Quaternionic analytic torsion

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April 10, 2001

Abstract

We define an (equivariant) quaternionic analytic torsion for antiselfdual vector bundles on quaternionic Kähler manifolds, using ideas by Leung and Yi. We compute this torsion for vector bundles on quaternionic homogeneous spaces with respect to any isometry in the component of the identity, in terms of roots and Weyl groups.

2000 Mathematics Subject Classification: 53C25, 58J52, 53C26, 53C35

Contents

1 Introduction

Analytic torsions were introduced by Ray and Singer as real numbers constructed using certain $\mathbf{Z}$–graduated complexes of elliptic differential operators acting on forms with coefficients in vector bundles on compact manifolds. The real analytic torsion was defined for the de Rham-operator associated to flat Hermitian vector bundles on Riemannian manifolds. It was proven by Cheeger and Müller to equal a topological invariant, the Reidemeister torsion, which can be defined using a finite triangulation of the manifold. This implies that the real analytic torsion is a homeomorphy invariant which is not invariant under homotopy. Lott and Rothenberg pointed out that an equivariant version of this torsion still is a diffeomorphy invariant.

The complex Ray-Singer (or holomorphic) torsion was defined for the Dolbeault–operator acting on antiholomorphic differential forms with coefficients in a holomorphic Hermitian vector bundle on a compact complex manifold. It turned out to play an important role in the Arakelov geometry of schemes over Dedekind rings. In fact it was shown by Bismut, Gillet and Soulé to provide a direct image in a K-theory of Hermitian vector bundles. This direct image verifies a Grothendieck-Riemann-Roch relation with Arakelov-Gillet-Soulé intersection theory, as was proven by Bismut, Lebeau, Gillet and Soulé. Later, Köhler and Roessler showed that an equivariant version of this direct image localizes on fixed point subschemes in Arakelov geometry. This had many applications in arithmetic geometry, algebra and global analysis.

Thus it seems natural to investigate torsions for other $\mathbf{Z}$–graded complexes occurring in geometry, in particular for quaternionic manifolds. A first attempt at a definition of analytic torsion for general quaternionic manifolds was made in an e–print by Leung and Yi [LY], using a complex first discussed by Salamon. We had problems understanding this very general, short and ambiguous definition. In the present paper, we first give a thorough definition of an (equivariant) quaternionic torsion for quaternionic Kähler manifolds M, with coefficients in the antiselfdual vector bundles $\mathcal{W}$. This is done by carefully decomposing the action of a natural Dirac operator on Salamon's complex on these manifolds, i.e. on the complex

$$\begin{aligned} 0 \longrightarrow \operatorname{Sym}^k H \otimes \mathcal{W} &\stackrel{d}{\longrightarrow} \operatorname{Sym}^{k+1} H \otimes \Lambda^{1,0} E^* \otimes \mathcal{W} \\ &\stackrel{d}{\longrightarrow} \dots \stackrel{d}{\longrightarrow} \operatorname{Sym}^{2n+k} H \otimes \Lambda^{2n,0} E^* \otimes \mathcal{W} \longrightarrow 0 \end{aligned}$$

for a parameter $k \in \mathbf{N}_0$ even and $TM \otimes_{\mathbf{R}} \mathbf{C} \cong H \otimes E$. The Laplace operator defining the torsion is the square of this Dirac operator. We detail the many traps to avoid in this construction.

In the third section, we compute the equivariant quaternionic torsion for all known quaternionic Kähler manifolds of positive curvature, i.e. for the quaternionic homogeneous spaces of the compact type, with respect to the action of any element of the associated Lie group and any equivariant antiselfdual vector bundle. These spaces are known to be symmetric, and for any simple compact Lie group there is exactly one quaternionic homogeneous space. This computation proceeds very similar to previous computations of the real analytic torsion and the holomorphic torsion for all appropriate symmetric spaces by one of the us. We regard this as further indication that the definition given here is a "good" definition of quaternionic torsion.

For the real analytic torsion, this computation led to a homeomorphy classification of quotients of some odd-dimensional symmetric spaces of the compact type. For the holomorphic torsion, this computation gave evidence for the fixed point formula mentioned above. In combination with the fixed point formula in Arakelov geometry, it provided a new proof of the Jantzen sum formula classifying the lattice representations of Chevalley group schemes except for the cases G_2, F_4, E_8. Thus one can reasonably hope for interesting applications of our result. Remarkably, the formula for quaternionic torsion happens to have the very same structure as the formula for the holomorphic torsion on Hermitian symmetric spaces (thus, on different manifolds). In a forthcoming paper, we intend to relate the torsion to holomorphic torsion on the twistor space, which should as an application of the computation done here provide a full proof of the Jantzen sum formula including the three exceptional cases.

