Equivariant analytic torsion on $\mathbb{P}^n\mathbb{C}$

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Abstract for the *Zentralblatt der Mathematik*: The subject of the paper is to calculate an equivariant version of the complex Ray-Singer torsion for all bundles on the $\mathbb{P}^1\mathbb{C}$ and for the trivial line bundle on $\mathbb{P}^n\mathbb{C}$, for isometries which have isolated fixed points. The result can for all n be expressed with a special function, which is very similar to the series defining the Gillet-Soul R-genus.

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Abstract: We calculate an equivariant version of the complex Ray-Singer torsion for all bundles on the $\mathbb{P}^1\mathbb{C}$ and for the trivial line bundle on $\mathbb{P}^n\mathbb{C}$, for isometries with isolated fixed points. The result gives for all n a part of the Gillet-Soul R-function.

Keywords: Determinants and determinant line bundles, Arakelov geometry, Homogeneous manifolds.

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Introduction

The analytic torsion was constructed by Ray and Singer [**RS**] as an analytic analogue to the Reidemeister torsion. Bismut, Gillet and Soul [**BGS**] proved as an extension of a result of Quillen important properties of the torsion in connection with vector bundles on fibrations:

Let $\pi : M \to B$ be a proper holomorphic map of compact complex manifolds and let ξ be a hermitian holomorphic vector bundle on M. Let $R\pi_*\xi$ be the right-derived direct image of ξ. Then the analytic torsion of the fibres of π induces a metric on the Knudsen-Mumford determinant $\lambda^{KM} := (\det R\pi_*\xi)^{-1}$ which is a holomorphic line bundle on B. The curvature of this Quillen metric as well as its behaviour under changes of the metrics on M and ξ was expressed in [**BGS**] explicitly by means of secondary Bott-Chern classes. In particular this gives a refinement of the Riemann-Roch theorem for families.

On the other hand let $i : Y \hookrightarrow X$ be an embedding of compact complex manifolds. Let η be a hermitian holomorphic vector bundle on Y and let ξ be a resolution of η by a complex of vector bundles on X. Bismut and Lebeau [**BL**] calculated the relation between the Quillen metrics of η and ξ. With the help of this result, Gillet and Soul [**GS2**] were able to prove a Riemann-Roch theorem in Arakelov geometry for the first Chern class of the direct image (see [**S**] for the theorem and some background information). This theorem was later proved by Faltings [**F**] for higher degrees.

The proof of the Riemann-Roch theorem uses a calculation of Gillet, Soul and Zagier [**GS1**] of the torsion for the trivial line bundle on the complex projective spaces $\mathbb{P}^n\mathbb{C}$. This led Gillet and Soul to conjecture this theorem, which was the initial motivation for [**BL**]. In particular this rather difficult calculation gives in particular the Gillet-Soul R-genus, which appears explicitly in the theorem. This is the additive genus associated to the series

$$R(x) = \sum_{\substack{\ell \geq 1 \\ \text{odd}}} \Big(2\zeta'(-\ell) + \zeta(-\ell) \sum_{j=1}^{\ell} \frac{1}{j} \Big) \frac{x^\ell}{\ell!} \,,$$

where ζ is the Riemann zeta function. To obtain this series, one has to caculate the torsion of $\mathbb{P}^n\mathbb{C}$ for every n.

Let us consider now a holomorphic isometry g of a hermitian vector bundle E over a compact Khler manifold M. One can define in a natural way an equivariant version of the torsion. This equivariant torsion appeared already in Ray's [**R**] calculation of the real analytic torsion for lens spaces.

In this paper we present the calculation of the equivariant analytic torsion for all holomorphic bundles on $\mathbb{P}^1\mathbb{C}$ and for the trivial line bundle on $\mathbb{P}^n\mathbb{C}$, where the projective spaces are equipped with the Fubini-Study metric. We consider only rotations with isolated fixpoints. For a rotation by angles $\in \pi \cdot \mathbb{Q}$, we obtain a closed expression involving the gamma function. For arbitrary angles a function R^{rot}, which is similar to the Gillet-Soul R-function, appears as an infinite series. This is relatively easy to calculate because the defining ζ-function Z has no singularities in contrast to the situation in [**GS1**].

The similarity of R^{rot} and R might help to find an equivariant Riemann-Roch formula in Arakelov geometry, where the two functions correspond to the extremal cases: isolated fixed points or identity map. In fact, Bismut [**B3**] found further evidence for such a formula: He constructed analytic torsion forms associated to a short exact sequence of hermitian holomorphic vector bundles equipped with a holomorphic unitary endomorphism g. In his result, a series $R(\varphi, x)$ appears with the properties

$$R(0,x) = R(x), \qquad R(\varphi, 0) = R^{\mathrm{rot}}(\varphi)\,.$$

As the appearance of the R-genus in [**B2**] gave evidence for the existence of the Riemann-Roch theorem, he now conjectures an equivariant Riemann-Roch formula.

