Equivariant Reidemeister torsion on symmetric spaces

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January 5, 1996

Abstract

We calculate explicitly the equivariant Ray-Singer torsion for all symmetric spaces G/K of compact type with respect to the action of G. We show that it equals zero except for the odd-dimensional Graßmannians and the space $\mathbf{SU}(3)/\mathbf{SO}(3)$. As a corollary, we classify up to diffeomorphism all isometries of these spaces which are homotopic to the identity; also, we classify their quotients by finite group actions up to homeomorphism.

Contents

1 Introduction

In 1935, Franz and Reidemeister [9] established the following classification of lens spaces :

Theorem 1 (Franz, Reidemeister) *Let* Γ_1, Γ_2 *be cyclic groups acting isometrically and freely on the spheres* S^{2n-1}, $n > 1$. *Then the quotients* $\Gamma_1 \backslash S^{2n-1}$, $\Gamma_2 \backslash S^{2n-1}$ *are diffeomorphic iff they are isometric, i.e. if* Γ_1 *and* Γ_2 *are conjugate in* $\mathbf{O}(2n)$.

To prove this theorem they invented a real-valued combinatorial invariant of CW-complexes which is fine enough to distinguish the lens spaces, the Reidemeister torsion. Their result was generalized by de Rham [8] in 1964:

Theorem 2 (de Rham) *Two isometries of* S^n *are diffeomorphic iff they are isometric.*

Here, two transformations g_1, g_2 of a manifold M are called diffeomorphic (resp. isometric) iff there exists a diffeomorphism (resp. an isometry) ϕ of M with $\phi g_1 = g_2 \phi$. In 1973, Chapman [5] proved that the Reidemeister torsion is in fact a homeomorphism invariant; thus theorem 1 holds with "diffeomorphic" replaced by "homeomorphic". In contrast, there exist counterexamples showing that this is not true for theorem 2 [4]. The first purpose of this article is to prove the following result:

Theorem 3 *Two isometries homotopic to the identity of an odd-dimensional Graßmannian* $G_{2m,2p-1}(\mathbf{R}) = \mathbf{SO}(2m)/\mathbf{SO}(2p-1) \times \mathbf{SO}(2m-2p+1)$ *or of* $\mathbf{SU}(3)/\mathbf{SO}(3)$ *are diffeomorphic iff they are isometric.*

Note that the connected component of the identity in the isometry group of $G_{2m,2p-1}$ and $\mathbf{SU}(3)/\mathbf{SO}(3)$ is given by $\mathbf{SO}(2m)$ and $\mathbf{SU}(3)$, resp. The isometry group consists of two or four copies of these groups [18].

The proof is given by calculating explicitly the equivariant Ray-Singer torsion for all compact symmetric spaces. The Ray-Singer torsion is defined as the derivative at zero of a certain zeta function associated to the spectrum of the Laplace operator on differential forms on a compact Riemannian manifold [17]. It has been determined by Ray for the lens spaces in an extensive calculation by determining first the eigenvalues and eigenspaces of the Laplacian on spheres. He found that the Reidemeister and Ray-Singer torsions are equal for these spaces. Using this result, Cheeger [6] and Müller [15] proved independently in 1978 the equality of the Reidemeister torsion and the Ray-Singer torsion. The second aim of our paper is to give a new, shorter proof of Ray‘s result.

Using the equality of Reidemeister torsion and Ray-Singer torsion and the topological invariance of the Reidemeister torsion, we show in the last section

Theorem 4 *Let $\Gamma_1, \Gamma_2 \subset \mathbf{SO}(2m)$ be finite groups acting isometrically and freely on an odd-dimensional Graßmannian $G_{2m,2p-1}(\mathbf{R})$ $(m>1)$. Then the quotients $\Gamma_1\backslash G_{2m,2p-1}(\mathbf{R})$ and $\Gamma_2\backslash G_{2m,2p-1}(\mathbf{R})$ are homeomorphic iff they are isometric (more precisely: iff Γ_1, Γ_2 are conjugate in $\mathbf{O}(2m)$).*

Let $\Gamma_1, \Gamma_2 \subset \mathbf{SU}(3)$ be finite groups acting isometrically and freely on $\mathbf{SU}(3)/\mathbf{SO}(3)$. Then the quotients are homeomorphic iff they are isometric (i.e. iff Γ_1 is conjugate in $\mathbf{SU}(3)$ to Γ_2 or $\overline{\Gamma_2}$).

Note that fixed-point free isometries of the above spaces are necessarily homotopic to the identity, except for $G_{4p-2,2p-1}$. Also, there are only a few sporadic fixed-point free isometries on even-dimensional Graßmannians, none of them homotopic to the identity [18, Th. 9.3.1]. Thus, the above theorem gives a rather complete topological classification of quotients of Graßmannians.

The equivariant Ray-Singer torsion associated to an isometry g acting on M has been investigated by Lott and Rothenberg. They compared it with an equivariant Reidemeister torsion for finite group actions. Using Ray‘s calculation, they found that the equivariant torsion for spheres is mainly given by sums of the digamma function; this enabled them to give a new proof of theorem 2 for orientation-preserving actions on odd-dimensional spheres. We shall apply their method to deduce theorem 3 from our result for the torsion.

Also, we shall show that the equivariant torsion equals zero for all symmetric spaces G/K with respect to the action of any $g \in G$, except for products of $G_{2m,2p-1}(\mathbf{R})$ or $\mathbf{SU}(3)/\mathbf{SO}(3)$ with some G'/K' so that G' and K' have the same rank. A similar result has been shown by Moscowici and Stanton for locally symmetric spaces of the noncompact type [14].

