

Holomorphic torsion on orbifolds and an equivariant arithmetic Hilbert-Samuel theorem

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Abstract

We compute the $\log p$ -part of the asymptotic expansion of the holomorphic torsion of the p -th power of a positive line bundle in terms of characteristic classes. This is done for orbifold and equivariant torsions. In the equivariant case the exponents in the expansion are less than or equal to the dimension of the fixed point set. We deduce a corresponding equivariant arithmetic Hilbert-Samuel theorem in Arakelov geometry.

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1 Introduction

Given a holomorphic vector bundle E equipped with an Hermitian metric h^E on a compact Hermitian manifold, Ray and Singer constructed the holomorphic torsion $T(M, E) \in \mathbb{R}$ as a regularized determinant of Kodaira-Laplace operators. This definition was extended to an equivariant setting ([K1]) and to orbifolds by Ma ([Ma1]). For powers L^p of a positive Hermitian line bundle (L, h^L) , Bismut and Vasserot ([BV]) determined the first terms of an asymptotic expansion of $T(M, E \otimes L^p)$ as $p \rightarrow +\infty$. Other extensions and applications to physics of asymptotics of holomorphic torsion have been given by Berman ([Ber]), Su ([Su]), Larraín-Hubach ([La]) and, for torsion forms, by Puchol ([P]). Finski ([Fi]) proved in the orbifold situation that this expansion has two kinds of terms, of the form p^ℓ and $p^\ell \log p$ for $-\infty < \ell \leq \dim M$. Also he showed in [Fi, Th. 1.5] that the coefficient of the $p^\ell \log p$ -terms had to be independent of the metrics on M , E and L . The first result of this article is the computation of the $p^\ell \log p$ -part of the asymptotic expansion in terms of explicitly given characteristic classes. Let $\mathbb{Z}/(m)$ denote the cyclic group of order $m \in \mathbb{Z}^+$. For each m we fix a primitive m -th root of unity ζ_m . For a compact effective orbifold M let $M \dot{\cup} \Sigma M = \bigcup_{j \in J} M_j$ denote the decomposition of M and its strata ΣM into connected components and let m_j be the multiplicity of M_j .

Theorem 1.1. *Let M be a compact effective Hermitian orbifold of dimension n and let (L, h^L) , $(E, h^E) \rightarrow M$ be proper Hermitian orbifold vector bundles with $\operatorname{rk} L = 1$ and L positive. Let $n_{\max} := \max_{M_j \subset M_{\text{sing}}} M_j$ denote the maximum of the dimensions of singular strata. Then there is a number $m \in \mathbb{Z}^+$ and families $\tilde{\beta}_\bullet : \mathbb{N}_0 \rightarrow \mathbb{R}$, $\tilde{\kappa}_\bullet : \mathbb{Z}/(m) \times \mathbb{N}_0 \rightarrow \mathbb{R}$ such that for any $k \in \mathbb{N}_0$ the orbifold holomorphic torsion satisfies for $p \rightarrow +\infty$*

$$T(M, E \otimes L^p) = \log p \cdot \sum_j \frac{1}{m_j} \int_{M_j} \left(n \cdot \operatorname{Td}^\Sigma(TM) - \operatorname{Td}'^\Sigma(TM) \right) \operatorname{ch}^\Sigma(E \otimes L^p)$$

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$$+ \sum_{j=0}^k p^{n-j} \tilde{\beta}_j + \sum_{j=0}^{k+n_{\max}-n} \sum_{r \in \mathbb{Z}/(m)} p^{n_{\max}-j} \zeta_m^{rp} \tilde{\kappa}_{r,j} + o(p^{n-k}).$$

In particular it turns out that the only non-vanishing $p^\ell \log p$ summands are those for which $\ell \geq 0$. In the special case where M is a manifold the theorem shows that the coefficient of $\log p$ of the asymptotic expansion of the classical non-orbifold holomorphic torsion is given by

$$\int_M (n \cdot \text{Td}(TM) - \text{Td}'(TM)) \text{ch}(E \otimes L^p).$$

From Th. 1.1, we deduce the analogous result for the equivariant torsion $T_g(M, E \otimes L^p)$ in the case of an action of finite order of an isometry g on M, E and L . In the equivariant case, however, the largest exponent of p is given by the maximum $n_{\max}$ of the dimensions of fixed point submanifolds instead of by $\dim M$.

Theorem 1.2. *Let M be a compact Hermitian manifold of dimension n and let $(L, h^L), (E, h^E) \rightarrow M$ be Hermitian vector bundles with $\text{rk } L = 1$ and L positive. Assume that M, E and L are invariant under the action of an holomorphic isometry g of finite order m and that E and L are equipped with a g -equivariant structure. Let $n_{\max}$ denote the largest dimension of the components of the fixed point submanifold M^g . Then there is a family $\kappa'_\bullet : \mathbb{Z}/(m) \times \mathbb{N}_0 \rightarrow \mathbb{R}[\zeta_m]$ such that for any $k \in \mathbb{N}_0$ the equivariant holomorphic torsion satisfies for $p \rightarrow +\infty$*

$$\begin{aligned} T_g(M, E \otimes L^p) &= \log p \cdot \int_{M^g} (n \cdot \text{Td}_g(TM) - \text{Td}'_g(TM)) \text{ch}_g(E \otimes L^p) \\ &\quad + \sum_{j=0}^k p^{n_{\max}-j} \sum_{r \in \mathbb{Z}/(m)} \zeta_m^{rp} \kappa'_{r,j} + o(p^{n_{\max}-k}). \end{aligned}$$

This formula has been proven in the case M symmetric, E trivial, $g \in G$ arbitrary in [K3, Th. 5.5] and for M homogeneous, $\dim M^g = 0$ in [K3, Th. 8.1]. Also, the main result [T, Th. 1] of an as of yet unpublished preprint by Teßmer states that for $p \rightarrow +\infty$ the part of differential form degree $2k$ of equivariant torsion forms is given by a linear combination of $p^{n_{\max}+k}$, $p^{n_{\max}+k} \log p$ and coefficients as in Th. 1.2 up to $o(p^{n_{\max}+k})$.

We also deduce corresponding anomaly formulae for the asymptotic expansion of L^2 -metrics on the Knudsen-Mumford determinant of cohomology in Cor. 4.2 and Cor. 4.4.

The main application of holomorphic torsion is the construction of the determinant of a direct image of Hermitian vector bundles within the context of Arakelov geometry. Similarly to the role played by the Kodaira vanishing theorem in algebraic geometry, describing the asymptotic behaviour of holomorphic torsion for increasing tensor powers of positive line bundles $L \rightarrow M$ yields arithmetic Hilbert-Samuel theorems in Arakelov geometry. The first result of this kind has been given by Gillet and Soulé in [GS3, Th. 8] using Bismut-Vasserot's theorem. Since then, numerous articles have been published that extend and apply this result, in particular by Abbes and Bouche [ABo], S.-W. Zhang [Z], Randriambololona [R], Yuan [Y], Moriwaki [Mo], Berman and Freixas i Montplet [BeFr], Botero, Burgos Gil, Holmes and de Jong [BoBuHoJo] and Chen and Moriwaki [ChMo].

We state the result for an equivariant arithmetic variety X , i.e. a regular integral scheme which is endowed with a μ_m -projective action, projective and flat over $\text{Spec } \mathbb{Z}$. Let $\widehat{\deg}(V, h^V) \in \mathbb{R}$ for a finitely generated Hermitian $\mathbb{Z}$ -module (V, h^V) denote Gillet-Soulé's arithmetic degree. We equip $H^\bullet(X(\mathbb{C}), E_{\mathbb{C}} \otimes L_{\mathbb{C}}^p)$ with the L^2 -metric induced by the Hodge embedding. The μ_m -action induces a natural orthogonal $\mathbb{Z}/(m)$ -grading $E|_{X^{\mu_m}} \simeq \bigoplus_{k \in \mathbb{Z}/(m)} E_k$ on the restriction of E to the fixed point scheme X^{μ_m} , and the same way a grading $H^\bullet(X(\mathbb{C}), E_{\mathbb{C}} \otimes L_{\mathbb{C}}^p) \simeq \bigoplus_{k \in \mathbb{Z}/(m)} H^\bullet(X(\mathbb{C}), E_{\mathbb{C}} \otimes L_{\mathbb{C}}^p)_k$.

