 1

Write the following subsets of $\mathbb{R}$ as union of intervals:

(a) $A := \{x \in \mathbb{R};\ |x+2| < |x+7|\}$,

(b) $B := \{x \in \mathbb{R};\ \frac{x}{x+2} > \frac{x+3}{3x+1}\}$,

(c) $C := \{x \in \mathbb{R};\ \forall n \in \mathbb{N} : x^{n+1} + \frac{1}{x^{n+1}} > x^n + \frac{1}{x^n}\}$,

(d) $D := \{\frac{x}{x+1};\ x \in \mathbb{R},\ x > -1\}$.

Determine also the supremum/infimum/maximum/minimum of the sets A to D, when they exist.

Question 2

Let $x, y, z \in \mathbb{R}$. Show that:

(a) $|x+y+z| \le |x| + |y| + |z|$

(b) $|x-y| \ge |x| - |y|$

(c) $x^2 + y^2 + z^2 \ge xy + yz + zx$

(d) $3\sqrt[3]{xyz} \le x + y + z$, falls $x, y, z > 0$

Question 3

(a) Show that for all $n \in \mathbb{N}$ the polynomial $x^{2n-1} + 1$ is divisible by $x + 1$.

(b) Let A_n be the arithmetic mean of the binomial coefficients

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}.$$

Show that $\lim_{n\to\infty} \sqrt[n]{A_n} = 2$ holds.

Question 4

Determine the limits of the sequences given below:

$$a_n := \frac{3n^2 - 5n}{6n^2 + 3n - 2},\ b_n := \frac{n(n+2)}{n+1} - \frac{n^3}{n^2+1},\ c_n := \sqrt{n+2} - \sqrt{n},$$
$$d_n := \left(\frac{2n+1}{3n-4}\right)^4,\ e_n := \frac{1}{n}(g_1 + \cdots + g_n) \text{ with } g_n := \frac{1}{2}(1 + (-1)^n),$$
$$f_{n+1} := \frac{1}{2}\Big(f_n + \frac{p}{f_n}\Big) \text{ with } p, f_1 > 0.$$

please turn over

Question 5

Show that:

(a) Every convergent real sequence is bounded.

(b) For every convergent real sequence $(a_n)_{n\in\mathbb{N}}$ with limit value $A \neq 0$ there is an $N \in \mathbb{N}$ such that $|a_n| > A/2$, for all $n \geq N$.

(c) Every convergent real sequence possesses either a maximum or a minimum, or both.

(d) For every real sequence $(a_n)_{n\in\mathbb{N}}$ the following statement holds: If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $(a_n)_{n\in\mathbb{N}}$ is a null-sequence.

Question 6

Determine the biggest domain $D \subset \mathbb{R}$ such that the following formulae define a function $f : D \to \mathbb{R}$.

(a) $\sqrt{(-x+3)(2x+4)}$, (b) $(x-2)(x^2-4)^{-1}$, (c) $\frac{\sin 3x}{\cos 2x}$, (d) $2^{\sqrt{x^2-1}}$.

Question 7

Let $f : D \to \mathbb{R}$ be a function and let $a \in \mathbb{R}$ be such that $\lim_{x\to a} f(x) = B \in \mathbb{R} \setminus \{0\}$. Show that: There exists a $\delta > 0$, so that $|f(x)| > \frac{1}{2} \cdot |B|$ holds for all $x \in D$ with $|x-a| < \delta$.

Question 8

For which values of the domain of definition are the following functions continuous?

(a) $f(x) = \dfrac{x}{x^2-1}$ (b) $f(x) = \dfrac{1+\cos x}{3+\sin x}$ (c) $f(x) = \dfrac{x-|x|}{x}$

(d) $f(x) = \begin{cases} \frac{x-|x|}{x}, & x < 0 \\ 2, & x = 0 \end{cases}$ (e) $f(x) = \dfrac{x}{\sin x}$ (f) $f(x) = \dfrac{x}{\sin x}$, $f(0) = 1$

Question 9

Let $f, g : \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}$ and suppose that the functions f/g and g are continuous at $x = a$. Show that then f is continuous at a, as well.

Question 10

(a) Construct a non-compact set $D \subset \mathbb{R}$ and an unbounded continuous function $f : D \to \mathbb{R}$.

(b) Construct a non-compact set $D \subset \mathbb{R}$ and a bounded continuous function $g : D \to \mathbb{R}$ that does not have a maximum.