The function R^{rot} can be obtained as follows: Let for $0 < \varphi < 2\pi$ and $s > 0$, $\zeta^{\mathrm{rot}}(\varphi, s)$ be the Dirichlet series

$$\zeta^{\mathrm{rot}}(\varphi, s) := \sum_{k\geq 1} \frac{\sin k\varphi}{k^s} .$$

Then ζ^{rot} can be seen as the imaginary part of a Lerch zeta function. We set $R^{\mathrm{rot}}(\varphi) := \frac{\partial}{\partial s}\zeta^{\mathrm{rot}}(\varphi, 0)$. The following is obtained by classical results:

Proposition 1. *R^{rot} is equal to*

$$R^{\mathrm{rot}}(\varphi) = \frac{C + \log \varphi}{\varphi} - \sum_{\substack{\ell\geq 1\\ \ell \text{ odd}}} \zeta'(-\ell)(-1)^{\frac{\ell+1}{2}} \frac{\varphi^\ell}{\ell!} .$$

If $\varphi = 2\pi\frac{p}{q}$ with $p, q \in \mathbb{N}$, $0 < p < q$, then

$$R^{\mathrm{rot}}(\varphi) = -\tfrac{1}{2} \log q \cdot \cot \frac{\varphi}{2} + \sum_{\ell=1}^{q-1} \log \Gamma\left(\frac{j}{q}\right) \cdot \sin j\varphi .$$

In the last chapter we give some other functional properties of R^{rot}. Let $E := \mathcal{O}(k_1) \oplus \ldots \oplus \mathcal{O}(k_n)$ be a holomorphic vector bundle on $\mathbb{P}^1\mathbb{C}$, equipped with the standard metric (i.e. the curvature of $\mathcal{O}(1)$ is the Fubini-Study Khler form). By a theorem of Grothendieck, each holomorphic vector bundle on $\mathbb{P}^1\mathbb{C}$ is of this form. Then we find

Theorem 2. *The equivariant analytic torsion $\tau(E, \varphi)$ with respect to a rotation by an angle $\varphi \in]0, 2\pi[$ is given by*

$$-2 \log \tau(E, \varphi) = \frac{2R^{\mathrm{rot}}(\varphi)}{\sin \frac{\varphi}{2}} \cdot \sum_{j=1}^{n} \cos(k_j+1)\frac{\varphi}{2} + \sum_{j=1}^{n} \sum_{m=1}^{|k_j+1|} \frac{\sin(2m - |k_j + 1|)\frac{\varphi}{2}}{\sin \frac{\varphi}{2}} \log j .$$

We see in particular that the equivariant torsion τ gives already for the trivial line bundle $\mathcal{O}$ on $\mathbb{P}^1\mathbb{C}$ the function

$$\log \tau(\mathcal{O}, \varphi) = \cot \frac{\varphi}{2} \cdot \Big(i \sum_{\substack{\ell\geq 1\\ \text{odd}}} \zeta'(-\ell) \frac{(i\varphi)^\ell}{\ell!} - \frac{C + \log \varphi}{\varphi} \Big) .$$

Let now $\Phi := \begin{pmatrix} i\varphi_1 & & 0 \\ & \ddots & \\ 0 & & i\varphi_{n+1} \end{pmatrix}$ be an element of the (canonical) maximal Cartan subalgebras of $\mathfrak{su}(n+1)$, hence an infinitesimal rotation on $\mathbb{P}^n\mathbb{C} \cong SU(n+1)/S(U(1) \times U(n))$. Assume that all the φ_j are distinct. Then we have

Theorem 3. *The equivariant torsion $\tau(\mathcal{O},e^{\Phi})$ for the trivial line bundle $\mathcal{O}$ on $\mathbb{P}^n\mathbb{C}$ is given by*

$$-2\ \log\,\tau(\mathcal{O},e^{\Phi}) = (-1)^n \sum_{\substack{j,k=1\\ j\neq k}}^{n+1} 2iR^{\mathrm{rot}}(\varphi_j-\varphi_k) \prod_{\substack{\ell=1\\ \ell\neq k}}^{n+1} (e^{i(\varphi_k-\varphi_\ell)}-1)^{-1} - \ \log n!\,.$$

I) Definition of the torsion

Let M be a Khler manifold of complex dimension n with holomorphic tangent bundle TM and Khler form ω_M, ξ a hermitian vector bundle on M and $\bar\partial$ the Dolbeault operator acting on sections of $\Lambda^q T^{*(0,1)}M\otimes\xi$. We define a hermitian product on the vector space of smooth sections of $\Lambda^q T^{*(0,1)}M\otimes\xi$ by

$$(\eta,\eta') := \int_M (\eta(x),\eta'(x))\frac{\omega^n}{(2\pi)^n n!}$$

as in [**GS1**]. Consider the adjoint operator $\bar\partial^*$ relative to this product and the Kodaira-Laplace operator

$$\Box_q := (\bar\partial+\bar\partial^*)^2 : \Gamma(\Lambda^q T^{*(0,1)}M\otimes\xi)\to\Gamma(\Lambda^q T^{*(0,1)}M\otimes\xi)\,.$$

Let g be a holomorphic isometry of M. Assume that the bundle and its hermitian metric are holomorphically invariant under the induced action of g. Let $\mathrm{Eig}_\lambda(\Box_q)$ be the eigenspace of $\Box_q$ corresponding to the eigenvalue λ and g^* the of g induced action on $\Gamma(\Lambda^q T^{*(0,1)}M\otimes\xi)$.