Theorem 1.3. *Let $f : X \rightarrow \operatorname{Spec} \mathbb{Z}$ be a flat projective arithmetic variety of dimension $n + 1$, equipped with a Kähler metric g^{TX} over $X(\mathbb{C})$. Let $(L, h^L), (E, h^E) \rightarrow X$ be Hermitian vector bundles with $\operatorname{rk} L = 1$ and L positive in the sense that $H^q(X, E \otimes L^p) = 0$ for $q > 0$ and $p \gg 0$ and that $c_1(L_{\mathbb{C}}, h^L) > 0$. Assume that X, E and L carry a μ_m action for $m \in \mathbb{Z}^+$ and that the metrics are invariant under this action. Let $n_{\max} + 1$ denote the largest dimension of the components of the fixed point scheme X^{μ_m} and let N denote the normal bundle for the embedding $X^{\mu_m} \hookrightarrow X$. Then there is a family $\hat{\kappa}_{\bullet} : \mathbb{Z}/(m) \times \mathbb{N}_0 \rightarrow \mathbb{R}[\zeta_m]$ such that for $p \rightarrow +\infty$ the following identity of complex numbers holds:*

$$\begin{aligned}
& - \sum_{k \in \mathbb{Z}/(m)} \zeta_m^k \cdot \log \operatorname{covol} (H^0(X, E \otimes L^p)_k, |\cdot|_{L^2}) \\
& = \widehat{\deg} f_*^{\mu_m} \left(\frac{p^{n - \operatorname{rk} N + 1}}{(n - \operatorname{rk} N + 1)!} \frac{\sum_{\ell \in \mathbb{Z}/(m)} \zeta_m^\ell \operatorname{rk} E_\ell}{\prod_{r \in \mathbb{Z}/(m)} (1 - \zeta_m^{-r})^{\operatorname{rk} N_r}} \sum_{k \in \mathbb{Z}/(m)} \zeta_m^{pk} \widehat{c}_1(L_k, h^L)^{n - \operatorname{rk} N + 1} \right) \\
& \quad + \log p \cdot \int_{X^{\mu_m}(\mathbb{C})} (n \cdot \operatorname{Td}_g(TX_{\mathbb{C}}) - \operatorname{Td}'_g(TX_{\mathbb{C}})) \operatorname{ch}_g(E_{\mathbb{C}} \otimes L_{\mathbb{C}}^p) \\
& \quad + \sum_{j=0}^k p^{n_{\max} - j} \sum_{r \in \mathbb{Z}/(m)} \zeta_m^{rp} \hat{\kappa}_{r,j} + o(p^{n_{\max} - k}),
\end{aligned}$$

where the coefficients $\hat{\kappa}_{r,j}$ for $j > n_{\max}$ depend only on the complex points and the metrics.

Thus a certain weighted logarithm of the covolume of the sections is given by a weighted height of the fixed point scheme, a topological term and an asymptotic expansion in powers of p , where the largest exponent of p is smaller than the dimension $n_{\max} + 1$ of the fixed point scheme. The analogue result for arithmetic varieties over general regular arithmetic rings D can be obtained by considering the result above for all complex embedding of D , as our contribution concerns the complex points. In contrast in previous versions of an arithmetic Hilbert-Samuel theorem the largest exponent is given by $\dim X$, and only the highest degree term of the $\log p$ -summand was known.

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2 Effective orbifolds

Throughout this article, a quotient $\frac{g'}{g}$ of two bilinear forms with g non-degenerate is understood to be the endomorphism A such that $g(A \cdot, \cdot) = g'$. We shall apply metrics h^E on bundles E to subbundles and to bundles constructed from E without changing the notation for the metric. For an element ω of a graded algebra we denote by $\omega^{[k]}$ the summand in degree k .

We consider the same (effective) orbifold setting as in [Kaw1], [Ma1, Section 1.1] and [Fi, section 5]. Let $\mathcal{M}_s$ denote the category such that the objects are pairs (G, M) , where G is a finite group acting effectively on a smooth connected manifold M . The morphisms $\Phi : (G, M) \rightarrow (G', M')$ shall be families of open embeddings $\varphi : M \rightarrow M'$ such that the following properties hold:

1. For every $\varphi \in \Phi$ there exists an group monomorphism $\lambda_\varphi : G \rightarrow G'$ such that φ is λ_φ -equivariant.
2. For all $\varphi \in \Phi$, $g' \in G'$ satisfying $g' \cdot (\varphi(M)) \cap \varphi(M) \neq \emptyset$, one finds $g' \in \lambda_\varphi(G)$.
3. Any $\varphi \in \Phi$ satisfies $\Phi = \{g' \varphi | g' \in G'\}$.

Definition 2.1. Consider a paracompact Hausdorff space M . Let $\mathcal{U}$ be a covering of M by connected open subsets which is dense, i.e. for all $U, U' \in \mathcal{U}$ and $x \in U \cap U'$ there is a $U'' \in \mathcal{U}$ such that $x \in U'' \subset U \cap U'$. An orbifold atlas $\mathcal{V}$ on M is a family of ramified coverings $\mathcal{V}(U) : (G_U, \tilde{U}) \rightarrow U$ for $U \in \mathcal{U}$ (the orbifold charts) such that $(G_U, \tilde{U}) \in \mathcal{M}_s$ and:

1. For any $U \in \mathcal{U}$ one has $U \cong \tilde{U}/G_U$.
2. For any $U, V \in \mathcal{U}$, $U \subset V$ there is a morphism $\varphi_{VU} : (G_U, \tilde{U}) \rightarrow (G_V, \tilde{V})$ that covers the inclusion $U \subset V$.
3. For any $U, V, W \in \mathcal{U}$, $U \subset V \subset W$ the equality $\varphi_{WU} = \varphi_{WV} \circ \varphi_{VU}$ holds.

A pair $(M, \mathcal{V})$ of M together with an orbifold atlas is called an (effective) orbifold.

Let $\#S$ denote the cardinality of a set S .

Definition 2.2. Consider an orbifold M . For $x \in M$ choose a chart $(G_x, \tilde{U}_x) \rightarrow U_x$ such that $x \in \tilde{U}_x$ is fix under the action of G_x . By the above definition G_x is uniquely determined up to isomorphism. The singular points of M are $M_{\text{sing}} := \{x \in M \mid \#G_x > 1\}$, the regular points are $M_{\text{reg}} := M \setminus M_{\text{sing}}$.

Definition 2.3. An orbifold vector bundle $\pi : E \rightarrow M$ is an orbifold E together with G_U, G_U^E -equivariant local vector bundles $\tilde{\pi}_U : \tilde{E}_U \rightarrow \tilde{U}$ for local charts $(G_U^E, \tilde{E}_U), (G_U, \tilde{U})$ of E and M , resp., such that the transition maps are equivariant, where $G_U = G_U^E/K_U^E$ with $K_U^E := \ker(G_U^E \rightarrow \text{Diffeo}(\tilde{U}))$. The vector bundle E is called proper if $\#K_U^E = 1$.

Let $\widetilde{E_U^{\text{pr}}}$ denote the maximal K_U^E -invariant subbundle of $\tilde{E}_U$. Let E^{pr} be the proper orbifold bundle given locally by $(G_U, \widetilde{E_U^{\text{pr}}})$.

Riemannian, complex, orientation structures etc. are defined by replacing the category $\mathcal{M}$ correspondingly.

A holomorphic (or smooth) section $s : M \rightarrow E$ is a map covered locally by holomorphic (or smooth) G_U^E -invariant sections $\tilde{s}_U : \tilde{U} \rightarrow \tilde{E}_U$. For a form α on M with $\text{supp } \alpha \subset U$ set $\int_M \alpha := \frac{1}{\#G_U} \int_{\tilde{U}} \tilde{\alpha}_U$. We extend this definition to forms with compact support by linearity.

By [MaM, Lemma 5.4.3], [Kaw1, p. 76], for any $x \in M$ there is a local chart $\tilde{U}_x \subset \mathbb{R}^n \rightarrow U_x$ such that G_x acts linearly on $\tilde{U}_x$. Let $\{[1], [h_x^1], \dots, [h_x^{\rho_x}]\}$ denote the set of conjugacy classes in G_x with representatives $h_x^j \in G_x$. Let $C_{G_x}(h_x^j)$ denote the centralizer of each h_x^j and let $\tilde{U}_x^{h_x^j} \subset \tilde{U}_x$ denote the fixed point set of h_x^j . Then there is a bijection ([Kaw1, p. 77]) to a disjoint union of orbifolds

$$\left\{ (y, [h_y^j]) \mid y \in U_x, j = 1, \dots, \rho_y \right\} \xrightarrow{\cong} \bigcup_{j=1}^{\rho_x} \tilde{U}_x^{h_x^j} / C_{G_x}(h_x^j).$$

The strata of the orbifold M are the elements of the set

$$\Sigma M := \left\{ (x, [h_x^j]) \mid x \in M, j = 1, \dots, \rho_x, x \in M_{\text{sing}} \right\}.$$

Set $K_x^j := \ker(C_{G_x}(h_x^j) \rightarrow \text{Diffeo}(U_x^{h_x^j}))$. ΣM can be equipped with the orbifold atlas

$$\left\{ (C_{G_x}(h_x^j)/K_x^j, \tilde{U}_x^{h_x^j}) \rightarrow \tilde{U}_x^{h_x^j} / C_{G_x}(h_x^j) \mid x \in M, j = 1, \dots, \rho_x, x \in M_{\text{sing}}, U_x \text{ chart} \right\}.$$

Let $\dot{\bigcup}_{j \in J} M_j$ denote the decomposition of $M \dot{\cup} \Sigma M$ into connected components and set $n_j := \dim_{\mathbb{C}} M_j$. The multiplicity of M_j is given by $m_j := \#K_x^r$ for any $(x, [h_x^r]) \in M_j \subset \Sigma M$ and

$m_j := 1$ for $M_j \subset M$. This is well-defined, as this integer is locally constant in x ([Kaw1, p. 77]).