Our method to obtain the value of the torsion is similar to the one used in a previous paper on holomorphic Ray-Singer torsion on Hermitian symmetric spaces [12]. For a symmetric space G/K an eigenvalue of the Laplacian is determined by its eigenspace as a G-representation. This reduces the problem of determining the zeta function to a problem in finite-dimensional representation theory. Nevertheless, there are big differences between the real and the holomorphic situation: In the complex case, the equivariant torsion is always non-trivial and depends in a rather subtle way on the fixed-point set of the isometry $g \in G$, in sharp contrast to our result theorem 11.

2 Equivariant Ray-Singer metrics

Let F be a complex flat hermitian vector bundle over a compact oriented Riemannian manifold M. Let

$$d : \Gamma(\Lambda^q T^* M \otimes F) \to \Gamma(\Lambda^{q+1} T^* M \otimes F)$$

denote the de Rham operator with coefficients in F and let d^* denote its formal adjoint with respect to the L^2-metric. Consider the Hodge-Laplacian $\Delta_q := (d + d^*)^2$ acting on q-forms with coefficients in F. We denote the eigenspace of Δ_q corresponding to an eigenvalue $\lambda \in \mathrm{Spec}\ \Delta_q$ by $\mathrm{Eig}_\lambda(\Delta_q)$. Let g be an isometry of M preserving the hermitian bundle F. Consider the zeta function

$$Z_g(s) := \sum_{q>0} (-1)^q q \sum_{\substack{\lambda \in \mathrm{Spec}\Delta_q \\ \lambda \neq 0}} \lambda^{-s}\, \mathrm{Tr}\, g^*_{|\mathrm{Eig}_\lambda(\Delta_q)}$$

for $\mathfrak{Re}\, s > \dim M/2$. Classically, this zeta function has a meromorphic continuation to the complex plane which is holomorphic at zero.

Definition 1 *The* **equivariant analytic torsion** *is defined as*

$$\tau_g(M, F) := e^{-\frac{1}{2} Z'_g(0)}.$$

This object has been defined by Ray [16]. We shall denote the torsion with coefficients in the trivial line bundle by $\tau_g(M)$. Ray showed the following property of τ_g: Consider a free action of a finite group Γ on M. Let $\rho : \Gamma \to \mathbf{U}(1)$ be an unitary representation, thus defining a flat hermitian line bundle F on $\Gamma\backslash M$. Then the usual non-equivariant Ray-Singer torsion with coefficients in F is given by

$$\log \tau(\Gamma\backslash M, F) = \frac{1}{\#\Gamma} \sum_{g \in \Gamma} \bar{\rho}(g) \log \tau_g(M).$$

This is known to be a topological invariant if the cohomology $H^*(M,F)$ with coefficients in F vanishes. The equivariant torsion has been investigated by Lott and Rothenberg [13] for flat metrics on F. They showed that it equals zero on even dimensional manifolds or for orientation reversing actions on odd dimensional manifolds. They proved the result

Theorem 5 (Lott, Rothenberg) *Assume that g is homotopic to the trivial action. Choose a sequence $(f_\nu) \in \coprod_{\nu\in\mathbf{Z}} \mathbf{R}$, $\sum f_\nu = 0$. Then the torsion of powers of g weighted with (f_ν)*

$$\sum f_\nu \tau_{g^\nu}(M)$$

is a diffeomorphism invariant.

This result has been generalized strongly by Bismut and Zhang [2].

3 Homogeneous and symmetric spaces

Let G be a connected compact Lie group and let K be a connected compact subgroup. Let $T_G \supset T_K$ denote the maximal tori of G and K. We denote the Lie algebras of G, K, T_G and T_K by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{t}_G$ and $\mathfrak{t}_K$, respectively. We fix compatible orderings on $\mathfrak{t}_G^*$ and $\mathfrak{t}_K^*$. The action of K on the homogeneous space G/K induces a representation $\mathrm{Ad}_{G/K}$ on the tangent space of G/K at the class of $[1] \in G$, i.e. on $\mathfrak{g}/\mathfrak{k}$. Let Ψ denote the set of weights of this representation, the isotropy representation, and let Δ_G and Δ_K be the sets of roots of G resp. K. Then the weights of the adjoint representation of G on $\mathfrak{g}$ are given by Δ_G and the weight $\{0\}$ with multiplicity the rank of G. The weights of the action of K on $\mathfrak{g}$ are given by Δ_K, Ψ and $\mathrm{rk}\,K$-times the $\{0\}$, thus

$$(\Delta_G \cup \{0\} \cdot \mathrm{rk}\,G)_{|\mathfrak{t}_K} = \Delta_K \cup \Psi \cup \{0\} \cdot \mathrm{rk}\,K \tag{1}$$

(counted with multiplicity). In particular, the dimension $\#\Psi$ of G/K is odd-dimensional iff $\mathrm{rk}\,G - \mathrm{rk}\,K$ is so. The space of forms $\Gamma(\Lambda^q T^* G/K)$ is an infinite dimensional G-representation which contains the space of its irreducible subrepresentations (V_π, π) as a L^2-dense subspace. Thus,

$$\Gamma(\Lambda^q T^* G/K) \overset{\text{dense}}{\supset} \bigoplus_\pi \mathrm{Hom}_G\left(V_\pi, \Gamma(\Lambda^q T^* G/K)\right) \otimes V_\pi. \tag{2}$$