Consider the ζ-function

$$Z(g,s) := \sum_{\substack{q>O\\ \lambda\in\mathrm{Spec}\Box_q\\ \lambda\neq 0}} (-1)^{q+1} q\lambda^{-s}\,\mathrm{Tr} g^*_{|\ \mathrm{Eig}_\lambda(\Box_q)}$$

for $s\gg 0$. The equivariant torsion of M relative to the action of g is then defined as an exponential of the derivative at zero $Z'(g,0)$ of the holomorphic continuation of $Z(g,\cdot)$,

$$\tau(g) := e^{-\frac{1}{2}Z'(g,0)}\,.$$

The eigenvalues and eigenspaces for the Kodaira Laplacian for the trivial line bundle on $\mathbb{P}^n\mathbb{C}$ were determined explicitly by Ikeda and Taniguchi [**IT**]. If one regards $\mathbb{P}^n\mathbb{C}$ as $SU(n+1)/S(U(1)\times U(n))$, the eigenspaces can be described by sums of irreducible representations of $SU(n+1)$. We are using their method and results in our proof; see also Malliavin and Malliavin [**MM**].

II) The Laplacian on $\mathcal{O}(k)$-bundles over $\mathbb{P}^1\mathbb{C}$

Let $\mathbb{P}^1\mathbb{C}$ be the one-dimensional complex projective space equipped with the usual Fubini-Study metric. That means, $\mathbb{P}^1\mathbb{C}$ is isometric to the 2-sphere with radius $1/2$. Take $G := SU(2)$ and $K := S(U(1)\times U(1))$ with the corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$. We equip G with the metric

$$\begin{aligned} \mathfrak{g}^2 &\to \mathbb{R} \\ (X,Y) &\mapsto -2\,\mathrm{tr}XY\,, \end{aligned}$$

which is minus one half of the Killing form. Then we may represent $\mathbb{P}^1\mathbb{C}$ as the homogeneous space G/K with the induced metric.

Let Λ be the weight of $\mathfrak{g}$ which acts on the Cartan subalgebras $\mathfrak{k}$ by $\mathrm{diag}(i\varphi,-i\varphi)\mapsto\frac{\varphi}{2\pi}$ and let

$$\begin{aligned} \rho_k^K &: \mathfrak{k}\to\mathbb{C} \\ \begin{pmatrix} i\varphi & 0 \\ 0 & -i\varphi \end{pmatrix} &\mapsto e^{ik\varphi} \end{aligned}$$

be the of $k\Lambda, k\in\mathbb{Z}$, induced representation of K. This gives an action of K on the right of $G\times\mathbb{C}$ as follows:

$$(g,x)\cdot h = (gh, \rho_k^K(h^{-1})x)$$

for $g\in G, x\in\mathbb{C}$ and $h\in K$. Then the holomorphic line bundle $\mathcal{O}(k)$ is the homogeneous vector bundle

$$\mathcal{O}(k) = G \underset{\rho_{-k}^K}{\times} \mathbb{C} := (G\times\mathbb{C})/K\,.$$

It is well known that $\mathcal{O}(2)\cong T\mathbb{P}^1\mathbb{C}\cong T^{*(0,1)}\mathbb{P}^1\mathbb{C}$. By a theorem of Grothendieck [**G**], each holomorphic vector bundle E on $\mathbb{P}^1\mathbb{C}$ is a direct sum

$$E = \mathcal{O}(k_1)\oplus\ldots\oplus\mathcal{O}(k_n)\,,$$

$k_1,\ldots,k_n\in\mathbb{Z}$, so it suffices to calculate the torsion for $\mathcal{O}(k)$. Obviously, $Z'(\cdot,0)$ behaves additively under direct sum of vector bundles.

We equip $\mathcal{O}(k)$ with the induced metric. If ∇ is the unique holomorphic hermitian connection on the bundle of forms with coefficients in $\mathcal{O}(k)$, $\Lambda T^{*(0,1)}\mathbb{P}^1\mathbb{C}\otimes\mathcal{O}(k)$, and (e_1,e_2) a real orthonormal frame in the real tangent bundle $T_{\mathbb{R}}\mathbb{P}^1\mathbb{C}$, we define the horizontal (or Bochner) Laplacian as

$$\Delta := \sum_1^2 (\nabla_{e_n})^2 - \sum_1^2 \nabla_{\nabla_{e_n} e_n} .$$

We know that the curvature tensor of $\mathcal{O}(1)$ is simply $-2i$ times the Khler form of $\mathbb{P}^1\mathbb{C}$. By applying Licherowicz's formula (cf. Bismut [**B1**, Prop. 1.2]), we find that the Kodaira Laplacian acting on $T^{*(0,1)}\mathbb{P}^1\mathbb{C}\otimes\mathcal{O}(k)$ is given by

$$\bar{\Box}^{0,1} = -\tfrac{1}{2}\Delta + \frac{k}{2} + 1 .$$

To find a better expression for Δ, we consider the Casimir Operators of G and K. For a given compact Lie algebra with Killing form B and orthonormal basis $\{X_1,\ldots,X_n\}$ with respect to B, its Casimir operator is defined as

$$\mathrm{Cas} := -\sum_i X_i \cdot X_i .$$

Cas is independent of the choice of the basis. Let Cas_G be the Casimir operator of G, acting on $C^\infty(G)$ by derivation, and Cas_K the Casimir operator of K, acting on $\mathbb{C}$ via the representation ρ^K_{-k-2}. Then it is easily verified (cf. for example [**BGV**, Prop. 5.6]) that

$$2\Delta = \mathrm{Cas}_G + \mathrm{Cas}_K$$

on sections of $T^{*(0,1)}\mathbb{P}^1\mathbb{C}\otimes\mathcal{O}(k) \cong G \underset{\rho^K_{-k-2}}{\times} \mathbb{C}$. The factor 2 appears because we take half of the negative Killing form as metric on G. For $X\in\mathfrak{k}$ we have $\rho^K_{-k-2}(X) = -i(k+2)$, so

$$\rho^K_{-k-2}(\mathrm{Cas}_K) = (k+2)^2 ,$$

hence

Lemma 4.