Let $\mathfrak{A}^{p,q}(M, E)$ denote the forms of holomorphic degree p and antiholomorphic degree q with coefficients in E and set $\mathfrak{A}^{p,q}(M) := \mathfrak{A}^{p,q}(M, \mathcal{O})$. Consider a holomorphic Hermitian vector bundle $E \rightarrow M$ over a complex manifold M . Let ∇^E be the associated canonical covariant derivative with curvature $\Omega^E \in \mathfrak{A}^{1,1}(M, \text{End } E)$. Let g be a biholomorphic map acting on M . Assume that E is g -invariant as a holomorphic Hermitian bundle and that E is equipped with an equivariant structure g^E . The Hermitian vector bundle E splits on the fixed point submanifold M^g into a direct sum $E|_{M^g} = \bigoplus_{\vartheta \in \mathbb{R}/(2\pi\mathbb{Z})} E_{\vartheta}$, where the equivariant structure g^E of E acts on E_{ϑ} as $e^{i\vartheta}$.

Definition 2.4. *The Chern character form on M^g is defined as*

$$\begin{aligned} \text{ch}_g(E, h^E) &:= \sum_{\vartheta} e^{i\vartheta} \text{ch}(E_{\vartheta}, h^{E_{\vartheta}}) := \sum_{\vartheta} e^{i\vartheta} \text{ch} \left(\frac{-\Omega^{E_{\vartheta}}}{2\pi i} \right) \\ &= \text{Tr } g^E + \sum_{\vartheta} e^{i\vartheta} c_1(E_{\vartheta}, h^{E_{\vartheta}}) + \sum_{\vartheta} e^{i\vartheta} \left(\frac{1}{2} c_1^2(E_{\vartheta}, h^{E_{\vartheta}}) - c_2(E_{\vartheta}, h^{E_{\vartheta}}) \right) + \dots \end{aligned}$$

The Todd form of an equivariant vector bundle is defined as

$$\text{Td}_g(E, h^E) := \frac{c_{\text{rk } E_0}(E_0, h^{E_0})}{\text{ch}_g(\sum_{j=0}^{\text{rk } E} (-1)^j \Lambda^j E^*, h^E)} = \text{Td} \left(\frac{-\Omega^{E_0}}{2\pi i} \right) \prod_{\vartheta \notin 2\pi\mathbb{Z}} \frac{\text{Td} \left(\frac{-\Omega^{E_{\vartheta}}}{2\pi i} + i\vartheta \right)}{c_{\text{top}}}.$$

Set as in [B, Def. 2.4]

$$\text{Td}'_t(E, h^E) := \frac{\partial}{\partial \varepsilon|_{\varepsilon=0}} \text{Td} \left(\frac{-\Omega^{E_0}}{2\pi i} + \varepsilon \text{id}_{E_0} \right) \prod_{\vartheta \notin 2\pi\mathbb{Z}} \frac{\text{Td} \left(\frac{-\Omega^{E_{\vartheta}}}{2\pi i} + i\vartheta + \varepsilon \text{id}_{E_{\vartheta}} \right)}{c_{\text{top}}}.$$

These are forms in $\bigoplus_{p=0}^{\dim M} \mathfrak{A}^{p,p}(M)$.

The associated characteristic classes are denoted by $\text{ch}_g(E)$, $\text{Td}_g(E)$ and $\text{Td}'_g(E)$. For $\zeta \in S^1$ and $s > 1$ consider the zeta function

$$L(\zeta, s) = \sum_{k=1}^{\infty} \frac{\zeta^k}{k^s}$$

and its meromorphic continuation to $s \in \mathbb{C}$. Consider the formal power series in x

$$\tilde{R}(\zeta, x) := \sum_{n=0}^{\infty} \left(\frac{\partial L}{\partial s}(\zeta, -n) + L(\zeta, -n) \sum_{j=1}^n \frac{1}{2^j} \right) \frac{x^n}{n!}.$$

Definition 2.5. *The Bismut equivariant R -class of an equivariant holomorphic vector bundle E is defined as the cohomology class*

$$R_g(E) := \left[\sum_{\vartheta} \left(\text{Tr } \tilde{R}(e^{i\vartheta}, -\frac{\Omega^{E_{\vartheta}}}{2\pi i}) - \text{Tr } \tilde{R}(e^{-i\vartheta}, \frac{\Omega^{E_{\vartheta}}}{2\pi i}) \right) \right].$$

This generalizes the non-equivariant *Gillet-Soulé R-class*. Given a characteristic class Φ , let

$$\tilde{\Phi}(\xi) \in \tilde{\mathfrak{A}}(M) := \bigoplus_{p \geq 0} \mathfrak{A}^{p,p}(M) / (\text{im } \partial + \text{im } \bar{\partial})$$

be the associated equivariant Bott-Chern secondary class as introduced in [KR, Th. 3.4], following the non-equivariant axiomatic definition in [BGS1, Section f]. Notice though that there are two sign conventions for Bott-Chern classes, one used by Gillet and Soulé and one used by Bismut, and in this article we work with Bismut's convention as used in [BGS1]: For any short exact sequence of g -equivariant holomorphic vector bundles $\xi : 0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$, equipped with any g -invariant Hermitian metrics $h^{E'}, h^E, h^{E''}$, the associated Chern-Weil differential forms satisfy

$$\frac{\bar{\partial}\partial}{2\pi i} \tilde{\Phi}(\xi) = \Phi(E, h^E) - \Phi(E' \oplus E'', (h^{E'}, h^{E''})).$$

In particular, for two Hermitian metrics h^E, h'^E on a holomorphic vector bundle E the class $\tilde{\Phi}$ provides the transgression

$$\frac{\bar{\partial}\partial}{2\pi i} \tilde{\Phi}_g(E, h^E, h'^E) = \Phi_g(E, h'^E) - \Phi_g(E, h^E).$$

For example, $\tilde{c}_1(E, h^E, h'^E) = \log \det \frac{h^E}{h'^E}$.

Following Kawasaki ([Kaw2, p. 153]), Ma defines the following characteristic classes for proper orbifold vector bundle $E \rightarrow M$ on an orbifold M in [Ma1, (1.7)]: Let E be represented by local charts $\tilde{E}_U \rightarrow \tilde{U}$ and consider the characteristic forms above as forms on $\tilde{U}^g$. Then the orbifold characteristic classes are represented locally by the corresponding g -equivariant characteristic forms on each connected component of $\tilde{U}^g/C_{G_U}(g)$. Hence they are given by

$$\text{ch}^\Sigma(E)_{|\tilde{U}^g/C_{G_U}(g)} := \text{ch}_g(\tilde{E}_U), \quad \text{Td}^\Sigma(E)_{|\tilde{U}^g/C_{G_U}(g)} := \text{Td}_g(\tilde{E}_U), \quad \text{Td}'^\Sigma(E)_{|\tilde{U}^g/C_{G_U}(g)} := \text{Td}'_g(\tilde{E}_U).$$

Analogously, orbifold versions of Bott-Chern classes are defined ([Ma1, Eq. (1.8)]).