In this imbedding, the tensor product $\mathrm{Hom}_G(V_\pi, \Gamma(\Lambda^q T^*G/K)) \otimes V_\pi$ is the direct sum of dim $\mathrm{Hom}_G(V_\pi, \Gamma(\Lambda^q T^*G/K))$ copies of the representations (V_π, π). By a Frobenious law due to Bott [3], there is a canonical isomorphism

$$\mathrm{Hom}_G(V_\pi, \Gamma(\Lambda^q T^*G/K)) \cong \mathrm{Hom}_K\left(V_\pi, \Lambda^q \mathrm{Ad}_{G/K}\right) \tag{3}$$

(Note that $(\mathfrak{g}/\mathfrak{k} \otimes \mathbf{C})^* \cong \mathfrak{g}/\mathfrak{k} \otimes \mathbf{C}$ via the metric). In particular, the representations (V_π, π) which occur are finite dimensional.

Let $(X_1, \ldots, X_N)$ be an orthonormal basis of $\mathfrak{g}$ with respect to the negative Killing form. The Casimir operator of $\mathfrak{g}$ is defined as the following element of the universal enveloping algebra of $\mathfrak{g}$

$$\mathrm{Cas} := -\sum X_j.X_j \ . \tag{4}$$

Ikeda and Taniguchi proved the following beautiful result [10]:

Theorem 6 (Ikeda, Taniguchi) *Assume that G/K is a symmetric space equipped with the metric induced by the Killing form. Then the Laplacian Δ acts on the V_π's as* Cas *with respect to the imbedding (2).*

The Casimir is known to act by multiplication with a constant on irreducible representations. Thus, the eigenspaces of the Laplacian correspond to the irreducible representations π with multiplicity dim $\mathrm{Hom}_K(V_\pi, \Lambda^q \mathrm{Ad}_{G/K})$ and its eigenvalue there depends only on π.

Let $\rho_G := \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha$ be half the sum of the positive roots of G and let W_G be the Weyl group of G. Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the metric and the norm on $\mathfrak{t}_G^*$ induced by the Killing form. We denote the sign of an element $w \in W_G$ by $(-1)^w$. As usual, we define

$$(\alpha, \rho_G) := \frac{2\langle\alpha, \rho_G\rangle}{\|\alpha\|^2} \tag{5}$$

for any weight α. For an irreducible representation π we denote by b_π the sum of its highest weight and ρ_G. Then, classically, the action of the Casimir is given by

$$\pi(\mathrm{Cas}) = \|b_\pi\|^2 - \|\rho_G\|^2.$$

To abbreviate we set

$$\mathrm{Alt}_G\{b\} := \sum_{w\in W_G} (-1)^w e^{2\pi i wb}.$$

Then the Weyl character formula for the character χ_{b_π} of the representation evaluated at $t \in T_G$ may be written as

$$\chi_{b_\pi}(t) = \frac{\mathrm{Alt}_G\{b_\pi\}(t)}{\mathrm{Alt}_G\{\rho_G\}(t)}.$$

This formula provides the definition of the so-called virtual (or formal) character χ_b for any b equal to ρ_G+some weight. This extends to all of G by setting χ_b to be invariant under the adjoint action. The corresponding virtual representation shall be denoted by V_b. Occasionally we shall use the notation χ_b^G, χ_b^K, V_b^G, V_b^K to distinguish G- and K-representations. From now on we shall consider the irreducible symmetric space G/K as being equipped with any G-invariant metric $\langle\cdot,\cdot\rangle_\diamond$. All these metrics are proportional to the metric induced by the Killing form [1, Th. 7.44] on the irreducible factors of G/K. We shall denote the dual metric and norm on $\mathfrak{t}_G^*$ by $\langle\cdot,\cdot\rangle_\diamond$, $\|\cdot\|_\diamond$, too.

4 The zeta function for symmetric spaces

By theorem 6, the equivariant zeta function defining the torsion is given by

$$Z(s) = \sum_{q=1}^{n}(-1)^q q \sum_\pi \frac{\dim \mathrm{Hom}_K(V_\pi, \Lambda^q \mathrm{Ad}_{G/K})}{(\|b_\pi\|_\diamond^2 - \|\rho_G\|_\diamond^2)^s}\chi_{b_\pi}. \tag{6}$$

In the case $K = \{1\}$, $\dim G = n > 1$, we observe that

$$\begin{aligned} Z(s) &= \left(\sum_{q=1}^{n}(-1)^q q \dim \Lambda^q \mathfrak{g}\right) \sum_\pi \frac{\dim V_\pi \cdot \chi_{b_\pi}}{(\|b_\pi\|_\diamond^2 - \|\rho_G\|_\diamond^2)^s} \\ &= -n(1-1)^{n-1} \sum_\pi \frac{\dim V_\pi \cdot \chi_{b_\pi}}{(\|b_\pi\|_\diamond^2 - \|\rho_G\|_\diamond^2)^s}. \end{aligned}$$

Thus the torsion $\tau_g(G)$ equals zero for all compact Lie groups except the circle. Our key result is the following

Lemma 7 *Let G/K be a n-dimensional homogeneous space.*

- *If* $\mathrm{rk}\, G > \mathrm{rk}\, K + 1$ *then the virtual representation*

  $$\sum_{q=1}^{n}(-1)^q q \Lambda^q \mathrm{Ad}_{G/K}$$

  is trivial.