$$\bar{\Box}^{0,1} = -\tfrac{1}{4}\,\mathrm{Cas}_G - \frac{k}{2}\Big(\frac{k}{2}+1\Big) .$$

III) Construction of the defining ζ-function

Let (ρ_ℓ^G, E_ℓ^G) be the irreducible representation $G \to \mathrm{End}(E_\ell^G)$ with highest weight $\ell\Lambda, \ell \in \mathbb{N}$. Then we have $\rho_\ell^G(\mathrm{Cas}_G) = -\ell(\ell+2) \cdot \mathrm{Id}_{E_\ell^G}$.

To determine the eigenspaces of $\bar{\square}^{0,1}$, we use as Ikeda and Taniguchi the following Frobenius law of Bott [**Bo**]:

Proposition 5. *For finite dimensional representations (ρ^K, E^K) and (ρ^G, E^G) of K and G, we have the canonical isomorphism of vector spaces*

$$\mathrm{Hom}_G(E^G, \Gamma(G \underset{\rho^K}{\times} E^K)) \cong \mathrm{Hom}_K(E^G, E^K)\,.$$

Now we know that the characters χ_ℓ^G of ρ_ℓ^G and χ_k^K of ρ_k^K are given by

$$\chi_\ell^G \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} = \frac{\sin(\ell+1)\varphi}{\sin\varphi}$$

(cf. Brcker, tom Dieck [**BD**, Ch. 5, p. 267]), and

$$\chi_k^K \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} = e^{ik\varphi}\,,$$

hence we find the decomposition

$$\chi_\ell^G = \begin{cases} \displaystyle\sum_{\substack{|n|\le\ell \\ n \text{ even}}} \chi_n^K & \text{when } \ell \text{ even} \\ \displaystyle\sum_{\substack{|n|\le\ell \\ n \text{ odd}}} \chi_n^K & \text{when } \ell \text{ odd}\,. \end{cases}$$

Now we can see by Proposition 5 that (ρ_ℓ^G, E_ℓ^G) occurs as irreducible subspace of $\Gamma(G \underset{\rho_n^K}{\times} \mathbb{C})$ iff $|n| \le \ell$ and $n \equiv \ell (\mathrm{mod}\ 2)$:

Lemma 6. $\Gamma(T^{*(0,1)}\mathbb{P}^1\mathbb{C} \otimes \mathcal{O}(k))$ *contains the L^2-dense subspace*

$$\bigoplus_{\ell\ge 0} E^G_{|k+2|+2\ell}\,.$$

The density of this subspace follows from the Peter-Weyl theorem (cf. [**Bo**]). By Lemma 4, the eigenvalues of $\bar{\square}^{0,1}$ for $\mathcal{O}(k)$ are given by

$$\begin{cases} \ell(\ell+k+1) \text{ on } E^G_{k+2\ell} \text{ for } \ell \ge 1 & \text{when } k \ge -1 \\ \ell(\ell-k-1) \text{ on } E^G_{-k-2+2\ell} \text{ for } \ell \ge 0 & \text{when } k < -1\,. \end{cases}$$

So we finally obtain the

Lemma 7. *Let* $g := \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \in G$, $\varphi \in]0,\pi[$, *be an element of the maximal torus* K *(which corresponds to the rotation of* S^2 *by the angle* 2φ*). Then the* ζ*-function* $Z_k(g,\cdot)$ *of the* $\mathcal{O}(k)$*-bundle on* $\mathbb{P}^1\mathbb{C}$ *is for* $s > \frac{1}{2}$ *given by*

$$Z_k(g,s) = \sum_{\substack{\ell \geq 0 \\ E^G_{|k+2|+2\ell} \not\subset \ker \bar{\Box}^{0,1}}} \chi^G_{|k+2|+2\ell} \cdot \left(\bar{\Box}^{0,1}\big|_{E^G_{|k+2|+2\ell}}\right)^{-s}$$
$$= \sum_{\ell \geq 1} \frac{\sin(2\ell + |k+1|)\varphi}{\sin\varphi} \cdot \ell^{-s}(\ell + |k+1|)^{-s}.$$

In particular, $Z_k(g,s) = Z_{-k-2}(g,s)$. This is in fact an immediate consequence of the Poincar duality.