3 Holomorphic torsion

Let M be a compact complex orbifold of dimension n equipped with an Hermitian metric with associated Kähler form ω . Consider an Hermitian holomorphic orbifold bundle $(E, h^E) \rightarrow M$ and the corresponding Dolbeault operator $\bar{\partial}^E : \mathfrak{A}^{0,q}(M, E) \rightarrow \mathfrak{A}^{0,q+1}(M, E)$ acting on antiholomorphic q -forms with coefficients in E . The space $\mathfrak{A}^{0,q}(M, E)$ is equipped with the L^2 -metric

$$(\eta, \eta')_{L^2} := \int_M \langle \eta|_x, \eta'|_x \rangle \frac{\omega^n}{(2\pi)^n n!}.$$

In [Ma1, Section 2.2] Ma extends Ray-Singer's definition of holomorphic torsion to orbifolds: Using the formal adjoint $\bar{\partial}^{E*}$ with respect to this L^2 -metric, let $\square^q := (\bar{\partial}^E + \bar{\partial}^{E*})^2$ denote the Kodaira-Laplace operator acting on $\mathfrak{A}^{0,q}(M, E)$. By [Ma1, p. 2214], $\square^q$ is a self-adjoint elliptic operator and thus the spectrum $\sigma(\square^q)$ on the compactum M is real and discrete. The zeta function

$$Z^\Sigma(s) := \sum_{q=0}^n (-1)^{q+1} q \sum_{\substack{\lambda \in \sigma(\square^q) \\ \lambda \neq 0}} \frac{\dim \text{Eig}_\lambda(\square^q)}{\lambda^s}$$

is well-defined for $\operatorname{Re} s \gg 0$ and its meromorphic continuation to the complex plane has a well-defined derivative $T(M, E) := (Z^\Sigma)'(0) \in \mathbb{R}$ at $s = 0$, the *orbifold holomorphic torsion*. Now consider an Hermitian holomorphic bundle $(E, h^E) \rightarrow M$ over an Hermitian manifold and assume that a holomorphic isometry g acts on M , leaves E invariant and that E is equipped with an equivariant structure g^E . This induces an action of g on the eigenspaces of $\square^q$ and we define

$$Z_g(s) := \sum_{q=0}^n (-1)^{q+1} q \sum_{\substack{\lambda \in \sigma(\square^q) \\ \lambda \neq 0}} \frac{\operatorname{Tr} g|_{\operatorname{Eig}_\lambda(\square^q)}}{\lambda^s}$$

for $\operatorname{Re} s \gg 0$ and the *equivariant holomorphic torsion* as $T_g(M, E) := Z'_g(0) \in \mathbb{C}$ as in [K1].

4 The asymptotic expansion

Let E, L, F be proper Hermitian holomorphic orbifold bundles over a compact effective Hermitian orbifold M . Assume that F is flat and that L is a positive line bundle. Let $\square_p^q$ denote the Kodaira-Laplace operator acting on antiholomorphic forms of degree q with coefficients in $E \otimes L^p \otimes F$. Let Z_p denote the zeta function defining the torsion $T(M, E \otimes L^p \otimes F) = Z'_p(0)$ of $E \otimes L^p \otimes F$. For M smooth Bismut and Vasserot prove in [BV, Eq. (14)] that there is an asymptotic expansion

$$p^{-n} \operatorname{Tr} \exp \left(-\frac{t}{p} \square_p^q \right) = \operatorname{rk} F \cdot \sum_{j=-n}^k a_{p,j}^q t^j + o(t^k), \quad (1)$$

where the $a_{p,j}^q$ do not depend on F . Using this equation ([Fi, Eq. (2.19)]) and the notation $b_{p,j} := \sum_{q=0}^n (-1)^q q a_{p,j}^q$, Finski shows in [Fi, Th. 2.7] that

$$Z'_p(0) = \sum_{j=0}^k p^{n-j} (\alpha_j \log p + \beta_j) + o(p^{n-k}) \quad (2)$$

where α_j, β_j are given in terms of $b_{p,j}$ such that they take the form $\operatorname{rk} F$ times a factor which is independent of F . For an orbifold M , [Fi, Th. 5.12] extends this result using coefficients $\tilde{\alpha}_j, \tilde{\beta}_j$:

Theorem 4.1. (*Finski, [Fi, Th. 1.5]*) *Let M be a compact effective Hermitian orbifold of dimension n and let $(L, h^L), (E, h^E) \rightarrow M$ be proper Hermitian orbifold vector bundles with $\operatorname{rk} L = 1$ and L positive. Then*

$$\begin{aligned} T(M, E \otimes L^p) &= \sum_{j=0}^k p^{n-j} (\tilde{\alpha}_j(\rho) \log p + \tilde{\beta}_j(\rho)) \\ &+ \sum_r \sum_{j=0}^{k+n_r-n} \frac{p^{n_r-j}}{m_r} e^{ip\vartheta_r} (\gamma_{r,j}(\rho) \log p + \kappa_{r,j}(\rho)) + o(p^{n-k}) \end{aligned}$$

where $e^{i\vartheta_r}$ is the action on L on the stratum M_r ([Fi, (5.5)]) and $\gamma_{r,j}, \kappa_{r,j}$ are integrals over local data ([Fi, Th. 5.12]).

As is detailed in [Fi, Rem. 1.7(3)], [Fi, Th. 5.12] shows that the $\tilde{\alpha}_j, \tilde{\beta}_j$ are given by integrals over the same local quantities as for α_j, β_j .

Proof of Th. 1.1. Let Z_p denote the zeta function defining the torsion $T(M, E \otimes L^p) = Z'_p(0)$ of $E \otimes L^p$ and set $\tilde{Z}_p(s) := p^{s-n} Z_p(s)$. Hence one gets the analogue of [BV, Eq. (40)]

$$Z'_p(0) = -p^n \log p \cdot \tilde{Z}_p(0) + p^n \tilde{Z}'_p(0).$$

Choose m as the lcm of the orders of local groups G_U of a finite atlas. [Fi, Th. 5.18] proves that $\tilde{Z}'_p(0)$ has an asymptotic expansion of the form

$$\sum_{j=0}^k p^{-j} \tilde{\beta}_j + \sum_{j=0}^{k+n_{\max}-n} \sum_{r \in \mathbb{Z}/(m)} p^{n_{\max}-n-j} \zeta_m^{rp} \tilde{\kappa}_{r,j} + o(p^{-k})$$

as in the statement. By definition, using the number operator $N : \mathfrak{A}^{0,q}(M, E) \rightarrow \mathfrak{A}^{0,q}(M, E)$, $\alpha \rightarrow q\alpha$ for $q \in \mathbb{N}_0$,

$$\tilde{Z}_p(s) = \frac{-p^{-n}}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \text{Tr}_s N e^{-\frac{t}{p} \square_p} dt.$$

This integral is well-defined for $\text{Re } s \gg 0$, as $\square_p^q$ is invertible for large p and $q \geq 1$ and the summand with $q = 0$ is erased by N . Now equip M with the Kähler metric induced by $c_1(L)$. For Kähler metrics [Ma1, Th. 2.4] states that

$$\text{Tr}_s N e^{-u \square_p} = \frac{1}{u} C_{-1} + C_0 + O(u)$$

as $u \searrow 0$ where

$$C_0 = \sum_j \frac{1}{m_j} \int_{M_j} \left(\dim M \cdot \text{Td}^\Sigma(TM) - \text{Td}'^\Sigma(TM) \right) \text{ch}^\Sigma(E \otimes L^p)$$

by [Ma1, Eq. (2.24)]. Replacing u by $\frac{t}{p}$ implies $\tilde{Z}_p(0) = -p^{-n} C_0$. Finally [Fi, p. 44] shows that $\tilde{Z}_p(0)$ is independent of the metric on M , using Ma's analogue [Ma1, Th 0.1] of the anomaly formula for orbifolds and [DLM, Th. 1.2]. Thus the formula holds for arbitrary Hermitian metrics. $\square$

Now we shall check that Th. 1.1 is compatible with the explicit formulae for the first coefficients given by Bismut and Vasserot in [BV, Th. 8] and Finski in [Fi, Th. 1.3, Th. 1.5, Rem. 1.7(3)]. In the case E trivial and M smooth this verification has already been done in [K3, Th. 6.5]. We shall use the expansions

$$\begin{aligned} \text{Td}(TM) &= 1 + \frac{c_1(TM)}{2} + \text{higher order}, \\ \frac{\text{Td}'(TM)}{\text{Td}(TM)} - n &= -\frac{n}{2} - \frac{c_1(TM)}{12} + \text{higher order}, \\ \text{ch}(L^p) &= \frac{p^n}{n!} c_1(L)^n + \frac{p^{n-1}}{(n-1)!} c_1(L)^{n-1} + O(p^{n-2}). \end{aligned}$$

Finski's formula for $\gamma_{r,0}$ in [Fi, Eq. (1.9)] follows immediately from the summands with $m_j > 1$ in Th. 1.1. If the singular strata have codimension at least 2, the coefficient of $\log p$ in Th. 1.1 equals

$$\log p \cdot \int_{M_{\text{reg}}} (n \cdot \text{Td}(TM) - \text{Td}'(TM)) \text{ch}(E \otimes L^p) + o(p^{n-1})$$

$$\begin{aligned}
&= \log p \cdot \int_{M_{\text{reg}}} \left(1 + \frac{c_1(TM)}{2}\right) \left(\frac{n}{2} + \frac{c_1(TM)}{12}\right) (\text{rk } E + c_1(E)) \\
&\quad \cdot \left(\frac{p^n}{n!} c_1(L)^n + \frac{p^{n-1}}{(n-1)!} c_1(L)^{n-1}\right) + o(p^{n-1}) \\
&= \log p \cdot \int_{M_{\text{reg}}} \left(\frac{p^n}{2(n-1)!} c_1(L)^n \text{rk } E \right. \\
&\quad \left. + \left(\frac{3n+1}{12} c_1(TM) \text{rk } E + \frac{n}{2} c_1(E)\right) \cdot \frac{p^{n-1}}{(n-1)!} c_1(L)^{n-1}\right) + o(p^{n-1}).
\end{aligned}$$

This corresponds exactly to the results by Bismut, Vasserot and Finski.