- *Assume that* $\operatorname{rk} G = \operatorname{rk} K + 1$ *and let* L *denote the line of those weights of* G *which restrict to zero on* $\mathfrak{t}_K$. *If* (V_π, π) *is an irreducible* G*-representation then the sum*

  $$\sum_{q=1}^{n} (-1)^q q \dim \operatorname{Hom}_K(V_\pi, \Lambda^q \mathrm{Ad}_{G/K}) \cdot \chi_{b_\pi}$$

  equals the sum of $-\chi_{\rho_G + w\alpha}$ *over those* $[w] \in W_G/W_K$ *and* $\alpha \in L$ *such that* b_π *lies in the* W_G*-orbit of* $\rho_G + w\alpha$.

By this lemma and theorem 6 we get the following expression for the equivariant zeta function Z:

Lemma 8 *For any odd-dimensional symmetric space* G/K*, the zeta function* $Z(s)$ *is given by*

$$Z(s) = \sum_{\substack{[w] \in W_G/W_K \\ \alpha \in L \\ \langle w\alpha, w\alpha + 2\rho_G \rangle > 0}} \frac{-\chi_{\rho_G + w\alpha}}{\langle w\alpha, w\alpha + 2\rho_G \rangle_\diamond^s} = \frac{1}{\# W_K} \sum_{\substack{w \in W_G \\ \alpha \in L \\ \langle w\alpha, w\alpha + 2\rho_G \rangle > 0}} \frac{-\chi_{\rho_G + w\alpha}}{\langle w\alpha, w\alpha + 2\rho_G \rangle_\diamond^s}$$

if $\operatorname{rk} G = \operatorname{rk} K + 1$ *and zero otherwise.*

In particular, one observes the following consequence:

Corollary 9 *Let* G/K_1, G/K_2 *be two symmetric spaces with* G*-conjugate tori* T_{K_1} *and* T_{K_2}. *Then the associated zeta functions* Z_1 *and* Z_2 *are proportional by the factor* $\# W_{K_1} / \# W_{K_2}$.

This shows already the desired classification of $\mathbf{SO}(2m)$-actions on odd-dimensional Graßmannians $G_{2m,2p-1}(\mathbf{R})$.

Theorem 10 *Two isometries* $g_1, g_2 \in \mathbf{SO}(2m)$ *of the Graßmannian* $G_{2m,2p-1}$ *are conjugate by a diffeomorphism iff they are conjugate in* $\mathbf{O}(2m)$. *If* g_1, g_2 *act fixed-point free, than they are conjugate by a homeomorphism iff they are conjugate in* $\mathbf{O}(2m)$.

Proof The maximal tori of $\mathbf{SO}(2p-1) \times \mathbf{SO}(2m-2p+1)$ are conjugate in $\mathbf{SO}(2m)$ for all p. By corollary 9, the equivariant torsion of $G_{2m,2p-1}$ at $g \in \mathbf{SO}(2m)$ equals the torsion of S^{2m-1} at g up to a non-zero constant. In [13], Lott and Rothenberg proved that an element of $\mathbf{SO}(2m)$ is determined up to conjugacy in $\mathbf{O}(2m)$ by certain linear combinations of this torsion (see the next section). Thus, the result for the spheres extends to all Graßmannians.

$\square$

Proof of lemma 7 Let χ^K denote the virtual character in the representation ring of K given by

$$\chi^K := \sum_{q=1}^{n} (-1)^q q \chi \left(\Lambda^q \mathrm{Ad}_{G/K} \right)$$

where $\chi \left(\Lambda^q \mathrm{Ad}_{G/K} \right)$ denotes the character of the K-representation $\Lambda^q \mathrm{Ad}_{G/K}$. By classical representation theory, one knows

$$\sum_{q=1}^{n} (-1)^q q \dim \mathrm{Hom}_K(V_\pi, \Lambda^q \mathrm{Ad}_{G/K}) = \int_K \overline{\chi^K} \cdot \chi_\pi \mathrm{dvol}_K. \tag{7}$$

Using the Weyl integral formula this transforms to

$$\frac{1}{\# W_K} \int_{T_K} \overline{\mathrm{Alt}_K\{\rho_K\}} \mathrm{Alt}_K\{\rho_K\} \overline{\chi^K} \cdot \chi_\pi \mathrm{dvol}_{T_K}$$

(where T_K is identified with the quotient of $\mathfrak{t}_K$ by the integral lattice). Classically the restriction of $\mathrm{Alt}_G\{\rho_G\}$ and $\mathrm{Alt}_K\{\rho_K\}$ to $\mathfrak{t}_K$ is given by

$$\mathrm{Alt}_G\{\rho_G\}_{|\mathfrak{t}_K} = \prod_{\alpha \in \Delta_G^+} 2i \sin \pi \alpha_{|\mathfrak{t}_K}$$

and

$$\mathrm{Alt}_K\{\rho_K\} = \prod_{\alpha \in \Delta_K^+} 2i \sin \pi \alpha.$$

Equation (1) shows that the restriction of χ^K is given by

$$\begin{aligned}
\chi^K &= \frac{\partial}{\partial s}_{|s=1} \det(1 - s\mathrm{Ad}_{G/K})_{|\mathfrak{t}_K} && (8)\\
&= \frac{\partial}{\partial s}_{|s=1} (1-s)^{\mathrm{rk}\, G - \mathrm{rk}\, K} \prod_{\alpha \in \Delta_G - \Delta_K} \left(1 - s e^{2\pi i \alpha}\right) && (9)\\
&= \begin{cases} -\prod_{\alpha \in \Delta_G - \Delta_K} \left(1 - e^{2\pi i \alpha}\right) & \text{if} \quad \mathrm{rk}\, G = \mathrm{rk}\, K + 1 \\ 0 & \phantom{\text{if}} \quad \mathrm{rk}\, G > \mathrm{rk}\, K + 1 \end{cases} && (10)
\end{aligned}$$