IV) The derivative at zero of the Lerch zeta function

Define for $0 < \varphi < 2\pi$, Re $s > 0$ the zeta function $\zeta^{\mathrm{rot}}(\varphi,s)$ by

$$\zeta^{\mathrm{rot}}(\varphi,s) := \sum_{\ell=1}^{\infty} \frac{\sin \ell\varphi}{\ell^s}.$$

ζ^{rot} continuous holomorphically to the whole complex plane. Let $\varphi = 2\pi\frac{p}{q}$, $p,q \in \mathbb{N}$, $0 < p < q$ be a rational angle and $\zeta(\cdot,\cdot)$ the Hurwitz zeta function. We obtain

$$\zeta^{\mathrm{rot}}(\varphi,s) = \sum_{j=1}^{q}\sum_{\ell=0}^{\infty} \frac{\sin(\ell q + j)\varphi}{(\ell q + j)^s} = \sum_{j=1}^{q} \frac{\sin j\varphi}{q^s} \sum_{\ell=0}^{\infty} \left(\ell + \frac{j}{q}\right)^{-s}$$
$$= \sum_{j=1}^{q} \frac{\sin j\varphi}{q^s} \zeta(s, \frac{j}{q}).$$

By using the equations (see for example [**WW**, Chap. XIII])

$$\zeta(0,x) = \tfrac{1}{2} - x \quad \text{and} \quad \frac{\partial}{\partial s}_{|\, s=0} \zeta(s,x) = \log \frac{\Gamma(x)}{\sqrt{2\pi}}$$

we find

$$\frac{\partial}{\partial s}\zeta^{\mathrm{rot}}(\varphi,0) = \sum_{j=1}^{q} \sin j\varphi \cdot \left(\log \frac{\Gamma(\frac{j}{q})}{\sqrt{2\pi}} - \log q \cdot \left(\tfrac{1}{2} - \frac{j}{q}\right)\right).$$

Because of $\sum_{j=1}^{q} \sin j\varphi = 0$ and $\sum_{j=1}^{q} \frac{j}{q} \sin j\varphi = -\frac{1}{2} \cot \frac{\varphi}{2}$ this is equal to

$$\frac{\partial}{\partial s}\zeta^{\mathrm{rot}}(\varphi,0) = -\tfrac{1}{2}\log q \cdot \cot\frac{\varphi}{2} + \sum_{j=1}^{q} \sin j\varphi \cdot \log \Gamma(\frac{j}{q}).$$

V) The derivative at zero for arbitrary angles

We are using Kummer's Fourier series for the logarithm of the Γ-function

$$\log\,\Gamma(x)=\tfrac12\log\,2\pi+\sum_{n\ge1}\Big(\frac{\cos 2\pi nx}{2n}+\frac{C+\log\,2\pi n}{n\pi}\sin 2\pi nx\Big)\ (0<x<1)\,.$$

With the orthogonal relations

$$\sum_{j=1}^{q}\sin\frac{2\pi jp}{q}\cos\frac{2\pi jn}{q}=0\,,$$

$$\sum_{j=1}^{q}\sin\frac{2\pi jp}{q}\sin\frac{2\pi jn}{q}=\frac q2\cdot\big(\delta_{p\equiv n(\bmod\ q)}-\delta_{p\equiv -n(\bmod\ q)}\big)$$

and the Fourier series of the identity function

$$x\ \log\,q=\frac{\log\,q}{2}-\sum_{n\ge1}\frac{\log\,q}{n\pi}\sin 2\pi nx\ (0<x<1)\,,$$

it follows that

$$\frac{\partial}{\partial s}\zeta^{\,\mathrm{rot}}(\varphi,0)=\frac q2\cdot\left[\frac{C+\log\,2\pi\frac pq}{p\pi}+\sum_{n\ge1}\left(\frac{C+\log\,(2\pi\frac{nq+p}{q})}{(nq+p)\pi}-\frac{C+\log\,(2\pi\frac{nq-p}{q})}{(nq-p)\pi}\right)\right]$$

$$=\frac{C+\log\,\varphi}{\varphi}+\sum_{n\ge1}\left(\frac{C+\log(2\pi n+\varphi)}{2\pi n+\varphi}-\frac{C+\log(2\pi n-\varphi)}{2\pi n-\varphi}\right).$$

We have the identities (see [**WW**] or Bismut and Soul [**B2**, Appendix])

$$\sum_{n\ge1}\Big(\frac{1}{n+x}-\frac{1}{n-x}\Big)=\pi\cot\pi x-\frac1x=-2\sum_{\substack{\ell\ge1\\ \text{odd}}}\zeta(\ell+1)x^{\ell}\,,$$

$$\sum_{n\ge1}\Big(\frac{\log\,n}{n+x}-\frac{\log\,n}{n-x}\Big)=2x\sum_{n\ge1}\frac{-\log\,n}{n^2}\sum_{\ell\ge0}\Big(\frac xn\Big)^{2\ell}=2\sum_{\substack{\ell\ge1\\ \text{odd}}}\zeta'(\ell+1)x^{\ell}$$

and

$$\sum_{n\ge1}\Big(\frac{\log(1+\frac xn)}{n+x}-\frac{\log(1-\frac xn)}{n-x}\Big)=\sum_{n\ge1}\frac2n\sum_{\substack{\ell\ge1\\ \text{odd}}}\frac{x}{n}^{\ell}\sum_{j=1}^{\ell}\frac1j$$

$$=2\sum_{\substack{\ell\ge1\\ \text{odd}}}\zeta(\ell+1)\sum_{j=1}^{\ell}\frac1j\cdot x^{\ell}\,,$$

