

A fixed point formula of Lefschetz type in Arakelov geometry IV: the modular height of C.M. abelian varieties

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Abstract

We give a new proof of a slightly weaker form of a theorem of P. Colmez ([C2, Par. 2]). This theorem (Corollary 5.8) gives a formula for the Faltings height of abelian varieties with complex multiplication by a C.M. field whose Galois group over \mathbf{Q} is abelian; it reduces to the formula of Chowla and Selberg in the case of elliptic curves. We show that the formula can be deduced from the arithmetic fixed point formula proved in [KR2]. Our proof is intrinsic in the sense that it does not rely on the computation of the periods of any particular abelian variety.

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1 Introduction

Let A be an abelian variety of dimension d defined over $\overline{\mathbf{Q}}$. Let $K \subseteq \mathbf{C}$ be a number field such that A is defined over K and such that the Néron model of A over \mathcal{O}_K has semi-stable reduction at all the places of K . Let Ω be the \mathcal{O}_K -module of global sections of the sheaf of differentials of A over \mathcal{O}_K and let α be a section of Ω^d . We write $A(\mathbf{C})_\sigma$ for the manifold of complex points of the variety $A \times_{\sigma(K)} \mathbf{C}$, where $\sigma \in \text{Hom}(K, \mathbf{C})$ is an embedding of K in \mathbf{C} . The modular (or Faltings) height of A is the quantity

$$h_{\text{Fal}}(A) := \frac{1}{[K : \mathbf{Q}]} \log(\#\Omega^d / \alpha \cdot \Omega^d) - \frac{1}{2[K : \mathbf{Q}]} \sum_{\sigma: K \rightarrow \mathbf{C}} \log \left| \frac{1}{(2\pi)^d} \int_{A(\mathbf{C})_\sigma} \alpha \wedge \overline{\alpha} \right|.$$

It does not depend on the choice of K or α . The modular height defines a height on some moduli spaces of abelian varieties and plays a key role in Falting's proof of the Mordell conjecture. The object of this article is to use higher dimensional Arakelov theory to prove a formula for the modular height of A , valid if A has complex multiplication by the ring of integers of an abelian extension of \mathbf{Q} . A full self-contained statement of the formula can in be found in corollary 5.8. This formula was first proved by a completely different method by P. Colmez and the remaining of the introduction is devoted to an exposition of his and our approach, followed by a plan of the paper.

If A is an abelian variety with complex multiplication, the modular height is related to the periods of A . So suppose that there exists a C.M. field E , of degree $2d$ over \mathbf{Q} and an embedding of rings $\mathcal{O}_E \rightarrow \text{End}(A)$ into the endomorphism ring of A . This is equivalent to saying that A has complex multiplication by \mathcal{O}_E (cf. [Sh]). We can suppose without loss of generality that the action of \mathcal{O}_E is defined over K and that K contains all the conjugates of E in \mathbf{C} . Let now $\tau \in \text{Hom}(E, \mathbf{C})$ be an embedding of E in \mathbf{C} and let ω_τ be an element of the first algebraic de Rham cohomology group $H_{\text{DR}}^1(A)$ of A over K (this is a K -vector space of rank $2d$), such that $a(\omega_\tau) = \tau(a) \cdot \omega_\tau$ for all $a \in \mathcal{O}_E$. If $\sigma \in \text{Hom}(K, \mathbf{C})$, let ω_τ^σ be the element of the complex cohomology group $H^1(A(\mathbf{C})_\sigma, \mathbf{C})$ obtained by base change. Let u_σ be an element of the rational homology group $H_1(A(\mathbf{C})_\sigma, \mathbf{Q})$. The *period* $P(A, \tau, \sigma) \in \mathbf{C}/K^*$ associated to τ and $\sigma \in \text{Hom}(K, \mathbf{C})$ is the complex number $\langle \omega_\tau^\sigma, u_\sigma \rangle_\infty$, where $\langle \cdot, \cdot \rangle_\infty$ is the natural pairing between cohomology and homology. Up to multiplication by an element of K^* , it is only dependent on τ and σ . Let Φ be the subset of $\text{Hom}(E, \mathbf{C})$ such that the subspace $\{t \in TA_0 : a(t) = \tau(a) \cdot t, \forall a \in \mathcal{O}_E\}$ contains a non-zero element (this is the *type* of the C.M. abelian variety A). The following lemma is a (very) weak form of a theorem of P. Colmez:

Lemma 1.1 *The equality*

$$(2\pi)^{-d/2} \cdot e^{-[K:\mathbf{Q}]h_{\text{Fal}}(A)} = \prod_{\tau \in \Phi} \prod_{\sigma \in \text{Hom}(K, \mathbf{C})} P(A, \tau, \sigma)$$

holds up to multiplication by an element of $\overline{\mathbf{Q}}$.

Furthermore, using a refinement of the above lemma, the theory of p -adic periods and explicit computation of periods of Jacobians of Fermat curves, he gives an explicit formula for $h_{\text{Fal}}(A)$ (see also [And], [Yo] and [Gr] [for elliptic curves] for mod. $\overline{\mathbf{Q}}$ versions of the latter formula). To describe it, suppose furthermore that E is Galois over \mathbf{Q} and let $G := \text{Gal}(E|\mathbf{Q})$. Identify Φ with its characteristic function $G \rightarrow \{0, 1\}$ and define Φ^\vee by the formula $\Phi^\vee(\tau) := \Phi(\tau^{-1})$.

Theorem 1.2 (Colmez) *If G is abelian, there exists $q \in \mathbf{Q}$, such that the identity*

$$\frac{1}{d}h_{\text{Fal}}(A) = - \sum_{\chi \text{ odd}} \langle \Phi * \Phi^\vee, \chi \rangle \left[2 \frac{L'(\chi_{\text{prim}}, 0)}{L(\chi_{\text{prim}}, 0)} + \log(f_\chi) \right] + q \log(2)$$

holds. If the conductor of E over \mathbf{Q} divides $8n$, where n is an odd natural number, then the identity holds with $q = 0$.

(it is conjectured that q always vanishes) Here $\langle \cdot, \cdot \rangle$ refers to the scalar product of complex valued functions on G and $*$ to the convolution product. The sum $\sum_{\chi \text{ odd}}$ is on all the odd characters of G (recall that χ is odd iff $\chi(h \circ c \circ h^{-1} \circ \tau) = -\chi(\tau)$ for all $\tau, h \in G$, where $c \in G$ is complex conjugation). The notation f_χ refers to the conductor of χ . Colmez conjectures that the theorem 1.2 holds even without the condition that G is abelian. This formula can be viewed as a generalisation of the formula of Chowla and Selberg (see [CS]), to which it reduces when applied to a C.M. elliptic curve.

It is the aim of this paper to provide a proof of 1.2 using higher dimensional Arakelov theory. More precisely, we shall show that a slightly weaker form of 1.2 can be derived from the fixed point formula in Arakelov theory proved in [KR2] (announced in [KR1]), when applied to abelian varieties with complex multiplication by a field generated over \mathbf{Q} by a root of 1. This proof has the advantage of being intrinsic, i.e. the right side of 1.2 is obtained directly from analytic invariants (the equivariant analytic torsion and the equivariant R -genus) of the abelian variety. It does not involve the computation of the periods of a particular C.M. abelian variety (e.g. Jacobians of Fermat curves). We shall prove:

Theorem 1.3 *Let f be the conductor of E over \mathbf{Q} ; if G is abelian there exist numbers $a_p \in \mathbf{Q}(\mu_f)$, where $p|f$, such that the identity*

$$\frac{1}{d}h_{\text{Fal}}(A) = - \sum_{\chi \text{ odd}} \langle \Phi * \Phi^\vee, \chi \rangle \left[2 \frac{L'(\chi_{\text{prim}}, 0)}{L(\chi_{\text{prim}}, 0)} + \sum_{p|f} a_p \log(p) \right]$$

holds.

Notice that the difference between the right sides of the equations in 1.2 and 1.3 is equal to $\sum_{p|f} b_p \log(p)$ for some $b_p \in \mathbf{Q}(\mu_f)$. To see this, let us write

$$\sum_{\chi \text{ odd}} \langle \Phi * \Phi^\vee, \chi \rangle \log(f_\chi) = \sum_{p|f} \left[\sum_{\chi \text{ odd}} \langle \Phi * \Phi^\vee, \chi \rangle n_{\chi,p} \right] \log(p)$$

where $n_{\chi,p}$ is the multiplicity of the prime number p in the number f_χ . By construction, the number $\sum_{\chi \text{ odd}} \langle \Phi * \Phi^\vee, \chi \rangle n_{\chi,p}$ is invariant under the action of $\text{Gal}(\overline{\mathbf{Q}}|\mathbf{Q})$ and is thus rational.

The paper is organised as follows. In section 2, we give the rephrasing of the main result of [KR2] which we shall need in the text. In section 3, we prove an equivariant version of Zarhin's trick (this is of independent interest) and we show that the fixed point scheme of the action of a finite group on an abelian scheme is well-behaved away from the fibers lying over primes dividing the order of the group. In section 4 we compute a term occurring in the fixed point formula, namely the equivariant holomorphic torsion of line bundles on complex tori; we adapt a method by Berthomieu to do so. In section 5, we apply the fixed point formula to the following setting: an abelian scheme with the action of a certain group of roots of unity and an equivariant ample vector bundle with Euler characteristic 1, which is provided by section 3; an expression for the Faltings height is then very quickly obtained (as a solution of the system of equations (7)) and the rest of the section is concerned with equating this expression with the linear combination of logarithmic derivatives of L -functions appearing in 1.2.

We shall in this paper freely use the definitions and terminology of section 4 of [KR2] (mainly up to def. 4.1 included).

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2 An arithmetic fixed point formula

In this section, we formulate the fixed point formula we shall apply to abelian varieties (more precisely, to their Néron models). It is an immediate consequence of the more general fixed point formula which is the main result [KR2, Th. 4.4] of [KR2]. To formulate it, we shall need the notions of arithmetic Chow theory, arithmetic degree and arithmetic characteristic classes; for these see [BoGS,

Par. 2.1]. To understand how it can be deduced from [KR2, Th. 4.4], one needs to read the last section of [KR2]. We shall nevertheless give a self-contained presentation (not proof) of the formula, when the scheme is smooth, and the associated fixed point scheme étale over the base scheme.