Let (g^{TM}, h^E, h^L) , (g'^{TM}, h'^E, h'^L) be two triples of metrics on TM, E, L . Set

$$\lambda_p := \det H^0(M, E \otimes L^p)^{-1}.$$

The Hodge embedding $H^0(M, E \otimes L^p) \cong \ker \square^0 \subset \Gamma^\infty(M, E \otimes L^p)$ induces associated L^2 -metrics $h'_{\lambda,p}, h_{\lambda,p}$ on λ_p .

Corollary 4.2. *Let g^{TM}, g'^{TM} be two Kähler metrics on a compact n -dimensional effective orbifold M and let $(L, h^L), (E, h^E) \rightarrow M$ be proper Hermitian orbifold vector bundles with $\text{rk } L = 1$ and L positive. Let h'^E denote a second Hermitian metric on E . Then there is a number $m \in \mathbb{Z}^+$ and families $\beta''_\bullet : \mathbb{N}_0 \rightarrow \mathbb{R}$, $\kappa''_\bullet : \mathbb{Z}/(m) \times \mathbb{N}_0 \rightarrow \mathbb{R}$ such that for any $k \in \mathbb{N}_0$ the L^2 -metrics on λ_p satisfy for $p \rightarrow +\infty$*

$$\begin{aligned}
\log \frac{h'_{\lambda,p}}{h_{\lambda,p}} &= p^n \int_M \left(\text{rk } E \cdot \log \det \frac{g^{TM}}{g'^{TM}} + \log \det \frac{h^E}{h'^E} \right) \cdot \frac{c_1(L, h^L)^n}{n!} \\
&\quad + \sum_{j=1}^k p^{n-j} \beta''_j + \sum_{j=0}^{k+n_{\max}-n} \sum_{r \in \mathbb{Z}/(m)} p^{n_{\max}-j} \zeta_m^{rp} \kappa''_{r,j} + o(p^{n-k}).
\end{aligned}$$

This extends the asymptotic anomaly formula [BV, Th. 10] to the Kähler orbifold case. Varying h^L adds a term that is proportional to p^{n+1} .

Proof. Ma shows in [Ma1, Th. 0.1] the following anomaly formula for two pairs of metrics $(h^E, h^L), (h'^E, h'^L)$ on E and L and two Kähler metrics g^{TM}, g'^{TM} on M :

$$\begin{aligned}
&\log \frac{h'_{\lambda,p}}{h_{\lambda,p}} - T(M, E \otimes L^p, (g'^{TM}, h'^E, h'^L)) + T(M, E \otimes L^p, (g^{TM}, h^E, h^L)) \\
&= \sum_j \frac{1}{m_j} \int_{M_j} \widetilde{\text{Td}}^\Sigma(TM, g^{TM}, g'^{TM}) \text{ch}^\Sigma(E \otimes L^p, (h^E, h^L)) \\
&\quad + \sum_j \frac{1}{m_j} \int_{M_j} \text{Td}^\Sigma(TM, g'^{TM}) \widetilde{\text{ch}}^\Sigma(E \otimes L^p, (h^E, h^L), (h'^E, h'^L)). \\
&\stackrel{h^L \equiv h'^L}{=} p^n \int_{M_{\text{reg}}} \left(\frac{1}{2} \widetilde{c}_1(TM, g^{TM}, g'^{TM}) \text{rk } E + \widetilde{c}_1(E, h^E, h'^E) \right) \frac{c_1(L, h^L)^n}{n!} \\
&\quad + \sum_{j=1}^n p^{n-j} \widetilde{\beta}''_j + \sum_{j=0}^{n_{\max}} \sum_{r \in \mathbb{Z}/(m)} p^{n_{\max}-j} \zeta_m^{rp} \widetilde{\kappa}''_{r,j} \\
&= p^n \int_{M_{\text{reg}}} \left(\frac{1}{2} \log \det \frac{g^{TM}}{g'^{TM}} \text{rk } E + \log \det \frac{h^E}{h'^E} \right) \frac{c_1(L, h^L)^n}{n!}
\end{aligned}$$

$$+ \sum_{j=1}^n p^{n-j} \tilde{\beta}_j'' + \sum_{j=0}^{n_{\max}} \sum_{r \in \mathbb{Z}/(m)} p^{n_{\max}-j} \zeta_m^{rp} \tilde{\kappa}_{r,j}''.$$

The right hand side is a polynomial in p of degree n with coefficients in functions of the form $e^{i\vartheta p}$. As the coefficient of $\log p$ in Th. 1.1 is independent of the metrics, the difference of the holomorphic torsions has an asymptotic expansion in powers of p as in the statement. In [Fi, Th. 1.5] it is shown that the coefficient $\tilde{\beta}_0$ in Th. 1.1 is equal to

$$\tilde{\beta}_0 = \frac{\text{rk } E}{2} \int_M \frac{c_1(L, h^L)^n}{n!} \log \det \frac{\Omega^L}{2\pi\omega^{TM}}.$$

Thus one finds when $h^L = h'^L$

$$\begin{aligned} & T(M, E \otimes L^p, (g'^{TM}, h'^E, h'^L)) - T(M, E \otimes L^p, (g^{TM}, h^E, h^L)) \\ &= p^n \frac{\text{rk } E}{2} \int_M \frac{c_1(L, h^L)^n}{n!} \log \det \frac{g^{TM}}{g'^{TM}} + \text{terms of lower order.} \end{aligned}$$

The corollary is obtained by combining these two expansions. $\square$

If Γ is a finite group and M/Γ is an effective global quotient, i.e. Γ acts smoothly effectively on M , the orbifold torsion and the equivariant torsion can be computed from each other. The equivariant torsion and the orbifold torsion are compared in [KR, p. 345] in the case where M/Γ is smooth, and in [Ma1, Eq. (0.6)] for ρ being the trivial representation.

Lemma 4.3. *Consider a right action by finite group Γ on an Hermitian manifold M by holomorphic isometries. Let (E, h^E) be a Γ -equivariant holomorphic Hermitian vector bundle. For a unitary representation (V_ρ, ρ) of Γ with character χ_ρ let $F_\rho := M \times_\rho V_\rho$ denote the associated flat Hermitian orbifold vector bundle on M/Γ . Then the orbifold holomorphic torsion on M/Γ and the equivariant torsion are Fourier transforms of each other. More precisely,*

$$T(M/\Gamma, E/\Gamma \otimes F_\rho) = \frac{1}{\#\Gamma} \sum_{g \in \Gamma} \chi_\rho(g) T_g(M, E)$$

and, equivalently, with the set $\Gamma^\vee$ of irreducible characters χ_ρ

$$T_g(M, E) = \sum_{\rho \in \Gamma^\vee} \overline{\chi_\rho(g)} T(M/\Gamma, E/\Gamma \otimes F_\rho) \quad \forall g \in \Gamma.$$

Proof. Consider a unitary representation $\rho : \Gamma \rightarrow V_\rho$, an equivariant structure with $g^E : E_{xg} \rightarrow E_x$ for any $g \in \Gamma$ and the induced action $g : \Gamma(M, E) \rightarrow \Gamma(M, E)$, $s \mapsto (x \mapsto g^E(s_{xg}))$. Let

$$P_\rho := \frac{1}{\#\Gamma} \sum_{g \in \Gamma} g^E \otimes \rho(g).$$

denote the projection on $\mathfrak{A}^{0,q}(M, E) \otimes V_\rho$ to the Γ -invariant part. Then

$$\begin{aligned} P_\rho(\mathfrak{A}^{0,q}(M, E) \otimes V_\rho) &= \{s \otimes v \in \mathfrak{A}^{0,q}(M, E) \otimes V_\rho \mid g^* s \otimes v = s \otimes \rho(g)v \text{ for all } g \in \Gamma\} \\ &= \mathfrak{A}^{0,q}(M/\Gamma, E/\Gamma \otimes F_\rho). \end{aligned}$$

Hence when $P^\perp$ denotes the projection on the orthogonal complement of $\ker \square$, one obtains for any q

$$\text{Tr}_s \square^{-s} P_{|\mathfrak{A}^{0,q}(M/\Gamma, E/\Gamma \otimes F_\rho)}^\perp = \text{Tr}_s P_\rho \square^{-s} P_{|\mathfrak{A}^{0,q}(M, E) \otimes V_\rho}^\perp$$

$$= \frac{1}{\#\Gamma} \sum_{g \in \Gamma} \text{Tr } \rho(g) \cdot \text{Tr}_s g^* \square^{-s} P_{|\mathfrak{A}^{0,q}(M,E)}^\perp.$$

Thus the first equality in the theorem follows. It is equivalent to the second one by the Schur orthogonality relations ([CR, p. 211, Prop. 2.3]). $\square$