so we obtain

$$\frac{\partial}{\partial s}\zeta^{\,\mathrm{rot}}(\varphi,0)=\frac{C+\ \log\,\varphi}{\varphi}+\frac1\pi\sum_{\substack{\ell\ge1\\ \text{odd}}}\Big(\frac{\zeta'(\ell+1)}{\zeta(\ell+1)}+\sum_{j=1}^{\ell}\frac1j-C-\ \log\,2\pi\Big)\cdot\zeta(\ell+1)\cdot\Big(\frac{\varphi}{2\pi}\Big)^{\ell}$$

$$=\frac{C+\ \log\,\varphi}{\varphi}-\sum_{\substack{\ell\ge1\\ \text{odd}}}\zeta'(-\ell)(-1)^{\frac{\ell+1}{2}}\frac{\varphi^{\ell}}{\ell!}\,.$$

This gives the Proposition 1 by continuity.

VI) The torsion on $\mathbb{P}^1\mathbb{C}$

Recall now the zeta function Z_k of Lemma 7 with $\varphi\neq 0$. By a Taylor expansion of the denominator with respect to $\frac{|k+1|}{\ell}$, we find for $s \searrow 0$

$$\begin{aligned}
\frac{\partial}{\partial s} Z_k(g,s) &= -\sum_{\ell\geq 1} \frac{\sin(2\ell+|k+1|)\varphi}{\sin\varphi} \cdot \Big(\frac{\log\,\ell}{\ell^s(\ell+|k+1|)^s} + \frac{\log(\ell+|k+1|)}{\ell^s(\ell+|k+1|)^s} \Big) \\
&= -\sum_{\ell\geq 1} \frac{\sin(2\ell+|k+1|)\varphi}{\sin\varphi} \cdot \frac{\log\,\ell}{\ell^{2s}} \cdot \Big(1+\frac{|k+1|}{\ell}\Big)^{-s} \\
&\quad - \sum_{\ell>|k+1|} \frac{\sin(2\ell-|k+1|)\varphi}{\sin\varphi} \cdot \frac{\log\,\ell}{\ell^{2s}} \cdot \Big(1-\frac{|k+1|}{\ell}\Big)^{-s} \\
&= -\sum_{\ell\geq 1} \frac{2\cos|k+1|\,\varphi\sin 2\ell\varphi}{\sin\varphi} \cdot \frac{\log\,\ell}{\ell^{2s}} + \sum_{\ell=1}^{|k+1|} \frac{\sin(2\ell-|k+1|)\varphi}{\sin\varphi} \cdot \frac{\log\,\ell}{\ell^{2s}} + \mathcal{O}(s) \\
&= \frac{2\cos|k+1|\,\varphi}{\sin\varphi} \frac{\partial}{\partial s}\zeta^{\mathrm{rot}}(2\varphi,2s) + \sum_{\ell=1}^{|k+1|} \frac{\sin(2\ell-|k+1|)\varphi}{\sin\varphi} \cdot \frac{\log\,\ell}{\ell^{2s}} + \mathcal{O}(s)\,,
\end{aligned}$$

hence for $s=0$

$$\frac{\partial}{\partial s} Z_k(g,0) = \frac{2\cos|k+1|\,\varphi}{\sin\varphi} R^{\mathrm{rot}}(2\varphi) + \sum_{\ell=1}^{|k+1|} \frac{\sin(2\ell-|k+1|)\varphi}{\sin\varphi}\ \log\,\ell\,.$$

Remark that this computation breaks down for $\varphi = 0$ because of the singularity of the Riemann ζ-function. The isomorphism g corresponds to a rotation of the sphere by an angle 2φ, so we obtain Theorem 2.

VII) The zeta function on $\mathbb{P}^n\mathbb{C}$

Now we regard as in **[IT]** the complex projective space $\mathbb{P}^n\mathbb{C}$ as the homogeneous space $SU(n+1)/S(U(1)\times U(n))$. Let

$$\mathfrak{h} := \left\{ \begin{pmatrix} i\varphi_1 & & 0 \\ & \ddots & \\ 0 & & i\varphi_{n+1} \end{pmatrix} \,\middle|\, \sum_1^{n+1} \varphi_j = 0 \right\}$$

be the canonical maximal Cartan subalgebra of the Lie algebra $\mathfrak{su}(n+1)$. Let Λ_j, $1\leq j\leq n$, be the fundamental weight

$$\Lambda_j : \mathrm{diag}(i\varphi_1,\dots,i\varphi_{n+1}) \mapsto \sum_1^j \frac{\varphi_k}{2\pi}\,.$$

