Letzte Notizen zur Ana 2:

$$f(\alpha) := \int_0^1 \cos(\alpha x)\,dx, \qquad f'(1) = ?$$

$u(x)=x \quad v'(x)=\sin(\alpha x)$

$$f'(\alpha) = \frac{d}{d\alpha}\int_0^1 \cos(\alpha x)\,dx = \int_0^1 \frac{\partial}{\partial\alpha}\cos(\alpha x)\,dx = -\int_0^1 x\sin(\alpha x)\,dx$$

- $\cos(\alpha x)$ ist stetig auf $[0,2]\times[0,1]$ ✓
- $\frac{\partial}{\partial\alpha}\cos(\alpha x)$ ist stetig ✓

part. $\int$

$$= + x\cdot\frac{1}{\alpha}\cos(\alpha x)\Big|_0^1 - \int_0^1 1\cdot\frac{1}{\alpha}\cos(\alpha x)\,dx = +\frac{1}{\alpha}\cos(\alpha) - \frac{1}{\alpha^2}\sin(\alpha).$$

- $f(x+\xi) = f(x) + A\cdot\xi + \varphi(\xi)$, z.z.: $\dfrac{\varphi(\xi)}{\|\xi\|} \xrightarrow{\xi\to 0} 0$

- $f(z) = f(a) + \underset{\substack{\uparrow\\ Df(a)}}{A}\cdot(z-a) + \varphi(z-a)$

  z.z.: $\dfrac{\varphi(z-a)}{\|z-a\|} \xrightarrow{z\to a} 0$

Bsp.:

$f: \mathbb{R}^2 \to \mathbb{R}^1$, $f(x,y) = x + 2xy^2$, $a = (1,2)$

Beh.: f ist total diff'bar in a, und

$Df(x,y) = (1+2y^2,\ 4xy)$, also $Df(1,2) = (9,8)$.

Bew.: Für z nahe a gilt: $\qquad f(1,2) = 1 + 2\cdot 4 = 9$

$$f(z) - 9 - (9,8)\cdot(z-(1,2))^T =: \varphi(z-a), \qquad z=(x,y)$$

Nun: $$\frac{\varphi(z-a)}{\|z-a\|} = \frac{x+2xy^2-9-(9,8)\cdot(x-1,y-2)^T}{\|(x-1,y-2)\|}$$

$$= \frac{x + 2xy^2 - 9 - 9(x-1) - 8(y-2)}{\|(x-1, y-2)\|} \quad \text{usw.}$$

$$\|(x-1, y-2)\| = \max\{|x-1|, |y-2|\} \to 0$$

$\rightsquigarrow \quad x \to 1$ und $y \to 2$

Beh.: $4x^3y^2 = o(|xy|^2)$

Bew.: $$\frac{4x^3y^2}{|xy|^2} = 4 \cdot \underbrace{\frac{x^3}{|x^2|}}_{\to 0} \cdot \underbrace{\frac{y^2}{|y|^2}}_{\pm 1 \text{ beschr.}} \xrightarrow[x,y \to 0]{} 0$$

$x \neq o(|x|^3)$, denn $\frac{x}{|x|^3} = \underbrace{\frac{x}{|x|}}_{\pm 1} \cdot \underbrace{\frac{1}{|x|^2}}_{\to \infty}$ div. für $x, y \to 0$

Bsp.: $f: U \to \mathbb{R}^1, U \subset \mathbb{R}^2$

$$f(x+\xi) = \sum_{|\alpha| \le k} \frac{D^\alpha f(x)}{\alpha!} \cdot \xi^\alpha + \sum_{|\alpha| = k+1} \frac{D^\alpha f(x + \theta\xi)}{\alpha!} \cdot \xi^\alpha \quad \leftarrow x \text{ ist Entw.-punkt}$$

$\alpha \in \mathbb{N}_0^n \rightsquigarrow |\alpha| = \alpha_1 + \ldots + \alpha_n$

Vor.: $\forall t \in [0,1]: a + t(z-a) \in U$.