This result extends to equivariant determinants equipped with Quillen metrics, as the isotypic decomposition shows that

$$\begin{aligned} \mathfrak{A}^{0,q}(M, E) &\cong \bigoplus_{\rho \in \Gamma^\vee} \text{Hom}_\Gamma(V_\rho^*, \mathfrak{A}^{0,q}(M, E)) \otimes V_\rho^* \\ &= \bigoplus_{\rho \in \Gamma^\vee} P_\rho(\mathfrak{A}^{0,q}(M, E) \otimes V_\rho) \otimes V_\rho^* = \bigoplus_{\rho \in \Gamma^\vee} \mathfrak{A}^{0,q}(M/\Gamma, E/\Gamma \otimes F_\rho) \otimes V_\rho^*. \end{aligned}$$

Proof of Th. 1.2. We continue to work with g being an element of a general finite group Γ rather than just the cyclic group generated by g , since this makes no difference to the proof. In the case of the orbifold M/Γ , for the bundles $E/\Gamma \otimes (L/\Gamma)^p \otimes F_\rho$ Th. 1.1 takes the form

$$\begin{aligned} T(M/\Gamma, E/\Gamma \otimes (L/\Gamma)^p \otimes F_\rho) \\ = \frac{\log p}{\#\Gamma} \sum_{h \in \Gamma} \int_{M^h} (n \cdot \text{Td}_h(TM) - \text{Td}'_h(TM)) \text{ch}_h(E \otimes L^p \otimes V_\rho) \\ + \sum_{j=0}^k p^{n-j} \tilde{\beta}_{j,\rho} + \sum_{j=0}^{k+n_{\max}-n} \sum_{r \in \mathbb{Z}/(m)} p^{n_{\max}-j} \zeta_m^{rp} \tilde{\kappa}_{r,j,\rho} + o(p^{n-k}) \end{aligned}$$

where we can choose $m := \#\Gamma$ and we added indices for the representation ρ . Since F_ρ is flat, $\text{ch}_h(F_\rho) = \chi_\rho(h)$ holds. The characteristic classes depend on h only up to conjugation in Γ . By the Schur orthogonality relations,

$$\sum_{\rho \in \Gamma^\vee} \overline{\chi_\rho(g)} \chi_\rho(h) = \begin{cases} \#C_\Gamma(g) & \text{if } g, h \text{ are conjugate,} \\ 0 & \text{else.} \end{cases}$$

Thus Lemma 4.3 implies

$$\begin{aligned} T_g(M, E \otimes L^p) \\ = \sum_{\rho \in \Gamma^\vee} \overline{\chi_\rho(g)} \left(\frac{\log p}{\#\Gamma} \sum_{h \in \Gamma} \int_{M^h} (n \cdot \text{Td}_h(TM) - \text{Td}'_h(TM)) \text{ch}_h(E \otimes L^p \otimes V_\rho) \right. \\ \left. + \sum_{j=0}^k p^{n-j} \tilde{\beta}_{j,\rho} + \sum_{j=0}^{k+n_{\max}-n} \sum_{r \in \mathbb{Z}/(m)} p^{n_{\max}-j} \zeta_m^{rp} \tilde{\kappa}_{r,j,\rho} \right) + o(p^{n-k}) \\ = \frac{\log p}{\#\Gamma} \sum_{h \in \Gamma} \left(\sum_{\rho \in \Gamma^\vee} \overline{\chi_\rho(g)} \chi_\rho(h) \right) \int_{M^h} (n \cdot \text{Td}_h(TM) - \text{Td}'_h(TM)) \text{ch}_h(E \otimes L^p) \\ + \sum_{\rho \in \Gamma^\vee} \overline{\chi_\rho(g)} \left(\sum_{j=0}^k p^{n-j} \tilde{\beta}_{j,\rho} + \sum_{j=0}^{k+n_{\max}-n} \sum_{r \in \mathbb{Z}/(m)} p^{n_{\max}-j} \zeta_m^{rp} \tilde{\kappa}_{r,j,\rho} \right) + o(p^{n-k}) \\ = \log p \cdot \frac{\#C_\Gamma(g)}{\#\Gamma} \sum_{\substack{h \text{ conjugate} \\ \text{to } g}} \int_{M^h} (n \cdot \text{Td}_h(TM) - \text{Td}'_h(TM)) \text{ch}_h(E \otimes L^p) \end{aligned}$$

$$\begin{aligned}
& + \sum_{\rho \in \Gamma^\vee} \overline{\chi_\rho(g)} \left(\sum_{j=0}^k p^{n-j} \tilde{\beta}_{j,\rho} + \sum_{j=0}^{k+n_{\max}-n} \sum_{r \in \mathbb{Z}/(m)} p^{n_{\max}-j} \zeta_m^{rp} \tilde{\kappa}_{r,j,\rho} \right) + o(p^{n-k}) \\
& = \log p \cdot \int_{M^g} (n \cdot \text{Td}_g(TM) - \text{Td}'_g(TM)) \text{ch}_g(E \otimes L^p) \\
& + \sum_{\rho \in \Gamma^\vee} \overline{\chi_\rho(g)} \left(\sum_{j=0}^k p^{n-j} \tilde{\beta}_{j,\rho} + \sum_{j=0}^{k+n_{\max}-n} \sum_{r \in \mathbb{Z}/(m)} p^{n_{\max}-j} \zeta_m^{rp} \tilde{\kappa}_{r,j,\rho} \right) + o(p^{n-k}).
\end{aligned}$$

As detailed after Eq. (2), the coefficients $\tilde{\alpha}_{j,\rho}, \tilde{\beta}_{j,\rho}$ in Th. 4.1 take the form $\text{rk } F_\rho$ times a factor which is independent of F_ρ . If $g = e_G$, then the powers of p start at $n = n_{\max}$. In the case $g \neq e_G$ one finds $\sum_{\rho \in \Gamma^\vee} \overline{\chi_\rho(g)} \text{rk } F_\rho = 0$ and thus $\sum_{\rho \in \Gamma^\vee} \overline{\chi_\rho(g)} \tilde{\beta}_{j,\rho} = 0$. $\square$

It would be interesting to have a general direct proof of Th. 1.2 for general holomorphic isometries g . In this equivariant case [B, Th. 8.3] computes the analogue of [Ma1, Th. 2.4]. In the non-equivariant case this formula has been shown in [BGS2, Th. 2.16] (see also [BV, Eq. (36)]). Classically (see [GS1, 1.2-1.3] and [K2, p. 9], which use the opposite sign convention for Bott-Chern classes though) the Bott-Chern forms associated to scaling of the metric by a constant are given by derivatives of the characteristic forms:

$$\begin{aligned}
\tilde{\text{ch}}_g(E, c^2 h^E, h^E) &= \log c^2 \cdot \frac{\partial}{\partial \varepsilon|_{\varepsilon=0}} \sum_{\vartheta} \text{ch} \left(-\frac{\Omega^{E_\vartheta}}{2\pi i} + i\vartheta + \varepsilon \text{id}_E \right) = \log c^2 \cdot \text{ch}_g(E, h^E), \\
\widetilde{\text{Td}}_g(E, c^2 h^E, h^E) &= \log c^2 \cdot \text{Td}'_g(E, h^E).
\end{aligned}$$

Hence

$$\begin{aligned}
& \log p \cdot \int_{M^g} (n \cdot \text{Td}_g(TM) - \text{Td}'_g(TM)) \text{ch}_g(E \otimes L^p) \\
& = \int_{M^g} \text{Td}_g(TM) \tilde{\text{ch}}_g(E \otimes L^p, p^n h^{E \otimes L^p}, h^{E \otimes L^p}) + \int_{M^g} \widetilde{\text{Td}}_g(TM, \frac{1}{p} g^{TX}, g^{TX}) \text{ch}_g(E \otimes L^p).
\end{aligned}$$

We also obtain an asymptotic anomaly formula like Cor. 4.2, [BV, Th. 10], for the equivariant Kähler case. As above let g act as a holomorphic isometry on M , L and E , such that g^m acts as the identity. Then $H^0(M, E \otimes L^p)$ splits into eigenspaces $H^0(M, E \otimes L^p)_r$ on which g acts as ζ_m^r for $r \in \mathbb{Z}/(m)$. Set $\lambda_{p,r} := \det H^0(M, E \otimes L^p)_r^{-1}$. Again the Hodge embedding induces L^2 -metrics $h'_{\lambda,p,r}, h_{\lambda,p,r}$ on $\lambda_{p,r}$.