In the following, $\Lambda(k,0,q)$ denotes the irreducible $SU(n+1)$ -representation with highest weight given by $(k-q)\Lambda_1+\Lambda_q+k\Lambda_n$ for all $k \geq q$, $n \geq q \geq 0$. Ikeda and Taniguchi found that the spaces

$$\bigoplus_{k\geq 0}\Lambda(k,0,0) \qquad\qquad (q=0)$$
$$\bigoplus_{k\geq q}\Lambda(k,0,q)\oplus\bigoplus_{k\geq q+1}\Lambda(k,0,q+1) \qquad (0<q<n)$$
$$\bigoplus_{k\geq n}\Lambda(k,0,n) \qquad\qquad (q=n)$$

can be regarded as L^2-dense subspaces of $\Gamma(\Lambda^q T^{*(0,1)}\mathbb{P}^n\mathbb{C})$, where the Laplacian acts on $\Lambda(k,0,q)$ by multiplication with $k(k+n+1-q)$. We denote by $\chi(k,0,q)$ the character to the representation $\Lambda(k,0,q)$. Hence we find for our zeta function

$$\begin{aligned}Z(\cdot,s) &= \sum_{q=1}^{n-1}(-1)^{q+1}q\Bigg(\sum_{k\geq q}\frac{\chi(k,0,q)}{k^s(k+n+1-q)^s}+\sum_{k\geq q+1}\frac{\chi(k,0,q+1)}{k^s(k+n-q)^s}\Bigg)\\ &\qquad +(-1)^{n+1}n\sum_{k\geq n}\frac{\chi(k,0,n)}{k^s(k+1)^s}\\ &= \sum_{q=1}^{n}(-1)^{q+1}\sum_{k\geq q}\frac{\chi(k,0,q)}{k^s(k+n+1-q)^s}\,.\end{aligned}$$

The "telescope" effect in the summation is not caused by accident, but by the natural splitting of each eigenspace $\mathrm{Eig}_\lambda(\Box)$ into $\mathrm{Eig}_\lambda(\Box)\cap\ker\bar\partial$ and $\mathrm{Eig}_\lambda(\Box)\cap\ker\bar\partial^*$, which are isomorphic. The character χ_Λ of an irreducible $SU(n+1)$-module with highest weight $\Lambda = m_1\Lambda_1 + m_2(\Lambda_2 - \Lambda_1) + \ldots + m_n(\Lambda_n - \Lambda_{n-1})$, $m_1 \geq \ldots \geq m_n \geq m_{n+1} = 0$, can classically be calculated by Weyl's character formula. One finds with $e_j := e^{i\varphi_j}$

$$\chi_\Lambda\begin{pmatrix} i\varphi_1 & & 0\\ & \ddots & \\ 0 & & i\varphi_{n+1}\end{pmatrix} = \frac{\det(e_j^{m_\ell+n+1-\ell})_{j,\ell=1}^{n+1}}{\det(e_j^{n+1-\ell})_{j,\ell=1}^{n+1}}\,.$$

In our case one gets after a rotation of the first q rows

$$\chi(k,0,q) = \begin{matrix}\text{exceptional}\\ q\text{-th row}\end{matrix}\ \rightarrow \begin{vmatrix} e_1^n & \cdots & e_{n+1}^n\\ \vdots & & \vdots\\ e_1^{n+1-(q-1)} & & e_{n+1}^{n+1-(q-1)}\\ e_1^{n+1-q+k} & & e_{n+1}^{n+1-q+k}\\ e_1^{n+1-(q+1)} & & e_{n1+}^{n+1-(q+1)}\\ \vdots & & \vdots\\ e_1 & & e_{n+1}\\ e_1^{-k} & \cdots & e_{n+1}^{-k}\end{vmatrix} : \begin{vmatrix} e_1^n & \cdots & e_{n+1}^n\\ \vdots & & \vdots\\ 1 & \cdots & 1\end{vmatrix}\cdot(-1)^{q+1}\,.$$

We see immediately

$$\chi(k-(n+1-q),0,q) = -\chi(-k,0,q)$$

and $\chi(k,0,q)=0$ for $k\in\{-n,\ldots,q-1\}\setminus\{0,q-n-1\}$.

VIII) The torsion on $\mathbb{P}^n\mathbb{C}$

Remark that $\chi(k,0,q)$ as a function in k can be regarded as a linear combination of exponentials $\exp\, ik(\varphi_j-\varphi_\ell)$ with $1\le j,\ell\le n+1$. So the function

$$\sum_{k\ge 1}\frac{\log\,k}{k^{2s}}\chi(k,0,q)$$

is a linear combination of Lerch ζ-functions. Hence it follows, if all the φ_j are distinct, for $s\searrow 0$

$$\begin{aligned}
Z'(\cdot,s) &= \sum_{q=1}^{n}(-1)^{q+1}\left(\sum_{k\ge q}\frac{\chi(k,0,q)\log k}{k^s(k+n+1-q)^s}-\sum_{k\ge n+1}\frac{\chi(-k,0,q)\log\,k}{k^s(k-n-1+q)^s}\right)\\
&= \sum_{q=1}^{n}(-1)^{q+1}\left(\sum_{k\ge 1}\big(\chi(k,0,q)-\chi(-k,0,q)\big)\frac{\log\,k}{k^{2s}}\right.\\
&\qquad\qquad \left.+\frac{\log(n+1-q)}{(n+1-q)^{2s}}\chi(q-n-1,0,q)\right)+\mathcal{O}(s)\\
&= \sum_{q=1}^{n}(-1)^{q+1}\sum_{k\ge 1}\big(\chi(k,0,q)-\chi(-k,0,q)\big)\frac{\log\,k}{k^{2s}}-\log\,n!+\mathcal{O}(s)\,,
\end{aligned}$$

because of $\chi(q-n-1,0,q)=(-1)^q$. The Laplace expansion theorem for determinants shows

$$\sum_{q=1}^{n}(-1)^{q+1}\chi(k,0,q) = 1-\sum_{j=1}^{n+1}e_j^k\begin{vmatrix} e_1^n & \ldots & e_{n+1}^n\\ \vdots & & \vdots\\ e_1 & \ldots & e_{n+1}\\ e_1^{-k} & \ldots & e_{n+1}^{-k}\end{vmatrix} : \begin{vmatrix} e_1^n & \ldots & e_{n+1}^n\\ \vdots & & \vdots\\ 1 & \ldots & 1\end{vmatrix}.$$