Let now K be a number field and \mathcal{O}_K its ring of integers. Let U be an open subset of $\text{Spec}\mathbf{Z}$ and let U_K be the open subset of $\text{Spec}\mathcal{O}_K$ lying above U . Let p_1, \dots, p_l be the prime numbers in the complement of U . Notice that U_K is the spectrum of an arithmetic ring (see the beginning of [KR2, Sec. 4] for the definition). Denote by $\widehat{\text{deg}} : \widehat{\text{CH}}^1(U_K) \rightarrow \widehat{\text{CH}}^1(U)$ the push-forward map in arithmetic Chow theory (this map coincides with the usual arithmetic degree map if $U_K = \text{Spec}\mathcal{O}_K$). By abuse of language, we shall write $\widehat{\text{deg}}(\overline{V})$ for the arithmetic degree $\widehat{\text{deg}}(\widehat{c}_1(\overline{V}))$ of the first arithmetic Chern class of a hermitian bundle \overline{V} on U_K . Recall that the formula

$$\begin{aligned} \widehat{\text{deg}}(\widehat{c}_1(\overline{V})) &= \widehat{\text{deg}}(\widehat{c}_1(\det(\overline{V}))) = \\ &= \log(\#(\det(V)/s.U_K)) - \sum_{\sigma \in \text{Hom}(K, \mathbf{C})} \log |s \otimes_{\sigma(U_K)} 1|_{\sigma} \end{aligned}$$

holds. Here s is any section of $\det(V)$ and $|\cdot|_{\sigma}$ is the norm arising from the hermitian metric of $\det(V) \otimes_{\sigma(U_K)} \mathbf{C}$. It is a consequence of [GS3, I, Prop. 2.2, p. 13] that there is a canonical isomorphism $\widehat{\text{CH}}^1(U) \simeq \mathbf{R}/\mathcal{R}$, where \mathcal{R} is the subgroup of \mathbf{R} consisting of the expressions $\sum_{k=1}^l n_k \log p_k$, where $n_k \in \mathbf{Z}$. We shall thus often identify both.

Let now $n \in \mathbf{N}^*$ and let \mathcal{C} be the subgroup of \mathbf{C} generated by the expressions $\sum_{k=1}^l q_k \alpha_k \log p_k$, where $q_k \in \mathbf{Q}$ is a rational number and α_k is an n -th root of unity. Let $f : Y \rightarrow U_K$ be a μ_n -equivariant arithmetic variety over U_K , of relative dimension d . As usual, we fix a primitive root of unity ζ_n . Recall also that g is the automorphism of the complex manifold $Y(\mathbf{C})$ associated to ζ_n via the action of $\mu_n(\mathbf{C})$ on $Y(\mathbf{C})$. Fix a g -invariant Kähler metric on $Y(\mathbf{C})$, with associated Kähler form ω_Y .

Theorem 2.1 *Let \overline{E} be a μ_n -equivariant hermitian vector bundle on Y . Suppose that $R^k f_* E = 0$ for $k > 0$. Then the equality*

$$\begin{aligned} & \sum_{k \in \mathbf{Z}/(n)} \zeta_n^k \widehat{\text{deg}}((R^0 f_* \overline{E})_k) = \\ &= \frac{1}{2} T_g(Y(\mathbf{C}), \overline{E}) - \frac{1}{2} \int_{Y_{\mu_n}(\mathbf{C})} \text{Td}_g(TY) \text{ch}_g(E) R_g(TY_{\mathbf{C}}) + \widehat{\text{deg}}(f_*^{\mu_n}(\widehat{\text{Td}}_{\mu_n}(\overline{Tf}) \widehat{\text{ch}}_{\mu_n}(\overline{E}))) \end{aligned}$$

holds in \mathbf{C}/\mathcal{C} .

Recall that $R^0 f_* \overline{E}$ refers to the U_K -module of global sections of E , endowed with the L_2 -metric inherited from the hermitian metric on E and the Kähler

metric on $Y(\mathbf{C})$. Recall that the L_2 -metric is defined as follows: if s, l are two holomorphic sections of $E_{\mathbf{C}}$, then

$$\langle s, l \rangle_{L_2} := \frac{1}{d!(2\pi)^d} \int_{Y(\mathbf{C})} \langle s, l \rangle_E \omega_Y^d.$$

Proof: The proof is similar to the proof of [KR2, Th. 6.14] and so we omit it. **Q.E.D.**

The above theorem is an immediate consequence of the arithmetic Riemann-Roch theorem of Bismut-Gillet-Soulé when the equivariant structure is trivial. Suppose now that Y is smooth and Y_{μ_n} étale over U_K . Let

$$L^{\text{Im}}(z, s) := \sum_{k \geq 0} \frac{\text{Im}(z^k)}{k^s}.$$

where $z \in \mathbf{C}$, $|z| = 1$ and $s \in \mathbf{C}$, $s > 1$. As a function of s , it extends to a meromorphic function of the whole plane, which is holomorphic at $s = 0$. We write $R^{\text{rot}}(\arg(z))$ for the derivative $\frac{\partial}{\partial s} L^{\text{Im}}(z, 0)$ of the function $L^{\text{Im}}(z, s)$ at 0. Write ζ for ζ_n . We let Ω be the sheaf of differentials of f , which is locally free. Define

$$D := \prod_{k \in \mathbf{Z}/n} (1 - \zeta^k)^{\text{rk}(\Omega_k)}, \quad T = \sum_{k \in \mathbf{Z}/n} \zeta^k \text{rk}(E_k)$$

which are locally constant complex-valued functions on Y_{μ_n} . In the just mentioned setting, the formula in theorem 2.1 becomes

$$\begin{aligned} & \sum_{k \in \mathbf{Z}/(n)} \zeta_n^k \widehat{\text{deg}}((R^0 f_* \bar{E})_k) = \\ &= \frac{1}{2} T_g(Y(\mathbf{C}), \bar{E}) - i \sum_{P \in Y_{\mu_n}(\mathbf{C})} \frac{\text{Trace}(g|_{E_P})}{\text{Det}(\text{Id} - g|_{\Omega_P})} \cdot \left[\sum_{k \in \mathbf{Z}/n} \text{rk}((TY(\mathbf{C})_P)_k) R^{\text{rot}}(\arg(\zeta^k)) \right] \\ &+ \widehat{\text{deg}}(f_*^{\mu_n} (\frac{1}{D} \sum_{k \in \mathbf{Z}/n} \zeta^k \widehat{c}_1(\bar{E}_k) + \frac{T}{D} \sum_{k \in \mathbf{Z}/n} \frac{\zeta^k}{1 - \zeta^k} \widehat{c}_1(\bar{\Omega}_k))) \end{aligned} \quad (1)$$

in \mathbf{C}/\mathcal{C} . Finally, let us recall the following lemma.

Lemma 2.2 *Let $n \geq 1$ be a natural number and let R be an entire ring of characteristic 0 in which the polynomial $X^n - 1$ splits. Let \mathbf{C}_n be the constant group scheme over \mathbf{Z} associated to \mathbf{Z}/n , the cyclic group of order n . For each primitive n -th root of unity in R , there is an isomorphism of group schemes*

$$\mu_n \times_{\mathbf{Z}} \text{Spec } R[\frac{1}{n}] \simeq \mathbf{C}_n \times_{\mathbf{Z}} \text{Spec } R[\frac{1}{n}]$$

over $R[\frac{1}{n}]$.

Proof: Both group schemes define isomorphic sheaves on the (small) étale site of $R[\frac{1}{n}]$ and they are both étale over $R[\frac{1}{n}]$ (see for instance [Tam, Par. 3.1, p. 100]). Hence they are isomorphic. **Q.E.D.**

Let as usual $\mathbf{Q}(\mu_n)$ refer to the number field generated by the complex n -th roots of unity. In view of the lemma, the constant group scheme \mathbf{C}_n and the group scheme μ_n become isomorphic over $\mathbf{Q}(\mu_n)[\frac{1}{n}]$. If we let U be the set of prime divisors of n , then $\mathbf{Q}(\mu_n)[\frac{1}{n}]$ corresponds to the open set $U_{\mathbf{Q}(\mu_n)}$. Hence we see that the formula above can be applied to the action of an automorphism of finite order of a (regular, integral, projective) scheme over $\mathbf{Q}(\mu_n)[\frac{1}{n}]$

3 Equivariant geometry on abelian schemes

The following two lemmata give an equivariant version of Zarhin's trick. Let \overline{K} be an algebraically closed field of characteristic zero. If X is a \mathbf{Z}/n -equivariant abelian variety over \overline{K} and L is a \mathbf{Z}/n -equivariant line bundle on X , we shall say that the action of \mathbf{Z}/n on L is *normalised*, if it induces the trivial action on the fiber $L|_0$ of L at the origin. In this section, we shall write $a(\cdot)$ for the action of $1 \in \mathbf{Z}/n$.

Lemma 3.1 *Let X, Y be \mathbf{Z}/n -equivariant abelian varieties defined over \overline{K} . Suppose that \mathbf{Z}/n acts by isogenies on X and Y . Let $f : X \rightarrow Y$ be an equivariant isogeny and let N be the kernel of f . Let M be a line bundle on X . Suppose that M is endowed with an N -equivariant structure and with a normalised \mathbf{Z}/n -equivariant structure. Let $\alpha \in \text{Hom}(N_{\mathbf{Z}/n}, \overline{K}^*) \simeq H^1(N_{\mathbf{Z}/n}, \overline{K}^*)$ be defined by the formula $\alpha(n) = a^{-1} \circ n \circ a \circ (-n)$. Then there exists a normalised \mathbf{Z}/n -equivariant line bundle L' on Y and a \mathbf{Z}/n -equivariant isomorphism $f^*L' \simeq L$, if and only if $\alpha = 1$.*

Note that we have used the identification $\text{Aut}(M) \simeq \overline{K}^*$ in our definition of α . Recall also that $N_{\mathbf{Z}/n}$ refers to the part of N fixed by every element of \mathbf{Z}/n .