Beh.: $$f(z) = \underbrace{\sum_{|\alpha| \le k} \frac{D^\alpha f(a)}{\alpha!} \cdot (z-a)^\alpha}_{} + \underbrace{\sum_{|\alpha| = k+1} \frac{D^\alpha f(a + \theta(z-a))}{\alpha!} \cdot (z-a)^\alpha}_{}$$

- $|\alpha| = 0 : \alpha = (0,0)$
- $|\alpha| = 1 : \alpha = (1,0), \alpha = (0,1)$
- $|\alpha| = 2 : \alpha = (2,0), (1,1), (0,2)$

- $|\alpha| = 0$, d.h.: $\alpha = (0,0)$: $\quad f(a)$

- $|\alpha| = 1$, $\quad \alpha = (1,0)$: $\quad \dfrac{D^{(1,0)} f(a)}{(1,0)!} (z-a)^{(1,0)} = \dfrac{D_1 f(a)}{1! \cdot 0!} \cdot (z_1 - a_1)^1$

  $\alpha = (0,1)$: $\quad \dfrac{D^{(0,1)} f(a)}{(0,1)!} (z-a)^{(0,1)} = \dfrac{D_2 f(a)}{1} \cdot (z_2 - a_2)^1$

- $|\alpha| = 2$: $\quad \alpha = (2,0)$: $\quad \dfrac{D^{(2,0)} f(a)}{(2,0)!} (z-a)^{(2,0)} = \dfrac{D_1^2 f(a)}{2! \cdot 0!} \cdot (z_1 - a_1)^2$

  $\alpha = (0,2)$: $\quad \dfrac{D^{(0,2)} f(a)}{(0,2)!} (z-a)^{(0,2)} = \dfrac{D_2^2 f(a)}{2! \cdot 0!} \cdot (z_2 - a_2)^2$

  $\alpha = (1,1)$: $\quad \dfrac{D^{(1,1)} f(a)}{(1,1)!} (z-a)^{(1,1)} = \dfrac{D_1 D_2 f(a)}{1! \, 1!} \cdot (z_1 - a_1)(z_2 - a_2)$

Also:

$$f(z) = f(a) + D_1 f(a) \cdot (z_1 - a_1) + D_2 f(a) \cdot (z_2 - a_2)$$
$$+ \frac{D_1^2 f(a)}{2} (z_1 - a_1)^2 + \frac{D_2^2 f(a)}{2} (z_2 - a_2)^2 + D_1 D_2 f(a) (z_1 - a_1) \cdot (z_2 - a_2)$$
$$+ \text{Rest}$$

Satz von Schwarz: $f: U \to \mathbb{R}$, $U \overset{\downarrow}{\subset} \mathbb{R}^2$, 2x stetig dbf

$\Rightarrow \forall\, 1 \leq i,j \leq m: \quad D_i D_j f(x) = D_j D_i (x)$

für alle $x \in U$

Bsp.: $f(x,y) = (\sin x, \; y + x^2 \cos y)$

$$f(x,y) = \underbrace{x^2y+y}, \quad x = x(t) = \sin t, \quad y = y(t) = 1 + \underbrace{t\cos t}$$

$$g(t) = f(x(t), y(t)), \qquad t=0 \rightsquigarrow x(0)=0,\ y(0)=1$$

$$g'(0) = \frac{d}{dt} f(x(t), y(t))\Big|_{t=0} = D_1 f(x,y)\cdot x'(0) + D_2 f(x,y)\cdot y'(0)$$

$$= 2x(0)y(0)\cdot \underbrace{\cos(0)}_{=1} + (x(0)^2+1)\cdot \underbrace{(\cos 0 - 0\sin 0)}_{1}$$

$$= 0 + (0+1)\cdot 1 = \underline{\underline{1}}$$