Corollary 4.4. *Let g^{TM}, g'^{TM} be two Kähler metrics on a compact n -dimensional manifold M and let $(L, h^L), (E, h^E) \rightarrow M$ be Hermitian vector bundles with $\text{rk } L = 1$ and L positive. Let h'^E denote a second Hermitian metric on E . Assume that M , E and L are invariant under the action of an holomorphic isometry g of finite order m and that E and L are equipped with a g -equivariant structure. Then there is a family $\tilde{\kappa}'_\bullet : \mathbb{Z}/(m) \times \mathbb{N}_0 \rightarrow \mathbb{R}[\zeta_m]$ such that for any $k \in \mathbb{N}_0$ the L^2 -metrics on $\lambda_{p,r}$ satisfy for $p \rightarrow +\infty$*

$$\sum_{r \in \mathbb{Z}/(m)} \zeta_m^r \log \frac{h'_{\lambda,p,r}}{h_{\lambda,p,r}} = \sum_{j=0}^k p^{n_{\max}-j} \sum_{r \in \mathbb{Z}/(m)} \zeta_m^{rp} \tilde{\kappa}'_{r,j} + o(p^{n_{\max}-k}).$$

Proof. Bismut shows in [B, Th. 0.1, Th. 2.5] that

$$\sum_{r \in \mathbb{Z}/(m)} \zeta_m^r \log \frac{h'_{\lambda,p,r}}{h_{\lambda,p,r}} = T_g(M, E \otimes L^p, (g'^{TM}, h'^E, h^L)) + T_g(M, E \otimes L^p, (g^{TM}, h^E, h^L))$$

$$\begin{aligned}
&= \int_{M^g} \widetilde{\text{Td}}_g(TM, g^{TM}, g'^{TM}) \text{ch}_g(E \otimes L^p, (h^E, h^L)) \\
&\quad + \int_{M^g} \text{Td}_g(TM, g'^{TM}) \widetilde{\text{ch}}_g(E \otimes L^p, (h^E, h^L), (h'^E, h'^L)).
\end{aligned}$$

which is a polynomial in p with coefficients as in the statement. By Th. 1.2 the difference of the holomorphic torsions also has the form of the right-hand side in the statement. $\square$

5 An application to Arakelov geometry

We consider the same setting as in [GS2] and [KR]. As in [GS2, Def. 3.1.1, p. 124] let a *regular arithmetic ring* D be an excellent regular Noetherian integral ring, together with a finite nonempty set $\mathcal{S}$ of ring monomorphisms $D \rightarrow \mathbb{C}$, which is invariant under complex conjugation. By μ_m we denote the diagonalisable group scheme over D associated to $\mathbb{Z}/(m)$. Extending [GS2, Def. 3.2.1] an *equivariant arithmetic variety* is defined in [KR, p. 357] as a regular integral scheme which is endowed with a μ_m -projective action over $\text{Spec } D$. Let $f : X \rightarrow \text{Spec } D$ be an equivariant arithmetic variety of dimension n . The set $X(\mathbb{C})$ of complex points of the variety $X \times_D \mathbb{C}^{\mathcal{S}}$ naturally carries the structure of a complex manifold. The group of m -th roots of unity acts on $X(\mathbb{C})$ by holomorphic automorphisms. We shall write g for the automorphism corresponding to ζ_m . By [KR, Prop. 2.12], the fixed point scheme X^{μ_m} is regular and by [KR, Cor. 2.11] and the GAGA principle, there are natural isomorphisms of complex manifolds $X^{\mu_m}(\mathbb{C}) \simeq X(\mathbb{C})^g$. Let f^{μ_m} denote the map $X^{\mu_m} \rightarrow \text{Spec } D$ induced by f . Complex conjugation induces antiholomorphic automorphisms F_∞ of $X(\mathbb{C})$ and $X^{\mu_m}(\mathbb{C})$.

An Hermitian equivariant vector bundle on X is a vector bundle $E \rightarrow X$ equipped with a μ_m -action which lifts the action of μ_m on X and an Hermitian metric h^E on $E_{\mathbb{C}}$, the bundle associated to E on the complex points, which is invariant under F_∞ and μ_m . There is a natural orthogonal $\mathbb{Z}/(m)$ -grading $E|_{X^{\mu_m}} \simeq \bigoplus_{k \in \mathbb{Z}/(m)} E_k$ on the restriction of E to the fixed point scheme X^{μ_m} .

In [GS2]) Gillet and Soulé associate an arithmetic Chow ring $\widehat{\text{CH}}(X)$ to each such variety, which carries a natural grading analogous to the grading of the classical Chow group. For every projective morphism $f : X' \rightarrow X$ which is smooth over $\mathbb{Q}$ between equivariant arithmetic varieties over D , there is a push-forward map $f_* : \widehat{\text{CH}}(X') \rightarrow \widehat{\text{CH}}(X)$.

For an Hermitian bundle E on X , Gillet and Soulé define arithmetic characteristic classes like an *arithmetic Chern character* $\widehat{\text{ch}}(E) \in \widehat{\text{CH}}(X)_{\mathbb{Q}}$ and an *arithmetic Todd class* $\widehat{\text{Td}}(E) \in \widehat{\text{CH}}(X)_{\mathbb{Q}}$. As in [KR, Def. 7.13] we set

$$\widehat{\text{ch}}_{\mu_m}(E, h^E) := \sum_{k \in \mathbb{Z}/(m)} \widehat{\text{ch}}(E_k, h^E) \otimes \zeta_m^k \in \widehat{\text{CH}}(X^{\mu_m}) \otimes_{\mathbb{Z}} \mathbb{C}.$$

To f as above Gillet and Soulé construct an element $\widehat{\text{Td}}(Tf) \in \widehat{\text{CH}}(X)_{\mathbb{Q}}$ to the map f and the Kähler form ω_X . In [KR, p. 393], using the normal bundle N to $X^{\mu_m} \hookrightarrow X$ the *equivariant arithmetic Todd class* is defined as

$$\widehat{\text{Td}}_{\mu_m}(E, h^E) := \left(\widehat{\text{ch}}_{\mu_m} \left(\sum_{q=0}^{\text{rk } N} (-1)^q \Lambda^q N^\vee, g^{TX} \right) \right)^{-1} \widehat{\text{Td}}(Tf^{\mu_m}, g^{TX}) \in \widehat{\text{CH}}(X^{\mu_m}) \otimes_{\mathbb{Z}} \mathbb{C}..$$

For simplicity's sake we shall suppose now that $D = \mathbb{Z}$. Let X be projective and flat over $\text{Spec } \mathbb{Z}$. Denote by A_{Tors} the torsion subgroup of an abelian group A . Gillet and Soulé show that there is a natural isomorphism $\widehat{\text{deg}} : \widehat{\text{CH}}^1(\mathbb{Z}) \rightarrow \mathbb{R}$, called the *arithmetic degree*. It is given

by $\widehat{\deg}(\widehat{\text{ch}}^{[1]}(V, h^V)) = -\log \frac{\text{covol}(V, h^V)}{\#V_{\text{Tors}}}$ for a finitely generated Hermitian $\mathbb{Z}$ -module (V, h^V) . In [KR, Th. 7.14] an equivariant refinement of Gillet-Soulés Riemann-Roch theorem for the arithmetic Chern character and the push-forward map in arithmetic Chow theory ([GS3]) is proven.

Theorem 5.1. ([KR, Th. 7.14]) *Let E be an equivariant Hermitian vector bundle on X . Then the equality*

$$\begin{aligned} & - \sum_{q \geq 0} (-1)^q \sum_{k \in \mathbb{Z}/(m)} \zeta_m^k \cdot \log \frac{\text{covol}(H^q(X, E)_k, |\cdot|_{L^2})}{\#H^q(X, E)_{k, \text{Tors}}} \\ &= \frac{1}{2} T_g(X(\mathbb{C}), E_{\mathbb{C}}) - \frac{1}{2} \int_{X^{\mu_m}(\mathbb{C})} \text{Td}_g(TX_{\mathbb{C}}) \text{ch}_g(E_{\mathbb{C}}) R_g(TX_{\mathbb{C}}) \\ & \quad + \widehat{\deg} \left(f_*^{\mu_m} (\widehat{\text{Td}}_{\mu_m}(Tf, g^{TX}) \widehat{\text{ch}}_{\mu_m}(E, h^E)) \right) \end{aligned}$$

of complex numbers holds.