Proof: If there exists a bundle L' satisfying the hypotheses of the lemma, then the equivariant structure of $f^*L'|_{N_{\mathbf{Z}/n}}$ is by construction trivial and thus $\alpha = 1$. So suppose that $\alpha = 1$. Note that N is sent into itself by the elements of \mathbf{Z}/n . Let ρ be the character of N defined by the formula $a^{-1} \circ a(n) \circ a \circ (-n)$ (this character extends α). Since $\alpha = 1$, ρ induces a character ρ' on the quotient group $N/N_{\mathbf{Z}/n}$. This quotient is naturally identified with the image of the endomorphism $a - \text{Id}$ of N . View ρ' as a character on $\text{Im}(a - \text{Id})$ and choose any character ρ'' extending $(\rho')^{-1}$ to N . Such a character always exists because \overline{K}^* is an injective abelian group. We modify the natural N -equivariant structure on M by multiplying by $\rho''(n)$ the automorphism of M given by n , for each $n \in N$. With this new structure, we have the identity $a(n) \circ a = a \circ n$ of automorphisms

of (the total space of) M . To see this consider the identities

$$\begin{aligned} & (a^{-1} \circ (a(n) \cdot \rho''(a(n))) \circ a) \circ (-n \rho''(-n)) \\ &= (a^{-1} \circ a(n) \circ a \circ (-n)) \cdot \rho''(a(n)) \cdot \rho''(-n) \\ &= (a^{-1} \circ a(n) \circ a \circ (-n)) \cdot \rho(n)^{-1} = \text{Id} \end{aligned}$$

Consider now the bundle $(f_*M)^N$, which is the quotient of the bundle M by the action of N . Using the identity $a(n) \circ a = a \circ n$, we see that the action of a descends to $(f_*M)^N$. Furthermore, it is shown in [Mu, Prop. 2, p. 70] that $(f_*M)^N$ is naturally isomorphic to the total space of a line bundle on $Y = X/N$. By [Mu, Prop. 2, p. 70] again, this bundle has the required properties. **Q.E.D.**

Lemma 3.2 *Let A be a \mathbf{Z}/n -equivariant abelian variety over \overline{K} , where \mathbf{Z}/n acts by isogenies. There exists a \mathbf{Z}/n -equivariant abelian variety B over \overline{K} and an ample \mathbf{Z}/n -equivariant bundle L on B , such that*

- (a) B is (non-equivariantly) isomorphic to $(A \times A^\vee)^4$;
- (b) the equation $\chi(L) = 1$ holds for the Euler characteristic of L ;
- (c) the group \mathbf{Z}/n acts by isogenies on B and there exists an equivariant isogeny p from A^8 to B .

Proof: We shall follow the steps of the proof of Zarhin's trick. Let P' be any ample line bundle on A . Let $P := \otimes_{g \in \mathbf{Z}/n} g^* P'$. The bundle P is ample and carries a natural \mathbf{Z}/n -equivariant structure. Endow A^4 with the induced \mathbf{Z}/n -equivariant structure and let $p_i : A^4 \rightarrow A$ be the i -th projection ($i = 1, \dots, 4$). Let $M' = p_1^* P \times p_2^* P \times p_3^* P \times p_4^* P$ be the fourth external tensor power of P . This is again an ample \mathbf{Z}/n -equivariant line bundle on A^4 . Let m be the order of the Mumford group $K(M')$ of M' (see [Mu, p. 60] for the definition). Now let a, b, c, d be integers such that $a^2 + b^2 + c^2 + d^2 = -1 \pmod{m^2}$. Consider the endomorphism $\alpha(m)$ of A^4 described by the matrix

$$\begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix}$$

(this is the endomorphism appearing in the proof of Zarhin's trick). This endomorphism commutes with all the elements of \mathbf{Z}/n , because \mathbf{Z}/n acts by isogenies. Let now N be the subgroup of $X = A^8$ given by the graph of $\alpha(m)|_{K(M')}$. This subgroup is sent into itself by all the elements of \mathbf{Z}/n . Let $B = X/N$ and let $p : X \rightarrow B$ be the quotient map. By construction B carries a natural \mathbf{Z}/n -equivariant structure such that p is equivariant. It is shown in [Mi, Rem. 6.12,

p. 136] that there exists a line bundle L on B and an (non-equivariant) isomorphism $M' \otimes_{\text{Ext}} M' \simeq p^*L$; we can thus endow $M' \otimes_{\text{Ext}} M'$ with an N -equivariant structure. The \mathbf{Z}/n -equivariant structure on $M' \otimes_{\text{Ext}} M'|_{X_{\mathbf{Z}/n}}$ is by construction trivial. By the last lemma, we can thus assume that L carries a \mathbf{Z}/n -equivariant structure and that there is an \mathbf{Z}/n -equivariant isomorphism $M' \otimes_{\text{Ext}} M' \simeq p^*L$. We claim that B and L are the objects required in the statement of the lemma. The fact that (a) holds is a step in the proof of Zarhin's trick and we refer to [Mi, Rem. 6.12, p. 136] for the details. To see that (b) holds, we use [Mu, Th. 2, p. 121] and compute $\chi(M) = \chi(M')^2 = \#N$ and $\chi(M) = \#N \cdot \chi(L)$, from which we deduce that $\chi(L) = 1$. To see that (c) holds, note that the map p defined above in the proof has the properties required of p in (c). **Q.E.D.**

We now quote the following results on extensions of line bundles from the generic fiber. If $X \rightarrow S$ is any S scheme, L is a line sheaf over X and $i : S \rightarrow X$ is an S -valued point, then a *rigidification* of L along i is an isomorphism $i^*L \simeq \mathcal{O}_S$. The bundle L together with a rigidification will be said to be *rigidified* along i . Once a point is given, the line bundles rigidified along that point form a category, where the morphisms are the sheaf morphisms that commute with the rigidification.

Proposition 3.3 *Let $A \rightarrow \text{Spec } K$ be an abelian variety over a number field. Suppose that A has good reduction at all the finite places of K . Let $\mathcal{A} \rightarrow \text{Spec } \mathcal{O}_K$ be its Néron model. Let $R_{\mathcal{A}}$ (resp. R_A) be the category of line bundles rigidified along the 0-section of \mathcal{A} (resp. A).*

- (1) (see [MB, 1.1, p. 40]) *The restriction functor $R_{\mathcal{A}} \rightarrow R_A$ is an equivalence of categories;*
- (2) (see [Ra, Th. VIII.2]) *if L is an ample line bundle on A , then any extension of L to \mathcal{A} is ample.*

Now let K be a number field that contains all the n -th roots of unity. Let A be an abelian variety over K that has good reduction at all the finite places of K . Let \mathcal{A} be its Néron model over \mathcal{O}_K and let $\mathcal{A}' := \mathcal{A} \times_{\text{Spec } \mathcal{O}_K} \text{Spec } \mathcal{O}_K[\frac{1}{n}]$. Let $f : \mathcal{A}' \rightarrow \text{Spec } \mathcal{O}_K[1/n]$ be the structure map and let $\Omega := \Omega_{\mathcal{A}'}$ be the sheaf of differentials of f .

Suppose that \mathcal{A}' is endowed with an action of \mathbf{Z}/n by $\text{Spec } \mathcal{O}_K[\frac{1}{n}]$ -group scheme automorphisms. Let as usual $f^{\mathbf{Z}/n} : \mathcal{A}'_{\mathbf{Z}/n} \rightarrow \text{Spec } \mathcal{O}_K[\frac{1}{n}]$ be the structure map of the fixed scheme. Let $i_0 : \text{Spec } \mathcal{O}_K[\frac{1}{n}] \rightarrow \mathcal{A}'$ be the zero section. We let $1 \in \mathbf{Z}/n$ act by $a(\cdot)$, as before.

Lemma 3.4 *If $i_0^*\Omega$ has no $\mathcal{O}_K[1/n]$ -submodule, which is fixed under the action of \mathbf{Z}/n , then the scheme $\mathcal{A}'_{\mathbf{Z}/n}$ is étale and finite over $\text{Spec } \mathcal{O}_K[\frac{1}{n}]$.*

Proof: We first prove that $f^{\mathbf{Z}/n}$ is étale. We have to show that $f^{\mathbf{Z}/n}$ is flat and that for any geometric point $\bar{x} \rightarrow \text{Spec } \mathcal{O}_K[1/n]$, the corresponding scheme obtained by base-change is regular. The second condition is verified because of [KR2, Cor. 2.9] (base-change invariance of the fixed scheme) and [KR2, Prop. 2.10] (regularity of the fixed-scheme). To show that $f^{\mathbf{Z}/n}$ is flat, we apply the criterion [Ha, Th. 9.9, p. 261]. Choose a very ample \mathbf{Z}/n -equivariant line bundle \mathcal{L} on \mathcal{A}' . Let \mathfrak{p} be a maximal ideal of $\text{Spec } \mathcal{O}_K[1/n]$ and denote by $k(\mathfrak{p})$ the corresponding residue field. The Hilbert-Samuel polynomial of the fiber $\mathcal{A}'_{\mathbf{Z}/n, k(\mathfrak{p})}$ of $\mathcal{A}'_{\mathbf{Z}/n}$ at \mathfrak{p} relatively to $\mathcal{L}|_{\mathcal{A}'_{\mathbf{Z}/n, k(\mathfrak{p})}}$ can be computed on the algebraic closure $\overline{k(\mathfrak{p})}$ of $k(\mathfrak{p})$. As $i_0^* \Omega$ has no fixed part, we see that $\mathcal{A}'_{\mathbf{Z}/n}(\overline{k(\mathfrak{p})})$ consists of isolated fixed points only. The number $F(\mathfrak{p})$ of these fixed points is the Hilbert-Samuel polynomial of $\mathcal{A}'_{\mathbf{Z}/n, k(\mathfrak{p})}$, which is of degree 0. We have to show that $F(\mathfrak{p})$ is independent of \mathfrak{p} . To prove this, consider first that $H^1(\mathcal{A}'_{k(\mathfrak{p})}, \mathcal{L}_{k(\mathfrak{p})}) = 0$ for all \mathfrak{p} ; this follows from the characterization of the cohomology of line bundles on abelian varieties [Mu, Par. 16, p. 150]. Thus we know that $f_* \mathcal{L}$ is locally free and that there are natural equivariant isomorphisms $(f_* \mathcal{L})_{k(\mathfrak{p})} \simeq f_{k(\mathfrak{p}),*} \mathcal{L}_{k(\mathfrak{p})}$. We may also assume that the action on $\mathcal{L}|_{\mathcal{A}'_{\mathbf{Z}/n}}$ is trivial; this might be achieved by replacing \mathcal{L} by its n -th tensor power. Using the Lefschetz trace formula of [BFQ], we see that $F(\mathfrak{p})$ depends only on the trace of a on $H^0(\mathcal{A}_K, \mathcal{L}_K)$ and on the determinant of $\text{Id} - a$ on $i_{0,K}^* \Omega_K$ (where $(\cdot)_K$ refers to the base change by $\text{Spec } K \rightarrow \text{Spec } \mathcal{O}_K[\frac{1}{n}]$). Thus $F(\mathfrak{p})$ is independent of \mathfrak{p} . To see that $f^{\mathbf{Z}/n}$ is finite, we only have to check that it is quasi-finite, as it is projective (see [Ha, Ex. 11.2]). Let again \mathfrak{p} be a prime ideal. As $f^{\mathbf{Z}/n}$ is étale, we know that $\mathcal{A}'_{\mathbf{Z}/n, k(\mathfrak{p})}$ is the spectrum of a direct sum of finite field extensions of $k(\mathfrak{p})$; furthermore this sum is finite, since the morphism is of finite type. Hence $\mathcal{A}'_{\mathbf{Z}/n, k(\mathfrak{p})}$ is a finite set and thus we are done. **Q.E.D.**