This shows the first part of lemma 7. Assume now that $\operatorname{rk} G = \operatorname{rk} K + 1$. Then

$$\chi^K{}_{|\mathfrak{t}_K} = -\frac{\prod_{\alpha\in\Delta_G} 2i\sin\pi\alpha_{|\mathfrak{t}_K}}{\prod_{\alpha\in\Delta_K} 2i\sin\pi\alpha} = -\frac{\overline{\mathrm{Alt}_G\{\rho_G\}}\mathrm{Alt}_G\{\rho_G\}}{\overline{\mathrm{Alt}_K\{\rho_K\}}\mathrm{Alt}_K\{\rho_K\}},$$

hence equation (7) yields

$$\sum_{q=1}^{n}(-1)^q q \dim \mathrm{Hom}_K(V_\pi, \Lambda^q \mathrm{Ad}_{G/K}) = -\frac{1}{\#W_K}\int_{T_K} \overline{\mathrm{Alt}_G\{\rho_G\}}\mathrm{Alt}_G\{b_\pi\}\mathrm{dvol}_{T_K}$$

which finishes the proof of lemma 7. $\square$

5 The torsion for symmetric spaces

Classically, a compact symmetric space decomposes as a Riemannian manifold into a product of irreducible symmetric spaces G_ν/K_ν, where the metric on G_ν/K_ν is induced by a negative real number times the Killing form of G_ν. By lemma 8 the equivariant torsion is non-zero only if $\operatorname{rk} G = \operatorname{rk} K+1$. Also it is zero for all Lie groups except the circle. By the classification of irreducible symmetric spaces, among them only the odd-dimensional Graßmannians

$$G_{2m,2p-1}(\mathbf{R}) = \mathbf{SO}(2m)/\mathbf{SO}(2p-1)\times\mathbf{SO}(2m-2p+1) \qquad (m,p\in\mathbf{N}, m\geq p).$$

and the 5-dimensional space $\mathbf{SU}(3)/\mathbf{SO}(3)$ can have non-zero torsion. Thus, the torsion is non-zero only for the spaces

$$G_{2m,2p-1}\times G'/K' \text{ and } \mathbf{SU}(3)/\mathbf{SO}(3)\times G'/K', \qquad (T)$$

where G'/K' is an arbitrary symmetric space with $\operatorname{rk} G' = \operatorname{rk} K'$. We imbed $\mathbf{SO}(2p-1)\times\mathbf{SO}(2m-2p+1)$ in $\mathbf{SO}(2m)$ as $K = \{\left(\begin{smallmatrix} A & 0\\ 0 & B\end{smallmatrix}\right) | A\in\mathbf{SO}(2p-1), B\in \mathbf{SO}(2m-2p+1)\}$. To diagonalize the standard maximal torus, we imbed $\mathbf{SO}(3)$ in $\mathbf{SU}(3)$ after conjugation by the matrix $\left(\begin{smallmatrix} -i & 1 & 0\\ i & 1 & 0\\ 0 & 0 & \sqrt{2}\end{smallmatrix}\right)/\sqrt{2}$. As maximal tori of G we choose

$$\mathfrak{t}_{\mathbf{SO}(2m)} := \left\{2\pi\begin{pmatrix} \begin{smallmatrix} 0 & -\lambda_1\\ \lambda_1 & 0\end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix} 0 & -\lambda_m\\ \lambda_m & 0\end{smallmatrix}\end{pmatrix} \middle| \lambda_1,\dots,\lambda_m\in\mathbf{R}\right\}$$

and

$$\mathfrak{t}_{\mathbf{SU}(3)} := \Big\{ 2\pi i \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \Big| \lambda_1 + \lambda_2 + \lambda_3 = 0 \Big\}.$$

Let $e_\nu \in \mathfrak{t}^*_{\mathbf{SO}(2m)}$, $1 \le \nu \le m$ resp. $f_\nu \in \mathfrak{t}^*_{\mathbf{SU}(3)}$, $1 \le \nu \le 3$ denote the weight mapping one of the above matrices to λ_ν, ordered according to their index. We set $\mathfrak{t}_K$ as the kernel of e_p resp. f_3; thus, these weights generate $L \cong \mathbf{Z}$. They shall be denoted by α_0. The orbit of α_0 under W_G/W_K is then given by $\{\pm e_\nu\}_{\nu=1}^m$ and $\{f_\nu\}_{\nu=1}^3$, respectively.

Let $\psi_{\mathrm{even}} : \mathbf{R}/\mathbf{Z} \to \mathbf{R}$ denote the map

$$[x] \mapsto C + \frac{1}{2}\Big(\psi_0([x]) + \psi_0([-x])\Big)$$

where C denotes Euler's constant and ψ_0 is the digamma function evaluated on the fundamental domain $]0,1]$, i.e. $\psi_0([x]) := \psi(x + \lceil 1 - x \rceil)$ with $\lceil x \rceil$ the integer part of x. We show now the following formula for the equivariant Ray-Singer torsion:

Theorem 11 *Let G/K be a space of type (T) which has no circle as irreducible component. Let α_0 be a generator of L. Then the logarithm of the equivariant torsion for $t \in T_G$ is given by*

$$-\frac{1}{2} Z_t'(0) = \log \prod_{[w] \in W_G/W_K} \frac{\langle w\alpha_0, \rho_G \rangle_\diamond}{\pi} - \sum_{[w] \in W_G/W_K} \psi_{\mathrm{even}}(w\alpha_0(t))$$

and for general $g \in G$ it is obtained by conjugating g into T_G.