Hence we obtain some Vandermonde determinants:

$$\sum_q (-1)^{q+1}\big(\chi(k,0,q)-\chi(-k,0,q)\big) =$$

$$- \sum_{\substack{j,\ell=1\\ j\neq\ell}}^{n+1} \left(\left(\frac{e_j}{e_\ell}\right)^k - \left(\frac{e_\ell}{e_j}\right)^k\right)(-1)^{n+\ell} \begin{vmatrix} e_1^n & \dots & \widehat{e_\ell^n} & \dots & e_{n+1}^n \\ \vdots & & & & \vdots \\ e_1 & \dots & \widehat{e_\ell} & \dots & e_{n+1} \end{vmatrix} : \begin{vmatrix} e_1^n & \dots & e_{n+1}^n \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{vmatrix}$$

$$= (-1)^n \sum_{j,\ell=1}^{n+1} \left(\left(\frac{e_j}{e_\ell}\right)^k - \left(\frac{e_\ell}{e_j}\right)^k\right) \prod_{\substack{k=1\\ k\neq\ell}}^{n+1} \left(\frac{e_\ell}{e_k}-1\right)^{-1}$$

(the $\frown$ indicates that the ℓ-th column is missing). By using

$$\left(\frac{e_j}{e_\ell}\right)^k - \left(\frac{e_\ell}{e_j}\right)^k = 2i \sin k(\varphi_j - \varphi_\ell)$$

and the definition of $R^{\mathrm{rot}}(\varphi)$, we find Theorem 3.

IX) Remarks about the function R^{rot}

The function R^{rot} has a rather simple definition and hence a lot of special properties. Here we only give a few of them.

Theorem 8. *The following identities hold*

(1) $R^{\mathrm{rot}}(\varphi) = -R^{\mathrm{rot}}(2\pi-\varphi) \qquad (0<\varphi<2\pi)$,

(2) $2R^{\mathrm{rot}}(2\varphi) = R^{\mathrm{rot}}(\varphi) + R^{\mathrm{rot}}(\pi+\varphi) + \log 2 \cdot \cot\varphi \qquad (0<\varphi<\pi)$,

(3) $3R^{\mathrm{rot}}(3\varphi) = R^{\mathrm{rot}}(\varphi) + R^{\mathrm{rot}}\left(\frac{2\pi}{3}+\varphi\right) - R^{\mathrm{rot}}\left(\frac{2\pi}{3}-\varphi\right) + \frac{3}{2}\log 3 \cdot \cot\frac{3\varphi}{2} \qquad (0<\varphi<\frac{2\pi}{3})$,

(4) $R^{\mathrm{rot}}(\pi+\varphi) = \int_0^\infty \log x \frac{\sinh \varphi x}{\sinh \pi x} dx \qquad (-\pi<\varphi<\pi)$.

Proof: 1) is trivial by the definition of R^{rot}. 2) follows from

$$2^{1-s}\zeta^{\mathrm{rot}}(2\varphi,s) = \zeta^{\mathrm{rot}}(\varphi,s) + \zeta^{\mathrm{rot}}(\pi+\varphi,s).$$

We see by the formulas of § IV that $\zeta^{\mathrm{rot}}(\varphi,0) = \frac{1}{2}\cot\frac{\varphi}{2}$. The result follows then by derivation. In the same way, one gets 3) from

$$3^{1-s}\zeta^{\mathrm{rot}}(3\varphi,s) = \zeta^{\mathrm{rot}}(\varphi,s) + \zeta^{\mathrm{rot}}\left(\frac{2\pi}{3}+\varphi\right) - \zeta^{\mathrm{rot}}\left(\frac{2\pi}{3}-\varphi\right).$$

To see the integral formula 4) we are using the Fourier series

$$-\frac{\pi}{2}\frac{\sinh \varphi x}{\sinh \pi x} = \sum_1^\infty \frac{(-1)^\ell \ell}{x^2+\ell^2}\sin \ell\varphi \qquad (|\varphi|<\pi)$$

and the definite integral

$$\int_0^\infty \frac{x^{-s}dx}{x^2+\ell^2} = \frac{\pi}{2\ell^{1+s}\cos\frac{s\pi}{2}} \qquad (|s|<1)\,.$$

We have for $|s|<1$.

$$\zeta^{\rm rot}(\pi+\varphi,s) = \sum_1^\infty \frac{(-1)^\ell \sin \ell\varphi}{\ell^s} = \frac{2}{\pi}\cos\frac{\pi s}{2}\sum_1^\infty \int_0^\infty \frac{(-1)^\ell \ell x^{-s}dx}{x^2+\ell^2}\sin \ell\varphi$$
$$= -\cos\frac{\pi s}{2}\int_0^\infty x^{-s}\frac{\sinh\, \varphi x}{\sinh\, \pi x}dx\,.$$

The desired result follows. □

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