4 The equivariant analytic torsion of line bundles on abelian varieties

Let (V, g^V) be a d -dimensional Hermitian vector space and let $\Lambda \subset V$ be a lattice of rank $2d$. The quotient $A := V/\Lambda$ is a flat complex torus. According to the Appell-Humbert theorem [LB, Ch. 2.2], the holomorphic line bundles on A can be described as follows: Choose an Hermitian form H on V such that $E := \text{Im } H$ takes integer values on $\Lambda \times \Lambda$. Choose furthermore $\alpha : \Lambda \rightarrow S^1$ such that

$$\alpha(\lambda_1 + \lambda_2) = \alpha(\lambda_1)\alpha(\lambda_2)e^{\pi i E(\lambda_1, \lambda_2)}$$

for all $\lambda_1, \lambda_2 \in \Lambda$. Then there is an associated line bundle $L_{H, \alpha}$ defined as the quotient of the trivial line bundle on V by the action of Λ given by

$$\lambda \circ (v, t) := (v + \lambda, \alpha(\lambda)e^{\pi H(v, \lambda) + \pi H(\lambda, \lambda)/2} t).$$

There is a canonical Hermitian metric h^L on $L_{H,\alpha}$ given by

$$h^L((v, t_1), (v, t_2)) := t_1 \bar{t}_2 e^{-\pi H(v,v)} .$$

Define $C \in \text{End}(V)$ by $H(v_1, v_2) = g^V(v_1, Cv_2)$. C is Hermitian with respect to g^V . Consider an automorphism g of (V, g^V) , leaving Λ , H and α invariant. Then g and C commute, thus they may be diagonalized simultaneously. Denote their eigenvalues by $(e^{i\phi_j})$ and (ν_j) , respectively, and let (e_j) be a corresponding set of g^V -orthonormal eigenvectors. Assume that for all j , $\phi_j \notin 2\pi\mathbf{Z}$, i.e. that g acts on A with isolated fixed points. Note that the isometric automorphism g of A has finite order.

Let $L_{\text{Tr}}(L_{H,\alpha})$ denote the trace of the action of g on $H^0(A, L_{H,\alpha})$.

Lemma 4.1 *Assume that $\nu_j = 0$ for all j . Then the equivariant analytic torsion of $\bar{L}_{H,\alpha}$ on (A, g^V) with respect to g vanishes.*

Proof: Let $V_{\mathbf{R}}^{\vee}$ be the dual of the underlying real vector space of V . Consider the dual lattice $\Lambda^{\vee} := \{\mu \in V_{\mathbf{R}}^{\vee} | \mu(\lambda) \in 2\pi\mathbf{Z} \forall \lambda \in \Lambda\}$. Represent $\alpha = e^{i\mu_0}$ with $\mu_0 \in V_{\mathbf{R}}^{\vee}$. It is shown in [K4, section V] that the eigenfunctions are given by the functions $f_{\mu} : V_{\mathbf{R}}/\Lambda \rightarrow \mathbf{C}$, $x \mapsto e^{i(\mu+\mu_0)(x)}$ with corresponding eigenvalue $\frac{1}{2}\|\mu + \mu_0\|^2$. The eigenforms are given by the product of these eigenfunctions times the pullback of elements of Λ^*V^{\vee} . The eigenvalue of such a form $f_{\mu} \cdot \eta$ is the eigenvalue of f_{μ} .

As $L_{H,\alpha}$ is g -invariant, we get $g^*e^{i\mu_0} = e^{i(\mu_0+\mu_1)}$ for some $\mu_1 \in \Lambda^{\vee}$. Also, g maps Λ^{\vee} to itself, thus for any $\mu \in \Lambda^{\vee}$ there is some $\mu' \in \Lambda^{\vee}$ with $g^*f_{\mu} = f_{\mu'}$. As g acts fixed point free on $V \setminus \{0\}$, it maps a function f_{μ} to a multiple of itself iff $\mu + \mu_0 = 0$, i.e. iff f_{μ} represents an element in the cohomology. Thus g acts diagonal free on the complement of the cohomology, and the zeta function defining the torsion vanishes. **Q.E.D.**

Let $\psi : \mathbf{R}/\mathbf{Z} \rightarrow \mathbf{R}$ denote the function $\psi([x]) := (\log \Gamma)'(x)$ for $x \in]0, 1[$.

Theorem 4.2 *Assume that $\nu_j > 0$ for all j . Then the equivariant analytic torsion of $\bar{L}_{H,\alpha}$ on (A, g^V) with respect to g is given by*

$$\begin{aligned} T_g(A, \bar{L}_{H,\alpha}) &= \sum_{j=1}^d \left[\frac{\log(2\pi\nu_j)}{e^{-i\phi_j} - 1} + iR^{\text{rot}}(\phi_j) \right. \\ &\quad \left. - \frac{1}{4} \left(2\log(2\pi) - 2\Gamma'(1) + \psi\left(\left[\frac{\phi_j}{2\pi}\right]\right) + \psi\left(\left[1 - \frac{\phi_j}{2\pi}\right]\right) \right) \right] L_{\text{Tr}}(L_{H,\alpha}) . \end{aligned}$$

Proof: Assume first that the ν_j are pairwise linear independent over \mathbf{Q} . Then as is shown in [Ber, p. 3], the spectrum of the Kodaira-Laplace operator on A

is given by

$$\sigma(\square) = \left\{ 2\pi \sum_{j=1}^d n_j \nu_j \mid n_j \in \mathbf{N}_0 \ \forall j \right\} .$$

Set $e^i := g^V(e_i, \cdot)$. Let E_ν^q denote the eigenspace corresponding to the eigenvalue ν of \square on $\Gamma^\infty(A, \Lambda^q T^* A \otimes L_{H,\alpha})$. We shall prove by induction that the trace of g on $E_{2\pi \sum_j n_j \nu_j}^q$ is given by

$$\binom{\#\{j \mid n_j \neq 0\}}{q} L_{\text{Tr}}(L_{H,\alpha}) \cdot \prod_j e^{in_j \phi_j} . \quad (2)$$

First this formula holds for E_0^0 . Now the eigenspaces verify the relations

$$E_\nu^q = \bigoplus_{j_1 < \dots < j_q} \left(\Lambda_{k=1}^q e^{j_k} \otimes E_{\nu - 2\pi \sum_{k=1}^q \nu_{j_k}}^0 \right) \quad (3)$$

[Ber, eq. (7)] and the complex

$$0 \rightarrow E_\nu^0 \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} E_\nu^d \rightarrow 0 \quad (4)$$

is acyclic. Thus we can determine the trace on E_ν^q for $q > 0$ by the trace on some $E_{\nu'}^0$ with $\nu' < \nu$. Then the trace on E_ν^0 is given by the trace on the E_ν^q for $q > 1$ by the sequence 4. As the relations (3), (4) are compatible with (2), equation (2) is proven.

Hence the zeta function defining the torsion is given by

$$Z(s) = \sum_{q=0}^d \sum_{\nu \in \sigma(\square)} \frac{(-1)^{q+1} q}{\nu^s} \binom{\#\{j \mid n_j \neq 0\}}{q} L_{\text{Tr}}(L_{H,\alpha}) \cdot \prod_j e^{in_j \phi_j} .$$

Notice that for any $0 \leq k \leq n$

$$\sum_{q=0}^d (-1)^{q+1} q \binom{k}{q} = \begin{cases} 1 & \text{for } k = 1 \\ 0 & \text{for } k \neq 1 \end{cases} .$$

Consider for $z \in \mathbf{C}$, $|z| = 1$ and for $s \in \mathbf{C}$, $\text{Re } s > 1$ the zeta function

$$L(z, s) := \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

and its meromorphic continuation to $s \in \mathbf{C}$. Thus

$$Z(s) = \sum_{j=1}^d \sum_{k=1}^{\infty} \frac{e^{ik\phi_j}}{(2\pi k \nu_j)^s} L_{\text{Tr}}(L_{H,\alpha}) = \sum_{j=1}^d (2\pi \nu_j)^{-s} L(e^{i\phi_j}, s) L_{\text{Tr}}(L_{H,\alpha})$$

and

$$Z'(0) = \sum_{j=1}^d (-\log(2\pi\nu_j)L(e^{i\phi_j}, 0) + L'(e^{i\phi_j}, 0)) L_{\text{Tr}}(L_{H,\alpha}) .$$

Using the definition of $R^{\text{rot}}(\phi) := \frac{\partial}{\partial s}|_{s=0} \frac{1}{2i}(L(e^{i\phi}, s) - L(e^{-i\phi}, s))$ in [K1] (compare section 2), the formula for $\frac{\partial}{\partial s}|_{s=0} \frac{1}{2}(L(e^{i\phi}, s) + L(e^{-i\phi}, s))$ in [K3, Lemma 13] and the formula $L(e^{i\phi}, 0) = (e^{-i\phi} - 1)^{-1}$ (e.g. in [K2, p. 108]) we find the theorem for ν_j pairwise linear independent over \mathbf{Q} . If the cohomology does not change, the torsion varies continuously with the metric ([BGS3, section (d)]), thus the result holds for any nonzero ν_j . Alternatively, one can show that formulas (3), (4) hold more general by arguing as in [Ber, Remark p. 4]) or by continuity. **Q.E.D.**

Remark. More general, assume only that all ν_j are non-zero. Again, by the results of [Ber] and a proof similar to the above one we get

$$\begin{aligned} T_g(A, \bar{L}_{H,\alpha}) &= \sum_{j=1}^d \text{sign}(\nu_j) \left[\frac{\log(2\pi|\nu_j|)}{e^{-i\phi_j} - 1} + iR^{\text{rot}}(\phi_j) \right. \\ &\quad \left. - \frac{1}{4} \left(2\log(2\pi) - 2\Gamma'(1) + \psi\left(\left[\frac{\phi_j}{2\pi}\right]\right) + \psi\left(\left[1 - \frac{\phi_j}{2\pi}\right]\right) \right) \right] L_{\text{Tr}}(L_{H,\alpha}) . \end{aligned}$$

Combining this with Lemma 4.1 as in [Ber, section 4] by splitting A (see also [LB, chapter 3,§3]) and using the product formula for equivariant torsion [K2, Lemma 2], one obtains the value of the equivariant torsion for any $L_{H,\alpha}$. Similarly, for automorphisms g having a larger fixed point set one can split A accordingly to obtain the value of the torsion.