Using the formula for the values of ψ at rational numbers this reproves Ray's result for the lens spaces [16]. Before proving the theorem we show two auxiliary lemmas. Let S_β denote the reflection of $\mathfrak{t}_G^*$ in the hyperplane orthogonal to $\beta \in \mathfrak{t}_G^*$. We need the following symmetry:

Lemma 12 *Let G/K be a space of the type (T). Then, for all $[w] \in W_G/W_K$ there exists $\tilde{w} \in W_G$ with $(-1)^{\tilde{w}} = 1$ such that*

$$S_{w\alpha_0} \rho_G = \tilde{w} \rho_G.$$

In particular, $(w\alpha_0, \rho_G) = -(\tilde{w}^{-1} w\alpha_o, \rho_g)$ is an integer and the map $[w] \mapsto [\tilde{w}^{-1} w] \in W_G/W_K$ is bijective.

Proof The proof reduces to that for the cases $G_{2m,2p-1}$ and $\mathbf{SU}(3)/\mathbf{SO}(3)$. Classically, $\rho_{\mathbf{SO}(2m)} = \sum_{\nu=1}^{m}(m-\nu)e_\nu$ and $\rho_{\mathbf{SU}(3)} = f_1 - f_3$. Hence, $S_{w\alpha_0}\rho_G$ is a weight and thus given by $\tilde{w}\rho_G$ for some unique $\tilde{w} \in W_G$. One observes immediately that $(-1)^{\tilde{w}} = 1$. By the invariance of $\langle\cdot,\cdot\rangle$ under reflections, one shows

$$\langle w\alpha_0, \rho_G\rangle = \langle S_{w\alpha_0}w\alpha_0, S_{w\alpha_0}\rho_G\rangle = -\langle w\alpha_0, \tilde{w}\rho_G\rangle = -\langle \tilde{w}^{-1}w\alpha_0, \rho_G\rangle;$$

as $S_{w\alpha_0}\rho_G = \rho_G - (w\alpha_0, \rho_G)w\alpha_0$, one notices that $(w\alpha_0, \rho_G)$ is an integer. As $S_{\tilde{w}^{-1}w\alpha_0}\rho_G = \tilde{w}^{-1}S_{w\alpha_0}(\tilde{w}\rho_G) = \tilde{w}^{-1}\rho_G$, the map $[w] \mapsto [\tilde{w}^{-1}w]$ is an involution. $\square$

Also, we need a lemma about values of zeta functions at zero. For $p, n \in \mathbf{Z}$ and $h, \phi \in \mathbf{R}$, $h > 0$, let $\tilde{\zeta}_{n,p,h}(s,\phi)$ denote the zeta function

$$\tilde{\zeta}_{n,p,h}(s,\phi) := \sum_{\substack{k\in\mathbf{Z}\\ k(k+p)>0}} \frac{k^n e^{ik\phi}\log(hk)^2}{(k(k+p))^s} + \sum_{\substack{k\in\mathbf{Z}\setminus\{0\}\\ k(k+p)\leq 0}} \frac{k^n e^{ik\phi}\log(hk)^2}{|k|^{2s}}$$

for $\Re e\, s > \frac{n+1}{2}$. This zeta function has a meromorphic extension to the complex plane.

Lemma 13 *The value at zero of $\tilde{\zeta}$ is independent of p. For $n = 0$ it takes the value*

$$\frac{1}{2}\tilde{\zeta}_{0,p,h}(0,\phi) = \log\frac{2\pi}{h} + \psi_{\text{even}}([\frac{\phi}{2\pi}]).$$

Proof The proof may be given easily by applying the more general result of [12, Lemma 7]; instead, for the case $\phi \not\equiv 0 \bmod 2\pi$ we shall give an alternative proof which is better adapted to this particular situation. Choose $\phi \in]0, 2\pi[$. First, observe that in this case

$$\tilde{\zeta}_{n,p,h}(s,\phi) = i^{-n-1}\left(\frac{\partial}{\partial\phi}\right)^{n+1}\tilde{\zeta}_{-1,p,h}(s,\phi)$$

for $n \geq 0$. For $n < 0$, the series defining $\tilde{\zeta}_{n,p,h}(s,\phi)$ converges at $s = 0$. Clearly its value there does not depend on p. Using Kummer's Fourier series for the logarithm of the Gamma function [11, sect. 5] one finds

$$\begin{aligned}\tilde{\zeta}_{-1,p,h}(0,\phi) &= \sum_{k>0}\frac{4i\sin k\phi\,\log hk}{k}\\ &= 2i(C + \log\frac{2\pi}{h})(\phi-\pi) + 2\pi i\log\Gamma(\frac{\phi}{2\pi}) - 2\pi i\log\Gamma(1-\frac{\phi}{2\pi}).\end{aligned}$$

To prove the lemma for the case $\phi \equiv 0$ one may use the Taylor expansion for $|k+p|^{-s} = |k|^{-s}|1+p/k|^{-s}$ as in [11, sect. 6] or [12]. $\square$