Proof of Th. 1.3. Let $X(\mathbb{C})^g =: \bigcup_{j \in J} X(\mathbb{C})_j^g$ denote the connected components of $X^{\mu_m}(\mathbb{C})$ of dimension $n_j := \dim X(\mathbb{C})_j^g$. As a function in p ,

$$\begin{aligned} & \int_{X(\mathbb{C})^g} \text{Td}_g(TX_{\mathbb{C}}) \text{ch}_g(E_{\mathbb{C}} \otimes L_{\mathbb{C}}^p) R_g(TX_{\mathbb{C}}) \\ &= \sum_{j \in J} \sum_{r=0}^{n_j} p^r (\text{ch}_g(L_{\mathbb{C}|X_j^g})^{[0]})^p \int_{X(\mathbb{C})_j^g} \text{Td}_g(TX_{\mathbb{C}}) \frac{c_1(L_{\mathbb{C}})^r}{r!} \text{ch}_g(E_{\mathbb{C}}) R_g(TX_{\mathbb{C}}) \end{aligned}$$

is a polynomial in p of degree less or equal to $n_{\max}$ with coefficients of the form ζ_m^{pk} , $k \in \mathbb{Z}/(m)$. Similarly one obtains

$$\begin{aligned} & \widehat{\deg} \left(f_*^{\mu_m} (\widehat{\text{Td}}_{\mu_m}(Tf) \widehat{\text{ch}}_{\mu_m}(E \otimes L^p)) \right) \\ &= \widehat{\deg} f_*^{\mu_m} \left(\widehat{\text{Td}}_{\mu_m}(N, g^{TX})^{[0]} \widehat{\text{ch}}_{\mu_m}(E, h^E)^{[0]} \widehat{\text{ch}}_{\mu_m}(L)^{[n - \text{rk } N + 1]} \right) + O(p^{n_{\max}}) \\ &= \widehat{\deg} f_*^{\mu_m} \left(\frac{p^{n - \text{rk } N + 1}}{(n - \text{rk } N + 1)!} \frac{\sum_{\ell \in \mathbb{Z}/(m)} \zeta_m^{\ell} \text{rk } E_{\ell}}{\prod_{r \in \mathbb{Z}/(m)} (1 - \zeta_m^{-r})^{\text{rk } N_r}} \sum_{k \in \mathbb{Z}/(m)} \zeta_m^{pk} \widehat{c}_1(L_k, h^L)^{n - \text{rk } N + 1} \right) + O(p^{n_{\max}}) \end{aligned}$$

where the $O(p^{n_{\max}})$ -part is again a polynomial in p with coefficients as above. As $H^0(X, E \otimes L^p)$ has no torsion (since f is flat) and the higher cohomology groups vanish for $p \rightarrow +\infty$, Th. 5.1 shows that

$$\begin{aligned} & - \sum_{k \in \mathbb{Z}/(m)} \zeta_m^k \cdot \log \text{covol}(H^0(X, E \otimes L^p)_k, |\cdot|_{L^2}) \\ &= \widehat{\deg} f_*^{\mu_m} \left(\frac{p^{n - \text{rk } N + 1}}{(n - \text{rk } N + 1)!} \frac{\sum_{\ell \in \mathbb{Z}/(m)} \zeta_m^{\ell} \text{rk } E_{\ell}}{\prod_{r \in \mathbb{Z}/(m)} (1 - \zeta_m^{-r})^{\text{rk } N_r}} \sum_{k \in \mathbb{Z}/(m)} \zeta_m^{pk} \widehat{c}_1(L_k, h^L)^{n - \text{rk } N + 1} \right) \\ & \quad + \frac{1}{2} T_g(X(\mathbb{C}), E_{\mathbb{C}} \otimes L_{\mathbb{C}}^p) + \sum_{j=0}^{n_{\max}} p^{n_{\max} - j} \sum_{r \in \mathbb{Z}/(m)} \zeta_m^{rp} \hat{\kappa}'_{r,j}. \end{aligned}$$

The asymptotic expansion of $T_g(X(\mathbb{C}), E_{\mathbb{C}} \otimes L_{\mathbb{C}}^p)$ from Th. 1.2 provides the remaining part of the right hand side in the statement. $\square$

Since $c_1(L_{\mathbb{C}}, h^L) > 0$ implies that the free summand of $H^q(X, E \otimes L^p)$ vanishes for $q > 0$ and $p \gg 0$, we could only assume that $H^q(X, E \otimes L^p)_{\text{Tors}} = 0$ holds for $q > 0$ and $p \gg 0$.

References

- [ABo] Abbes, A.; Bouche, T.: *Théorème de Hilbert-Samuel "arithmétique"*. Ann. Inst. Fourier (Grenoble) **45** (1995), 375–401.
- [BeFr] Berman, R. J.; Freixas i Montplet, G.: *An arithmetic Hilbert-Samuel theorem for singular hermitian line bundles and cusp forms*. Compos. Math. **150** (2014), 1703–1728.
- [B] Bismut, J.-M.: *Equivariant Immersions and Quillen Metrics*. J. Differ. Geom. **41** (1995), 53–157.
- [BGS1] Bismut, J.-M.; Gillet, H.; Soulé, C.: *Analytic torsion and holomorphic determinant bundles. I. Bott-Chern forms and Analytic Torsion*. Comm. Math. Phys. **115** (1988), 49–78.
- [BGS2] Bismut, J.-M.; Gillet, H.; Soulé, C.: *Analytic torsion and holomorphic determinant bundles. II. Direct images and Bott-Chern forms*. Comm. Math. Phys. **115** (1988), 79–126.
- [BV] Bismut, J.-M.; Vasserot, E.: *The asymptotics of the Ray-Singer analytic torsion associated with high powers of a positive line bundle*. Comm. Math. Phys. **125** (2) (1989) 355–367.
- [Ber] Berman, R.: *Tunneling, the Quillen metric and analytic torsion for high powers of a holomorphic line bundle*. Contemp. Math., **630**, AMS, Providence, RI, 2014, 15–31.
- [BoBuHoJo] Botero, A. M.; Burgos Gil, J. I.; Holmes, D.; de Jong, R. *Chern-Weil and Hilbert-Samuel formulae for singular Hermitian line bundles*. Doc. Math. **27** (2022), 2563–2624.
- [ChMo] Chen, H.; Moriwaki, A.: *Positivity in Arakelov geometry over adelic curves—Hilbert-Samuel formula and equidistribution theorem*. Progr. Math. **355**. Birkhäuser/Springer, Cham, 2024.
- [CR] Curtis, C. W.; Reiner, I.: *Methods of representation theory, with applications to finite groups and orders. Vol. I*. Wiley-Interscience, New York, 1981.
- [DLM] Dai, X.; Liu, K.; Ma, X.: *A remark on weighted Bergman kernels on orbifolds*. Math. Res. Lett. **19** (1) (2012) 143–148.
- [Fi] Finski, S.: *On the full asymptotics of analytic torsion*. J. Funct. Anal. **275** (2018), 3457–3503.
- [GS1] Gillet, H., Soulé, C.: *Characteristic classes for algebraic vector bundles with Hermitian metric. I*. Ann. Math. **131** (1990), 163–203.
- [GS2] Gillet, H., Soulé, C.: *Arithmetic intersection theory*. Inst. Hautes Etudes Sci. Publ. Math. **72** (1990).
- [GS3] Gillet, H., Soulé, C.: *An arithmetic Riemann-Roch theorem*. Invent. Math. **110** (1992), 473–543.
- [Kaw1] Kawasaki, T.: *The signature theorem for V-manifolds*. Top. **17** (1978) 75–83.
- [Kaw2] Kawasaki, T.: *The Riemann-Roch theorem for complex V-manifolds*. Osaka Math. J. **16** (1979), 151–159.

- [K1] Köhler, K.: *Equivariant analytic torsion on $\mathbf{P}^n(\mathbf{C})$* . Math. Ann. **297** (1993), 553–565.
- [K2] Köhler, K.: *Analytic torsion forms for fibrations by projective curves*. Math. Z. **305** (2023), 47.
- [K3] Köhler, K.: *The full asymptotic expansion of analytic torsion on homogeneous spaces*. arXiv:2602.08132.
- [KR] Köhler, K., Roessler, D. : *A fixed point formula of Lefschetz type in Arakelov geometry I: statement and proof*. Invent. Math. **145** (2001), 333–396
- [La] Larraín-Hubach, A.: *Asymptotics of a modified holomorphic analytic torsion*. Complex Manifolds **12** (2025), Paper No. 20250010.
- [Ma1] Ma, X.: *Orbifolds and analytic torsions*. Trans. Amer. Math. Soc. **357** (2005), 2205–2233.
- [MaM] Ma, X., Marinescu, G.: *Holomorphic Morse inequalities and Bergman kernels*. Progr. Math. **254**, Birkhäuser Verlag, Basel, 2007.
- [Mo] Moriwaki, A.: *Continuity of volumes on arithmetic varieties*. J. Algebraic Geom. **18** (2009), 407–457.
- [P] Puchol, M.: *The asymptotics of the holomorphic torsion forms*. J. Lond. Math. Soc. (2) **108** (2023), 80–140.
- [R] Randriambololona, H.: *Métriques de sous-quotient et théorème de Hilbert-Samuel arithmétique pour les faisceaux cohérents*. J. Reine Angew. Math. **590** (2006), 67–88.
- [Su] Su, G.: *The asymptotics for Cappell-Miller holomorphic torsion*. Manuscripta Math. **154** (2017), 411–428.
- [T] Teßmer, P.: *Asymptotic equivariant holomorphic torsion forms*. arXiv:2408.12559.
- [Y] Yuan, X.: *Big line bundles over arithmetic varieties*. Invent. Math. **173** (2008), 603–649.
- [Z] Zhang, S.-W.: *Positive line bundles on arithmetic varieties*. J. Amer. Math. Soc. **8** (1995), 187–221.