5 Application of the fixed point formula to abelian varieties with C.M. by a cyclotomic field

Let now $n > 0$ be a natural number and let $\phi(n) := \#(\mathbf{Z}/(n))^\times$. Let A be an abelian variety of dimension $d = \phi(n)/2$ defined over a number field K . Suppose that $A_{\bar{\mathbf{Q}}}$ has complex multiplication by $\mathcal{O}_{\mathbf{Q}(\mu_n)}$ and fix a ring embedding $\mathcal{O}_{\mathbf{Q}(\mu_n)} \rightarrow \text{End}(A_{\bar{\mathbf{Q}}})$ (see [Sh] for the general theory). As before we choose a primitive n -th root of unity $\zeta := \zeta_n$; this root defines an isomorphism $\mu_n(\mathbf{C}) \simeq \mathbf{Z}/n$ and thus a canonical \mathbf{Z}/n -action on $A_{\bar{\mathbf{Q}}}$. We may suppose that $\mathbf{Q}(\mu_n) \subseteq K$ and that A has good reduction at all the finite places of K . The latter hypothesis is possible in view of [ST, Th. 7, p. 505]. Let B be the abelian variety obtained from $A_{\bar{\mathbf{Q}}}$ via lemma 3.2. We can suppose without loss of generality that B , as well as the line bundle L promised in 3.2, are defined over K ; we may also suppose that the map p appearing in (c) is defined over K .

The existence of p shows that B has good reduction at the finite places of K as well. Furthermore, we normalize the action of \mathbf{Z}/n on L so that its restriction to $L|_0$ becomes trivial (one might achieve this by multiplying the action by some character of \mathbf{Z}/n). Choose some rigidification of L along the 0-section. The action of \mathbf{Z}/n will then leave this rigidification invariant.

We let $\pi : \mathcal{B} \rightarrow \text{Spec } \mathcal{O}_K$ be the Néron model of B . By the universal property of Néron models, the action of \mathbf{Z}/n on B extends to \mathcal{B} . Also, by (1) of Proposition 3.3, there exists a line bundle \mathcal{L} on \mathcal{B} , endowed with a \mathbf{Z}/n -action and a \mathbf{Z}/n -invariant rigidification along the 0-section, which both extend the corresponding structures on L . By [MB, Prop. 2.1, p. 48], there exists a unique metric on $\mathcal{L}_{\mathbf{C}}$ whose first Chern form is translation invariant and such that the rigidification is an isometry. We endow $\mathcal{L}_{\mathbf{C}}$ with this metric. It is \mathbf{Z}/n -invariant by unicity. We endow $B(\mathbf{C})$ with the translation invariant Kähler metric given by the Riemann form of $\mathcal{L}_{\mathbf{C}}$. We let Ω_{π} be the sheaf of differentials of π and endow it with the metric induced by the Kähler metric. If we let $i_0 : \text{Spec } \mathcal{O}_K \rightarrow \mathcal{B}$ be the 0-section, we then have a canonical isometric equivariant isomorphism $\pi^* i_0^* \overline{\Omega}_{\pi} \simeq \overline{\Omega}_{\pi}$. It is shown in [Bo, Par. 4, 4.1.10] that $\widehat{\text{deg}}(i_0^* \overline{\Omega}_{\pi}) = [K : \mathbf{Q}] h_{\text{Fal}}(B_{\overline{\mathbf{Q}}})$.

We now apply the equivariant arithmetic Riemann-Roch formula (1) to $\overline{\mathcal{L}}$. We shall work over $\mathcal{O}_K[1/n]$, over which \mathbf{Z}/n - and μ_n -action are equivalent concepts because of lemma 2.2. Let $d_B = 8d$ be the relative dimension of \mathcal{B} . Let p_1, \dots, p_l be the prime factors of n and let U be the complement of the set $\{p_1, \dots, p_l\}$ in $\text{Spec } \mathbf{Z}$. We then have the identification $U_K \simeq \text{Spec } \mathcal{O}_K[1/n]$. We identify μ_n and \mathbf{Z}/n over U_K via the root of unity ζ . We let \mathcal{S} be the subgroup of \mathbf{R} generated by the expressions $\sum_{k=1}^l q_k \text{Im}(\alpha_k) \log p_k$, where $q_k \in \mathbf{Q}$ is a rational number and α_k is an n -th root of unity.

Proposition 5.1 *Let $e^{i\phi_1}, \dots, e^{i\phi_{d_B}}$ be the eigenvalues of the action of $1 \in \mathbf{Z}/n$ on $T\mathcal{B}|_0$ (with multiplicities). The equality*

$$\frac{1}{[K : \mathbf{Q}]} \sum_{k \in \mathbf{Z}/n} \text{Im}\left(\frac{\zeta^k}{1 - \zeta^k}\right) \widehat{\text{deg}}(i_0^* \overline{\Omega}_{\pi, k}) = \frac{1}{2} \sum_{j=1}^{d_B} R^{\text{rot}}(\phi_j) \quad (5)$$

holds in \mathbf{R}/\mathcal{S} .

Proof: For any $l \in \mathbf{Z}$, let $[l]$ denote the l -plication map from a group scheme into itself. First notice that by [Bo, (4.1.23)] (i.e. a hermitian refinement of the theorem of the square), the identity $[l]^* \widehat{c}_1(\overline{\mathcal{L}} \otimes [-1]^* \overline{\mathcal{L}}) = l^2 \widehat{c}_1(\overline{\mathcal{L}} \otimes [-1]^* \overline{\mathcal{L}})$ holds in $\widehat{\text{CH}}^1(\mathcal{B})$, for every $l \in \mathbf{Z}$. Now consider that by lemma 3.4, the scheme \mathcal{B}_{μ_n} is a finite commutative group scheme over $\mathcal{O}_K[1/n]$; it thus has an order $l_0 \geq 0$ and the l_0 -plication map is the 0-map on \mathcal{B}_{μ_n} (see [Ta]). Now we compute in $\widehat{\text{CH}}^1(\mathcal{B}_{\mu_n})$:

$$[l_0]^* \widehat{c}_1(\overline{\mathcal{L}}|_{\mathcal{B}_{\mu_n}} \otimes [-1]^* \overline{\mathcal{L}}|_{\mathcal{B}_{\mu_n}}) = l_0^2 \widehat{c}_1(\overline{\mathcal{L}}|_{\mathcal{B}_{\mu_n}} \otimes [-1]^* \overline{\mathcal{L}}|_{\mathcal{B}_{\mu_n}}) = 0$$

and thus we get $\widehat{\deg}(f_*^{\mu_n}(l_0^2 \widehat{c}_1(\overline{\mathcal{L}}|_{\mathcal{B}_{\mu_n}}) + l_0^2 \widehat{c}_1([-1]^* \overline{\mathcal{L}}|_{\mathcal{B}_{\mu_n}}))) = \widehat{\deg}(f_*^{\mu_n}(2.l_0^2 \widehat{c}_1(\overline{\mathcal{L}}|_{\mathcal{B}_{\mu_n}}))) = 0$ and finally $\widehat{\deg}(f_*^{\mu_n}(\widehat{c}_1(\overline{\mathcal{L}}_{\mathcal{B}_{\mu_n}}))) = 0$.

Let us now write down the equation (1) in our situation. First notice that in view of the equivariant isomorphism $\pi^* i_0^* \Omega_\pi \simeq \Omega_\pi$, the functions T and D are constant. Until the end of the proof, let N be the number of fixed points of the action of 1 on an arbitrary connected component of $B(\mathbf{C})$. The holomorphic Lefschetz trace formula (see [BFQ] or [ASe, III, (4.6), p. 566]) shows that $N.T/D = \sum_{k \in \mathbf{Z}/n} \zeta_n^k \text{rk}((R^0 f_* \mathcal{L})_k)$ and thus N is independent of the choice of the component. We obtain

$$\begin{aligned} & \sum_{k \in \mathbf{Z}/(n)} \zeta^k \cdot \widehat{\deg}((R^0 f_* \overline{\mathcal{L}})_k) = \\ & = \frac{1}{2} T_g(B(\mathbf{C}), \overline{E}) - i[K : \mathbf{Q}] \frac{N.T}{D} \cdot \sum_{k=1}^{d_B} R^{\text{rot}}(\phi_k) \\ & + \widehat{\deg}(f_*^{\mu_n}(\frac{T}{D} \cdot \sum_{k \in \mathbf{Z}/n} \frac{\zeta^k}{1 - \zeta^k} \widehat{c}_1(\overline{\Omega}_{\pi,k}))) \end{aligned}$$

in \mathbf{C}/\mathcal{C} . As $\text{rk}(R^0 \pi_*(\mathcal{L})) = 1$, we also have the equality of complex numbers

$$\sum_{k \in \mathbf{Z}/(n)} \zeta^k \cdot \widehat{\deg}((R^0 f_* \overline{\mathcal{L}})_k) = \widehat{\deg}((R^0 f_* \overline{\mathcal{L}})) \cdot \frac{N.T}{D}$$

And thus we get

$$\begin{aligned} & \widehat{\deg}((R^0 f_* \overline{\mathcal{L}})) \\ & = (\frac{N.T}{D})^{-1} \frac{1}{2} T_g(B(\mathbf{C}), \overline{E}) - i[K : \mathbf{Q}] \sum_{k=1}^{d_B} R^{\text{rot}}(\phi_k) \\ & + \widehat{\deg}(f_*^{\mu_n}(\sum_{k \in \mathbf{Z}/n} \frac{\zeta^k}{(1 - \zeta^k)} \widehat{c}_1(\overline{\Omega}_{\pi,k}))) \cdot N^{-1} \end{aligned} \quad (6)$$

in \mathbf{C}/\mathcal{C} . The theorem 4.2 shows that the imaginary part of $(\frac{N.T}{D})^{-1} \frac{1}{2} T_g(B(\mathbf{C}), \overline{E})$ is equal to $[K : \mathbf{Q}] R^{\text{rot}}(\phi_k)$. Furthermore, using the isomorphism $\pi^* i_0^* \overline{\Omega}_\pi \simeq \overline{\Omega}_\pi$ and the projection formula, we see that

$$\widehat{\deg}(f_*^{\mu_n}(\sum_{k \in \mathbf{Z}/n} \frac{\zeta^k}{1 - \zeta^k} \widehat{c}_1(\overline{\Omega}_{\pi,k}))) \cdot N^{-1} = \sum_{k \in \mathbf{Z}/n} \frac{\zeta^k}{1 - \zeta^k} \widehat{c}_1(i_0^* \overline{\Omega}_{\pi,k}).$$