Proof of theorem 11 The derivative of $Z(s)$ is given by

$$\begin{aligned} Z'(s) &= \sum_{\substack{[w]\in W_G/W_K \\ k\in\mathbf{Z} \\ k(k+(w\alpha_0,\rho_G))>0}} \frac{\chi_{\rho_G+kw\alpha_0}\left(\log|k| + \log|k+(w\alpha_0,\rho_G)| + \log\|\alpha_0\|_\diamond^2\right)}{\left(k(k+(w\alpha_0,\rho_G))\|\alpha_0\|_\diamond^2\right)^s} \\ &= \sum_{\substack{[w]\in W_G/W_K \\ k\in\mathbf{Z} \\ k(k+(w\alpha_0,\rho_G))>0}} \frac{\chi_{\rho_G+kw\alpha_0}\log|k|}{\left(k(k+(w\alpha_0,\rho_G))\|\alpha_0\|_\diamond^2\right)^s} \\ &\quad + \sum_{\substack{[w]\in W_G/W_K \\ k\in\mathbf{Z} \\ k(k-(w\alpha_0,\rho_G))>0}} \frac{\chi_{\rho_G+(k-(w\alpha_0,\rho_G))w\alpha_0}\log|k|}{\left(k(k-(w\alpha_0,\rho_G))\|\alpha_0\|_\diamond^2\right)^s} + Z(s)\log\|\alpha_0\|_\diamond^2. \quad (11) \end{aligned}$$

By lemma 12 one obtains

$$\rho_G + (k-(w\alpha_0,\rho_G))w\alpha_0 = S_{w\alpha_0}(\rho_G - kw\alpha_0) = \tilde{w}(\rho_G + k\tilde{w}^{-1}w\alpha_0)$$

with $(-1)^{\tilde{w}} = 1$. Hence the second sum in line (11) equals

$$\sum_{\substack{[w]\in W_G/W_K \\ k\in\mathbf{Z} \\ k(k+(\tilde{w}^{-1}w\alpha_0,\rho_G))>0}} \frac{\chi_{\rho_G+k\tilde{w}^{-1}w\alpha_0}\log|k|}{\left(k(k+(\tilde{w}^{-1}w\alpha_0,\rho_G))\|\alpha_0\|_\diamond^2\right)^s}.$$

By the bijectivity of $[w] \mapsto [\tilde{w}^{-1}w]$ one finds

$$Z'(s) = \sum_{\substack{[w]\in W_G/W_K \\ k\in\mathbf{Z} \\ k(k+(w\alpha_0,\rho_G))>0}} \frac{\chi_{\rho_G+kw\alpha_0}\log\|k\alpha_0\|_\diamond^2}{\left(k(k+(w\alpha_0,\rho_G))\|\alpha_0\|_\diamond^2\right)^s}.$$

Fix $g \in G$ and consider the character $\chi_{\rho_G+kw\alpha_0}(g)$ as a function in $k \in \mathbf{Z}$. One shows easily that $\chi_{\rho_G+kw\alpha_0}(g)$ is a linear combination of functions of the type $k \mapsto k^n e^{i\phi k}$ ($n \in \mathbf{Z}$, $\phi \in \mathbf{R}$) [12, eq. 81]. The characters $\chi_{\rho_G+kw\alpha_0}$ are zero for $0 > k(k+(w\alpha_0,\rho_G)) = (\|\rho_G + k\alpha_0\|_\diamond^2 - \|\rho_G\|_\diamond^2)/\|\alpha_0\|_\diamond^2$. For $k = -(w\alpha_0,\rho_G)$ we have by lemma 12 the result

$$\chi_{\rho_G-(w\alpha_0,\rho_G)w\alpha_0} = \chi_{S_{w\alpha_0}\rho_G} = (-1)^{\tilde{w}} = 1.$$

Thus, we may write $Z'(s)$ as

$$\begin{aligned} Z'(s) &= \sum_{[w]\in W_G/W_K} \left(\sum_{\substack{k\in\mathbf{Z} \\ k(k+(w\alpha_0,\rho_G))>0}} \frac{\chi_{\rho_G+kw\alpha_0} \log \|k\alpha_0\|_\diamond^2}{(k(k+(w\alpha_0,\rho_G))\|\alpha_0\|_\diamond^2)^s} \right. \\ &\quad \left. + \sum_{\substack{k\in\mathbf{Z}\setminus\{0\} \\ k(k+(w\alpha_0,\rho_G))\leq 0}} \frac{\chi_{\rho_G+kw\alpha_0} \log \|k\alpha_0\|_\diamond^2}{\|k\alpha_0\|_\diamond^{2s}} - \frac{\log \|(w\alpha_0,\rho_G)\alpha_0\|_\diamond^2}{\|(w\alpha_0,\rho_G)\alpha_0\|_\diamond^{2s}} \right). \end{aligned}$$

Lemma 13 states that the value at zero of $Z'(s)$ equals the value at zero of the zeta function

$$\tilde{Z}(s) = \sum_{[w]\in W_G/W_K} \left(\sum_{k\in\mathbf{Z}\setminus\{0\}} \frac{\chi_{\rho_G+kw\alpha_0} \log \|k\alpha_0\|_\diamond^2}{\|k\alpha_0\|_\diamond^{2s}} - \frac{\log \|(w\alpha_0,\rho_G)\alpha_0\|_\diamond^2}{\|(w\alpha_0,\rho_G)\alpha_0\|_\diamond^{2s}} \right).$$