Taking these two facts into account, we can take the imaginary part of both sides in (6) to conclude. **Q.E.D.**

Let now $\Phi := \{\Phi_1, \dots, \Phi_d\}$ be the type of $A_{\overline{\mathbf{Q}}}$. Define $\zeta^{P(k)} = \Phi_k(\zeta)$ for $k = 1, \dots, d$. In view of 3.2 (c), the identity (5) can be rewritten as

$$\frac{1}{[K : \mathbf{Q}]} \sum_{k=1}^d \operatorname{Im} \left(\frac{\Phi_k(\zeta)}{1 - \Phi_k(\zeta)} \right) \widehat{\deg}(i_0^* \overline{\Omega}_{\pi, P(k)}) = -4 \sum_{k=1}^d \frac{\partial}{\partial s} L^{\operatorname{Im}}(\Phi_k(\zeta), 0)$$

(in \mathbf{R}/\mathcal{S}). Now notice that we can change our choice of ζ in the latter equation and replace it by $\sigma(\zeta)$, where $\sigma \in \operatorname{Gal}(\mathbf{Q}(\mu_n)|\mathbf{Q})$ (this corresponds to applying the fixed point formula to a power of the original automorphism), thus obtaining a system of linear equations in the $\widehat{\deg}(i_0^* \overline{\Omega}_{\pi, k})$. Notice also the equations obtained from $\sigma(\zeta)$ and $\overline{\sigma}(\zeta)$ are equivalent. With this remark in mind, we see that the just mentioned system of equations is equivalent to the following one:

$$\frac{1}{[K : \mathbf{Q}]} \sum_{k=1}^d \operatorname{Im} \left(\frac{\Phi_k^{-1} \circ \Phi_l(\zeta)}{1 - \Phi_k^{-1} \circ \Phi_l(\zeta)} \right) X_k = -4 \sum_{k=1}^d \frac{\partial}{\partial s} L^{\operatorname{Im}}(\Phi_k^{-1} \circ \Phi_l(\zeta), 0) + E_l \quad (7)$$

(in \mathbf{R}) where $l = 1, \dots, d$ and the coefficients of the vector $E := {}^t[E_1, \dots, E_d]$ lie in \mathcal{S} . We shall show that the matrix $M := [\operatorname{Im}(\frac{\Phi_k^{-1} \circ \Phi_l(\zeta)}{1 - \Phi_k^{-1} \circ \Phi_l(\zeta)})]_{l,k}$ of this system is invertible as a matrix of real numbers. Since the coefficients of M all lie in $\mathbf{Q}(\mu_n)$, the coefficients of M^{-1} lie in $\mathbf{Q}(\mu_n)$ as well and thus we see that the coefficients of the vector $M^{-1}E$ lie in \mathcal{C} . Thus we can determine the quantities X_k up to an element of \mathcal{S} . Now recall that by construction

$$h_{\operatorname{Fal}}(\mathcal{B}_{\overline{\mathbf{Q}}}) = \frac{1}{[K : \mathbf{Q}]} \left(\sum_{k=1}^d \widehat{\deg}(i_0^* \overline{\Omega}_{\pi, P(k)}) \right).$$

in \mathbf{R}/\mathcal{S} . Furthermore, by a result of Raynaud in [SCM, Exp. VII, Cor. 2.1.3, p. 207], the modular height of an abelian variety and the modular height of the dual of the latter are equal. Thus, using (a) 3.2, we see that $h_{\operatorname{Fal}}(\mathcal{B}_{\overline{\mathbf{Q}}}) = 8 \cdot h_{\operatorname{Fal}}(A_{\overline{\mathbf{Q}}})$ (in \mathbf{R}). Thus we can determine $h_{\operatorname{Fal}}(A_{\overline{\mathbf{Q}}})$ up to an element of \mathcal{S} .

Before we proceed to solve the system (7), we need two lemmata that relate the quantities appearing in the system with Dirichlet L -functions.

Warning. In what follows, in contradiction with classical usage, the notation $L(\chi, s)$ will always refer to the **non-primitive** L -function associated with a Dirichlet character. We write $\chi_{\operatorname{prim}}$ for the primitive character associated with χ and accordingly write $L(\chi_{\operatorname{prim}}, s)$ for the associated primitive L -function.

First take notice of the elementary fact that $\operatorname{Im}(z/(1-z)) = \frac{1}{2} \cot(\frac{1}{2} \arg(z))$ if $|z| = 1$. Let G be the Galois group of the extension $\mathbf{Q}(\mu_m)|\mathbf{Q}$. From now on, for simplicity we fix $\zeta = e^{2\pi i/n}$. The following lemma is a variation on the functional equation of the Dirichlet L -functions. When χ is a primitive character, it can be derived directly from the functional equation and classical results on Gauss sums.

Lemma 5.2 *Let χ be an odd character of G . The equality*

$$\langle L^{\text{Im}}(\sigma(\zeta), s), \chi \rangle := \frac{1}{2d} \sum_{\sigma \in G} \overline{\chi(\sigma)} L^{\text{Im}}(\sigma(\zeta), s) = \frac{1}{2d} n^{1-s} \frac{\Gamma(1-s/2)}{\Gamma((s+1)/2)} \pi^{s-1/2} L(\overline{\chi}, 1-s)$$

holds for all $s \in \mathbf{C}$.

In the expression $\langle L^{\text{Im}}(\sigma(\zeta), s), \chi \rangle$, the symbol σ is considered as a variable in G .

Proof: We prove the equality for $0 < s < 1$. The full equality then follows by analytic continuation. We compute

$$\begin{aligned} & n^{\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) \pi^{-\frac{s+1}{2}} \frac{1}{2d} \sum_{\sigma \in G} \overline{\chi(\sigma)} L^{\text{Im}}(\sigma(\zeta), s) \\ &= n^{\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) \pi^{-\frac{s+1}{2}} \frac{1}{2d} \sum_{\sigma \in G} \sum_{k \geq 0} \frac{\text{Im}(\sigma(\zeta)^k)}{k^s} \overline{\chi(\sigma)} \\ &= n^{\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) \pi^{-\frac{s+1}{2}} \frac{1}{2d} \sum_{k \geq 0} \frac{1}{k^s} \sum_{\sigma \in G} \overline{\chi(\sigma)} \text{Im}(\sigma(\zeta)^k) \\ &= \sum_{k \geq 0} \int_0^\infty \frac{1}{2d} k \cdot e^{-k^2 \pi u/n} u^{\frac{1}{2}(s+1)-1} \left[\sum_{\sigma \in G} \overline{\chi(\sigma)} \text{Im}(\sigma(\zeta)^k) \right] \mathbf{d}\mathbf{u} \\ &= -i \cdot \int_0^\infty \frac{1}{2d} \sum_{k \geq 0} k \cdot e^{\frac{-k^2 \pi}{nu}} u^{-1-\frac{1}{2}(s+1)} \left[\sum_{\sigma \in G} \overline{\chi(\sigma)} \sigma(\zeta)^k \right] \mathbf{d}\mathbf{u} \quad (8) \end{aligned}$$

$$= \int_0^\infty \frac{1}{2d} \sum_{k \geq 0} \chi(k) u^{-1-\frac{1}{2}(s+1)} \frac{k}{n} e^{-\pi k^2 u/n} (n \cdot u)^{\frac{3}{2}} \mathbf{d}\mathbf{u} \quad (9)$$

$$= \frac{1}{2d} \sum_{k \geq 0} \chi(k) \int_0^\infty u^{-1-\frac{1}{2}(s+1)} \frac{k}{n} e^{-\pi k^2 u/n} (n \cdot u)^{\frac{3}{2}} \mathbf{d}\mathbf{u} \quad (10)$$

$$= \frac{1}{2d} n^{3/2-s/2} \Gamma(1-s/2) \pi^{s/2-1} L(\overline{\chi}, 1-s)$$

For the equality (9), we used the Poisson summation formula. To obtain the equality (8), we first exchange the summation and integration symbols and then make the change of variable $u \mapsto \frac{1}{u}$. The exchange of summation and integration symbols is justified by the following estimates. Let $\sigma_0 \in G$ and let $t_k := \text{Im}(\sigma_0(\zeta)^k)$ and $v_k := k \cdot e^{k^2 \pi u/n}$. Since $\sum_{k=0}^{ln} t_k = 0$ for all $l \geq 0$, the sequence $T_k := \sum_{j=0}^k t_j$ is bounded above by $C > 0$. Consider now that

$$\left| \sum_{k=0}^N k \cdot e^{-k^2 \pi u/n} \text{Im}(\sigma_0(\zeta)^k) \right| = \left| \sum_{k=0}^N t_k v_k \right| = \left| \sum_{k=0}^{N-1} T_k (v_k - v_{k+1}) + T_N v_N \right| \quad (11)$$

where we used partial summation for the last equality. The function (of k) $k \cdot e^{-k^2 \pi u/n}$ is increasing on the interval $[0, \sqrt{\frac{n}{2\pi u}}]$ and decreasing on the interval

$[\sqrt{\frac{n}{2\pi u}}, \infty[$. Let k_0 be the largest integer less or equal to $\sqrt{\frac{n}{2\pi u}}$. The expression (11) can be bounded above by

$$\begin{aligned} & Cv_N + C \sum_{k=k_0+1}^{N-1} (v_k - v_{k+1}) + C|v_{k_0} - v_{k_0+1}| + C \sum_{k=0}^{k_0-1} (v_{k+1} - v_k) \\ &= Cv_N + C(v_{k_0+1} - v_N) + C(v_{k_0} - v_0) + C|v_{k_0} - v_{k_0+1}| = C[v_{k_0+1} + v_{k_0} + |v_{k_0} - v_{k_0+1}|] \\ &= 2C \max\{v_{k_0+1}, v_{k_0}\} \leq 2C \sqrt{\frac{n}{2\pi e}} u^{-1/2}. \end{aligned}$$

Hence

$$\left| \sum_{k=0}^N k \cdot e^{-k^2 \pi u/n} u^{\frac{1}{2}(s+1)-1} \operatorname{Im}(\sigma_0(\zeta)^k) \right| \leq 2C \sqrt{\frac{n}{2\pi e}} |u^{-1/2} u^{\frac{1}{2}(s+1)-1}| = 2C \sqrt{\frac{n}{2\pi e}} |u^{\frac{1}{2}s-1}|$$

On the other hand, for $u > 1$, the classical estimate

$$\begin{aligned} \left| \sum_{k=0}^N k \cdot e^{-k^2 \pi u/n} u^{\frac{1}{2}(s+1)-1} \operatorname{Im}(\sigma_0(\zeta)^k) \right| &\leq \left| \sum_{k=0}^N k \cdot e^{-k\pi u/n} u^{\frac{1}{2}(s+1)-1} \right| \leq \\ &\leq e^{-\pi u/n} |u^{\frac{1}{2}(s+1)-1}| \frac{1}{(1 - e^{-\pi u/n})^2} \end{aligned}$$

holds. The first estimate show that for $u \in]0, 1[$ and $s > 0$, the function of u

$$\left| \sum_{k=0}^N k \cdot e^{-k^2 \pi u/n} u^{\frac{1}{2}(s+1)-1} \operatorname{Im}(\sigma_0(\zeta)^k) \right|$$

is bounded by an element of $L^1([0, 1])$. The second estimate shows that for $u \in]1, \infty[$ and $s \in \mathbf{C}$ the same function is bounded by an element of $L^1(]1, \infty[)$. By the dominated convergence theorem, we might thus exchange summation and integration symbols. Completly similar estimates justify the equality (10).

Q.E.D.

It is shown in [K1, Proof of Th. 8, p. 564] that $L^{\operatorname{Im}}(\sigma(\zeta), 0) = \frac{1}{2} \cot(\frac{1}{2} \arg(\sigma(\zeta)))$. Thus, we see that the last lemma has the following corollary.

Corollary 5.3 *For any odd character χ on G , the equality*

$$\sum_{\sigma \in G} \cot\left(\frac{1}{2} \arg(\sigma(\zeta))\right) \chi(\sigma) = \frac{2n}{\pi} L(\chi, 1)$$

holds.

To tackle with the system (7), we shall need the following result from linear algebra. We say that a complex-valued function f on G is *odd* if $f(c \circ x) = -f(x)$ for all $x \in G$, where c denotes complex conjugation.

Lemma 5.4 Let $\vec{X} := (X_1, \dots, X_d)$ and $\vec{Y} := (Y_1, \dots, Y_d)$. Let f be any odd function on G and let M_f be the matrix $[f(\Phi_k^{-1} \circ \Phi_l)]_{l,k}$. If $\langle \chi, f \rangle \neq 0$ for all the odd characters χ , then the system $M\vec{X} = \vec{Y}$ is maximal and

$$X_j = \sum_{\chi \text{ odd}} \frac{\chi(\Phi_j) \sum_l \bar{\chi}(\Phi_l) Y_l}{d^2 \langle f, \chi \rangle}$$

Proof: Let ϕ_k be the function defined on G such that $\phi_k(x) = 1$ if $x = \Phi_1 \circ \Phi_k$ and 0 otherwise ($k = 1, \dots, d$). on G . Let V^- be the complex vector space of odd functions on G . An ordered basis B_Φ of V^- is given by $\phi_1 - \phi_1 \circ c, \dots, \phi_d - \phi_d \circ c$. Another basis B_χ is given by the odd characters on G . We view M as a linear endomorphism of V^- , via B_Φ . We now proceed to find the matrix of M in the basis B_χ . We compute

$$M \cdot {}^t[\chi(\Phi_1 \circ \Phi_1), \dots, \chi(\Phi_1 \circ \Phi_d)] = {}^t \left[\sum_k f(\Phi_k^{-1} \circ \Phi_l) \chi(\Phi_1 \circ \Phi_k) \right]_l$$

and

$$\begin{aligned} \sum_k f(\Phi_k^{-1} \circ \Phi_l) \chi(\Phi_1 \circ \Phi_k) &= \frac{1}{2} \sum_{\sigma \in G} f(\sigma^{-1} \circ \Phi_l) \chi(\Phi_1 \circ \sigma) \\ &= \frac{1}{2} \sum_{\sigma \in G} f(\sigma^{-1}) \chi(\sigma \circ \Phi_l \circ \Phi_1) = d \langle f, \chi \rangle \chi(\Phi_1 \circ \Phi_l). \end{aligned}$$

Thus M is represented by the diagonal matrix $\text{Diag}[d \langle f, \chi \rangle]_\chi$ in the basis B_χ . The vector $\vec{Y}_\chi := \frac{1}{d} [\sum_l Y_l \bar{\chi}(\Phi_1 \circ \Phi_l)]_\chi$ represents the vector \vec{Y} in B_χ . Thus the solution of $M\vec{X} = \vec{Y}$ in B_χ is the vector $\vec{X}_\chi := [\frac{\sum_l Y_l \bar{\chi}(\Phi_1 \circ \Phi_l)}{d^2 \langle f, \chi \rangle}]_\chi$ and we thus obtain

$$X_j = \sum_{\chi \text{ odd}} \chi(\Phi_1 \circ \Phi_j) \frac{\sum_l Y_l \bar{\chi}(\Phi_1 \circ \Phi_l)}{d^2 \langle f, \chi \rangle} = \sum_{\chi \text{ odd}} \frac{\chi(\Phi_j) \sum_l \bar{\chi}(\Phi_l) Y_l}{d^2 \langle f, \chi \rangle} \quad (12)$$

Q.E.D.

In the following proposition, we apply the lemma 5.4 to the system (7) and use the lemmata 5.3 and 5.2 to evaluate the resulting expression in terms of logarithmic derivatives of L -functions.

Proposition 5.5 *The Faltings height of $A_{\overline{\mathbf{Q}}}$ is given by the identity*

$$\frac{1}{d} h_{\text{Fal}}(A) = - \sum_{\chi \text{ odd}} \langle \Phi * \Phi^\vee, \chi \rangle 2 \frac{L'(\chi_{\text{prim}}, 0)}{L(\chi_{\text{prim}}, 0)} + \sum_{p|n} a_p \log(p)$$

where $a_p \in \mathbf{Q}(\mu_n)$.

Proof: We apply the lemma 5.4 to the system (7) (with $E_l = 0$). Define $f : G \rightarrow \mathbf{C}$ by the formula $f(\sigma) := \cot(\frac{1}{2} \arg(\sigma(\zeta)))$. The fact that the system is maximal is implied by the fact that $L(\chi, 1) \neq 0$, when χ is a non-principal Dirichlet character (see for instance [CaFr, Th. 2, p. 212]). We compute

$$\begin{aligned} \frac{1}{8} \sum_j X_j &= - \sum_j \sum_{\chi \text{ odd}} \frac{\chi(\Phi_j) \sum_{k,l} \frac{\partial}{\partial s} L^{\text{Im}}(\Phi_k^{-1} \circ \Phi_l(\zeta), 0) \overline{\chi}(\Phi_l)}{d^2 \langle f, \chi \rangle} \\ &= - \sum_{\chi} \frac{\sum_j \chi(\Phi_j) \sum_{k,l} \frac{\partial}{\partial s} L^{\text{Im}}(\Phi_k^{-1} \circ \Phi_l(\zeta), 0) \overline{\chi}(\Phi_l)}{d^2 \langle f, \chi \rangle} \end{aligned}$$

Using scalar and convolution products, we can write

$$\begin{aligned} & \frac{\sum_j \chi(\Phi_j) \sum_{k,l} \frac{\partial}{\partial s} L^{\text{Im}}(\Phi_k^{-1} \circ \Phi_l(\zeta), 0) \overline{\chi}(\Phi_l)}{d^2 \langle f, \chi \rangle} \\ &= \frac{2d \langle \chi, \Phi \rangle \cdot 2d^2 \langle \chi, \frac{\partial}{\partial s} L^{\text{Im}}(\sigma(\zeta), 0) * \Phi * (\cdot)^{-1} \rangle}{d^2 \langle f, \chi \rangle} \\ &= \frac{4d^3 \langle \chi, \Phi \rangle \langle \chi, \frac{\partial}{\partial s} L^{\text{Im}}(\sigma^{-1}(\zeta), 0) * \Phi^\vee \rangle}{d^2 \langle f, \chi \rangle} \\ &= 8d^2 \langle \chi, \Phi * \Phi^\vee \rangle \cdot \frac{\langle \frac{\partial}{\partial s} L^{\text{Im}}(\sigma(\zeta), 0), \chi \rangle}{\frac{2n}{\pi} L(\overline{\chi}, 1)} \\ &= \frac{4d^2 \pi}{n} \langle \Phi * \Phi^\vee, \chi \rangle \cdot \frac{\langle \frac{\partial}{\partial s} L^{\text{Im}}(\sigma(\zeta), 0), \chi \rangle}{L(\overline{\chi}, 1)} \end{aligned} \quad (13)$$

where $\sigma \in G$ is a variable and the equalities are in \mathbf{R} . Now by lemma 5.2, the sum over all odd χ of (13) is equal to

$$\sum_{\chi \text{ odd}} \langle \Phi * \Phi^\vee, \chi \rangle \left(\frac{4d^2 \pi}{n} \frac{\frac{\partial}{\partial s} [L(\overline{\chi}, 1 - s) n^{1-s} \frac{\Gamma(1-s/2)}{\Gamma((s+1)/2)} \pi^{s-1/2}]_{s=0} \frac{1}{2d}}{L(\overline{\chi}, 1)} \right) \quad (14)$$

Furthermore, from the existence of Euler product expansions for L functions, we deduce that

$$\frac{L'(\overline{\chi}, 1)}{L(\overline{\chi}, 1)} = \frac{L'(\overline{\chi}_{\text{prim}}, 1)}{L(\overline{\chi}_{\text{prim}}, 1)} + \sum_{p|n} \frac{\overline{\chi}_{\text{prim}}(p)/p}{1 - \overline{\chi}_{\text{prim}}(p)/p} \log(p).$$