The sum of the characters over W_G/W_K equals

$$\begin{aligned} \sum_{[w]\in W_G/W_K} \chi_{\rho_G+kw\alpha_0} &= \frac{1}{\#W_K} \sum_{w\in W_G} \frac{\sum_{w'\in W_G} (-1)^{w'} e^{2\pi i(w'\rho_G+kw'w\alpha_0)}}{\mathrm{Alt}_G\{\rho_G\}} \\ &= \frac{1}{\#W_K} \frac{\sum_{w\in W_G} e^{2\pi ikw\alpha_0} \sum_{w'\in W_G} (-1)^{w'} e^{2\pi iw'\rho_G}}{\mathrm{Alt}_G\{\rho_G\}} \\ &= \sum_{[w]\in W_G/W_K} e^{2\pi ikw\alpha_0}. \end{aligned}$$

Hence we may apply the formula in lemma 13 to $\tilde{Z}(s)$. This proves the theorem. □

6 The classification

To proof theorem 3 for the case $\mathbf{SU}(3)/\mathbf{SO}(3)$ we apply the method used by Lott and Rothenberg for the spheres. In [13, Prop. 32] they proved

Lemma 14 (Franz, Lott, Rothenberg) *Let $x_\nu \in \mathbf{R}/\mathbf{Z}$, $1 \leq \nu \leq N$ be elements of the circle. Then the x_ν are determined up to order and sign by the sequence*

$$\left(\sum_{\nu=1}^{N} \psi_{\mathrm{even}}(nx_\nu) \right)_{n\in\mathbf{Z}}.$$

This lemma has been proven by Franz for rational x_ν. For the action of a torus element $t := 2\pi i \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ this means that λ_1, λ_2 and λ_3 are determined up to order and sign by the torsion τ_{t^n} of powers of t. As $\lambda_1 + \lambda_2 + \lambda_3 = 0$, all λ_ν have to change sign if one of them does so. Now a change of the order is an element of $W_{\mathbf{SU}(3)}$, thus given by conjugation with an element of $\mathbf{SU}(3)$. A change of the sign of all λ_ν is obtained by the symmetry around $[1] \in \mathbf{SU}(3)/\mathbf{SO}(3)$ composed with interchanging λ_1 and λ_2 (recall our non-standard imbedding of $\mathbf{SO}(3)$). Thus, all isometries which have the same torsion are actually isometric.

Theorem 4 follows by the same method which has been applied by de Rham for the spheres [7, sect. 8]: First let $\Gamma_1, \Gamma_2 \subset \mathbf{SO}(2m)$ act freely on $G_{2m,2p-1}$ and assume that they are conjugate by a homeomorphism. Thus we may consider them as two euclidean representations $\rho_1, \rho_2 : \Gamma \to \mathbf{SO}(2m)$ of one single group Γ. Now theorem 10 implies that for each $\gamma \in \Gamma$ there is an element $g \in \mathbf{O}(2m)$ such that

$$g\rho_1(\gamma)g^{-1} = \rho_2(\gamma).$$

In particular, $\mathrm{Tr}\,\rho_1(\gamma) = \mathrm{Tr}\,\rho_2(\gamma)$. Hence ρ_1 and ρ_2 have the same character, thus they are isomorphic as euclidean representations and conjugate in $\mathbf{O}(2m)$.

Now consider two finite groups $\Gamma_1, \Gamma_2 \subset \mathbf{SU}(3)$ acting freely on $\mathbf{SU}(3)/\mathbf{SO}(3)$. Let Γ_1, Γ_2 be conjugate by a homeomorphism. Consider them as unitary representations $\rho_1, \rho_2 : \Gamma \to \mathbf{SU}(3)$. For each $\gamma \in \Gamma$ there exists $\tilde{g} \in \mathbf{SU}(3) \rtimes \mathbf{Z}/2\mathbf{Z} = \mathrm{Aut}(\mathbf{SU}(3))$ such that

$$\tilde{g}\rho_1(\gamma)\tilde{g}^{-1} = \rho_2(\gamma).$$

In other terms, there is some $g \in \mathbf{SU}(3)$ such that

$$g\rho_1(\gamma)g^{-1} = \rho_2(\gamma) \text{ or } \overline{\rho_2(\gamma)}.$$

Consider the (normal) subgroups of Γ

$$\Gamma^+ := \{\gamma \in \Gamma \,|\, \mathrm{Tr}\,\rho_1(\gamma) = \mathrm{Tr}\,\rho_2(\gamma)\}, \tag{12}$$

$$\Gamma^- := \{\gamma \in \Gamma \,|\, \mathrm{Tr}\,\rho_1(\gamma) = \overline{\mathrm{Tr}\,\rho_2(\gamma)}\}. \tag{13}$$

Then $\Gamma = \Gamma^+ \cup \Gamma^-$, and for reasons of cardinality Γ has to equal at least one of them. Thus, ρ_1 is conjugate to ρ_2 or $\overline{\rho_2}$ via some $h \in \mathbf{U}(3)$. Hence they are conjugate by

$$\frac{1}{\sqrt[3]{\det h}}h \in \mathbf{SU}(3).$$

Acknowledgement *During the preparation of this paper I have been supported by the Max-Planck-Institut in Bonn and by the Bennigsen-Foerder-Preis. I want to thank Wolfgang Lück and Christian Kaiser for helpful discussions.*

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