The sum

$$\sum_{\chi \text{ odd}} \langle \Phi * \Phi^\vee, \chi \rangle \frac{\overline{\chi}_{\text{prim}}(p)/p}{1 - \overline{\chi}_{\text{prim}}(p)/p}$$

is an algebraic number. By construction, it is invariant under the action of any element of $\text{Gal}(\overline{\mathbf{Q}}|\mathbf{Q})$ and it is thus an element of \mathbf{Q} . Taking this into account

and using the functional equation of primitive Dirichlet L -functions, we obtain that (14) is equal to

$$\begin{aligned} \sum_{\chi \text{ odd}} \langle \Phi * \Phi^\vee, \chi \rangle & \left(\frac{2d\pi}{n} \left[\frac{-n\Gamma'(1)}{2\pi} - \frac{n \log(n)}{\pi} + \frac{n \log(\pi)}{\pi} - \frac{n\Gamma'(1/2)}{2\Gamma(1/2)\pi} \right] \right. \\ & \left. + 2d \left[\frac{L'(\chi_{\text{prim}}, 0)}{L(\chi_{\text{prim}}, 0)} - \log\left(\frac{2\pi}{f_{\bar{\chi}}}\right) + \frac{\Gamma'(1)}{\Gamma(1)} \right] \right) \end{aligned} \quad (15)$$

in \mathbf{R}/\mathcal{S} . By Galois invariance again, the expressions $\sum_{\chi \text{ odd}} \langle \Phi * \Phi^\vee, \chi \rangle \log(n)$ and $\sum_{\chi \text{ odd}} \langle \Phi * \Phi^\vee, \chi \rangle \log(f_{\bar{\chi}})$ lie in \mathcal{S} . Furthermore, by [C2, (1), p. 363], we have the equality $\sum_{\chi \text{ odd}} \langle \Phi * \Phi^\vee, \chi \rangle = \frac{1}{2}$ of real numbers. Thus the two lines of the expression (15) are equal to

$$\begin{aligned} & \left[2d \sum_{\chi \text{ odd}} \langle \Phi * \Phi^\vee, \chi \rangle \frac{L'(\chi_{\text{prim}}, 0)}{L(\chi_{\text{prim}}, 0)} \right] + 2d[-\Gamma'(1)/4 + \frac{1}{2} \log(\pi) - \frac{1}{4} \frac{\Gamma'(1/2)}{\Gamma(1/2)} - \frac{1}{2} \log(2\pi) + \frac{1}{2} \frac{\Gamma'(1)}{\Gamma(1)}] \\ & = \left[2d \sum_{\chi \text{ odd}} \langle \Phi * \Phi^\vee, \chi \rangle \frac{L'(\chi_{\text{prim}}, 0)}{L(\chi_{\text{prim}}, 0)} \right] \end{aligned} \quad (16)$$

$$+ 2d[-\Gamma'(1)/4 - \frac{1}{2} \log(2) + \frac{1}{2} \frac{\Gamma'(1)}{\Gamma(1)} - \frac{1}{4} \frac{\Gamma'(1)}{\Gamma(1)} + \frac{1}{2} \log(2)] \quad (17)$$

$$= 2d \sum_{\chi \text{ odd}} \langle \Phi * \Phi^\vee, \chi \rangle \frac{L'(\chi_{\text{prim}}, 0)}{L(\chi_{\text{prim}}, 0)}$$

in \mathbf{R}/\mathcal{S} . We used the equality $2^{2z-1}\Gamma(z)\Gamma(z + \frac{1}{2}) = \sqrt{\pi}\Gamma(2z)$ ($z \in \mathbf{C}$) for the equality (17). This concludes the proof. **Q.E.D.**

The next lemma shows that the expression for the Faltings height appearing in the last proposition is invariant under number field extension (this lemma is implicit in the definition of h_{Fal} given in [C2]).

Lemma 5.6 *Let $E \subseteq E'$ be C.M. fields. Suppose that E and E' are both Galois extensions of \mathbf{Q} . Let Φ be a type of E and let Φ' the type induced by Φ on E' . The identity of real numbers*

$$\sum_{\chi_E} \langle \Phi * \Phi^\vee, \chi_E \rangle \frac{L'(\chi_{E,\text{prim}}, 0)}{L(\chi_{E,\text{prim}}, 0)} = \sum_{\chi_{E'}} \langle \Phi' * \Phi'^\vee, \chi_{E'} \rangle \frac{L'(\chi_{E',\text{prim}}, 0)}{L(\chi_{E',\text{prim}}, 0)}$$

holds, where the first sum is over the odd characters χ_E of E and the second sum is over the odd characters of E' .

Proof: Let \mathcal{G}_E (resp. $\mathcal{G}_{E'}$) be the Galois group of E (resp. of E') over \mathbf{Q} . Let $p: \mathcal{G}_{E'} \rightarrow \mathcal{G}_E$ be the natural map. By construction the identities

$$\langle f \circ p, g \circ p \rangle_{E'} = \langle f, g \rangle_E \quad \text{and} \quad (f \circ p) * (g \circ p) = (f * g) \circ p$$

hold for functions $f, g : \mathcal{G}_E \rightarrow \mathbf{C}$. By construction again, the equality $L((\chi \circ p)_{\text{prim}}, s) = L(\chi_{\text{prim}}, s)$ also holds, where χ is a character of \mathcal{G}_E . To prove the identity of the lemma, it is thus sufficient to show that $\langle \Phi' * \Phi'^{\vee}, \chi \rangle = 0$ if χ is an odd character of E' not induced from E . So let χ be such a character. By assumption, χ is non-trivial on the normal subgroup $H := p^{-1}(\text{Id}_E)$ of $\mathcal{G}_{E'}$ and $(\Phi' * \Phi'^{\vee})(h.g) = (\Phi' * \Phi'^{\vee})(g)$ for all $g \in \mathcal{G}_{E'}$ and $h \in H$. Let $h_0 \in H$ be such that $\chi(h_0) \neq 1$. We compute

$$\begin{aligned} \langle \Phi' * \Phi'^{\vee}, \chi \rangle &:= \frac{1}{\#\mathcal{G}_{E'}} \sum_{g \in \mathcal{G}_{E'}} (\Phi' * \Phi'^{\vee})(g) \bar{\chi}(g) \\ &= \frac{1}{\#\mathcal{G}_{E'}} \sum_{g \in \mathcal{G}_{E'}} (\Phi' * \Phi'^{\vee})(g.h_0) \bar{\chi}(g.h_0) = \left[\frac{1}{\#\mathcal{G}_{E'}} \sum_{g \in \mathcal{G}_{E'}} (\Phi' * \Phi'^{\vee})(g) \bar{\chi}(g) \right] \bar{\chi}(h_0) \\ &= \langle \Phi' * \Phi'^{\vee}, \chi \rangle \bar{\chi}(h_0) \end{aligned}$$

If $\langle \Phi' * \Phi'^{\vee}, \chi \rangle \neq 0$, then $\bar{\chi}(h_0) = 1$, a contradiction. **Q.E.D.**

A slightly weaker form of the following lemma (in the sense that only the isogeny classes of the varieties are determined) is contained in a lemma due to Shimura-Taniyama.

Lemma 5.7 *Let $E \subseteq \bar{\mathbf{Q}}$ be a C.M. field that is an abelian extension of \mathbf{Q} and let Φ be a type of E . Fix an integer n such that $E \subseteq \mathbf{Q}(\mu_n)$ and let Φ' be the type of $\mathbf{Q}(\mu_n)$ lifted from Φ via the inclusion. Let A be an abelian variety over $\bar{\mathbf{Q}}$, that admits a complex multiplication by \mathcal{O}_E of type Φ . The abelian variety $A^{[\mathbf{Q}(\mu_n):E]}$ admits a complex multiplication by $\mathcal{O}_{\mathbf{Q}(\mu_n)}$ of type Φ' .*

Proof: Let $r := [\mathbf{Q}(\mu_n) : E]$. From the fact that $\mathcal{O}_{\mathbf{Q}(\mu_n)}$ is generated as an E -algebra by a primitive root of 1, we deduce that there exists a basis of r elements of $\mathcal{O}_{\mathbf{Q}(\mu_n)}$ as an \mathcal{O}_E -module. Notice now that multiplication by a fixed element of $\mathcal{O}_{\mathbf{Q}(\mu_n)}$ defines an \mathcal{O}_E -module endomorphism of $\mathcal{O}_{\mathbf{Q}(\mu_n)}$, that is the identity iff the element is 1. We can thus use the basis to define a ring injection $\mathcal{O}_{\mathbf{Q}(\mu_n)} \rightarrow M_{r \times r}(\mathcal{O}_E)$ of $\mathcal{O}_{\mathbf{Q}(\mu_n)}$ into the ring of $r \times r$ matrices with coefficients in \mathcal{O}_E . This ring injection gives rise to a complex multiplication by $\mathcal{O}_{\mathbf{Q}(\mu_n)}$ on the product A^r . By construction, each element $\sigma \in \text{Hom}(\mathbf{Q}(\mu_n), \bar{\mathbf{Q}})$ of the corresponding type has the property that $\sigma|_E \in \Phi$. This implies that the type is lifted from Φ and concludes the proof. **Q.E.D.**

Corollary 5.8 *Let A be an abelian variety of dimension d defined over $\bar{\mathbf{Q}}$. Let E be a C.M. field, suppose that E is an abelian extension of \mathbf{Q} and that A has complex multiplication by \mathcal{O}_E . Let $\mathcal{O}_E \rightarrow \text{End}(A)$ be an embedding of rings and let Φ be the associated type. Then the identity*

$$\frac{1}{d} h_{\text{Fal}}(A) = - \sum_{\chi \text{ odd}} \langle \Phi * \Phi^{\vee}, \chi \rangle 2 \frac{L'(\chi_{\text{prim}}, 0)}{L(\chi_{\text{prim}}, 0)} + \sum_{p|f} a_p \log(p)$$

of real numbers holds, where f is the conductor of E over \mathbf{Q} and $a_p \in \mathbf{Q}(\mu_f)$.

Proof: By class field theory $E \subseteq \mathbf{Q}(\mu_f)$; furthermore the modular height h_{Fal} is additive for products of abelian varieties over $\overline{\mathbf{Q}}$. We can thus apply the two last lemmata to reduce the proof of the identity to the case $E = \mathbf{Q}(\mu_f)$. This case is covered by proposition 5.5 and so we are done. **Q.E.D.**

Remark. The equation (7) can also be deduced from the application of the arithmetic fixed point formula to the trivial bundle, instead of an ample bundle. To perform this alternative computation, one has to use the computation of the relative cohomology of the trivial bundle on an abelian scheme given in [BBM]. This cohomology can then be related to the sheaf of relative differentials via the equivariant principal polarisation provided by section 2.

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