In the last section, we comment briefly on the special case of hyperkähler manifolds, in which the quaternionic torsion can be expressed in terms of a Dolbeault–operator. Related work for this case has been done recently by Gerasimov and Kotov [GK1], [GK2].

Acknowledgements: We are indebted to Sebastian Goette for many fruitful discussions. The first author thanks the Deutsche Forschungsgemeinschaft for supporting him with a Heisenberg fellowship.

2 Quaternionic analytic torsion

Perhaps the most fundamental difference between quaternionic geometry and complex geometry is the lack of a plausible notion of quaternionic differentiability, any such notion leads inevitably to a finite dimensional space of quaternionic differentiable functions even on $\mathbf{H}^n$. Hence the stock of local transition functions is rather limited and it seems impossible to define a quaternionic manifold in terms of an atlas of holomorphic coordinate charts. Another way to express this difference between complex and quaternionic geometry is that each quaternionic manifold comes along with a distinguished "projective" equivalence class of torsion free connections, a feature unheard of in complex geometry but rather characteristic for so called parabolic geometries. In fact quaternionic geometry can be seen as an example for parabolic geometries and many of the aspects discussed below are more or less directly linked to this fact. The interested reader is referred to [BS] for this point of view.

A quaternionic manifold M is a manifold of dimension $4n$, $n \geq 2$, endowed with a smooth quaternionic structure on its tangent spaces admitting an adapted torsion free connection. In other words M is endowed with a reduction $\mathbf{Sp}(1) \cdot \mathbf{GL}_{\mathbf{H}}(M) \subset \mathbf{GL}(M)$ of its frame bundle to the bundle of quaternionic frames with structure group $\mathbf{Sp}(1) \cdot \mathbf{GL}_n(\mathbf{H}) \subset \mathbf{GL}_{4n}(\mathbf{R})$ tangent to some torsion free connection. The projective equivalence class of this connection is uniquely determined by the quaternionic structure in the sense that the adapted connections are parametrized by 1–forms on M. A guiding principle in the construction of differential complexes on quaternionic manifolds is hence to twist with a trivialisable line bundle in order to make the differential operators independent of the choice of connection following Fegan's approach to the construction of conformally invariant differential operators [F].

Note that dimension 4 is explicitly excluded from the definition given above, in fact the group $\mathbf{Sp}(1) \cdot \mathbf{GL}_n(\mathbf{H})$, $n = 1$, is exactly the conformal group and the existence of a torsion free connection imposes no integrability assumption whatsoever on the conformal structure. Consequently differential sequences like (3) and (4) below fail in general to be complexes in conformal geometry. However there is a geometry in dimension 4 analogous to quaternionic geometry in higher dimensions $4n$, $n > 1$, namely the so called half conformally flat geometry of conformal manifolds with vanishing self–dual Weyl tensor. Mutatis mutandis our considerations below are valid in half conformally flat geometry in dimension 4, in particular the differential sequences (3) and (4) become complexes under this integrability assumption.

Any representation of the group $\mathbf{Sp}(1) \cdot \mathbf{GL}_n(\mathbf{H})$ gives rise to a vector bundle on M associated to the quaternionic frame bundle $\mathbf{Sp}(1) \cdot \mathbf{GL}_{\mathbf{H}}(M)$. Consider the two defining representations $\pi_H = \mathbf{C}^2$ of $\mathbf{Sp}(1)$ and $\pi_E = \mathbf{C}^{2n}$ of $\mathbf{GL}_n(\mathbf{H})$ respectively, which both carry invariant quaternionic structures J by definition. Moreover the representation π_H carries an invariant symplectic form σ, which is real (i.e. $\sigma(Jh_1,\, Jh_2) = \overline{\sigma(h_1,\, h_2)}$) and positive (i.e. $\sigma(h,\, Jh) > 0$ for all $h \neq 0$). The existence of a non–degenerate bilinear form σ implies in particular that π_H is equivalent to its dual π_H^* as an $\mathbf{Sp}(1)$–representation via the musical isomorphism $\sharp: \ \pi_H \longrightarrow \pi_H^*,\, h \longmapsto h^\sharp$ or its inverse $\flat$ with $h^\sharp := \sigma(h, \cdot)$.

Notice that the complex determinant of an element of $\mathbf{GL}_n(\mathbf{H}) \subset \mathbf{GL}_{2n}(\mathbf{C})$ is always a real positive number. Hence the representation $(\det \pi_E)^s$ of $\mathbf{GL}_n(\mathbf{H})$ is defined for any $s \in \mathbf{R}$. Moreover $\mathbf{PGL}_n(\mathbf{H}) := \mathbf{GL}_n(\mathbf{H})/\mathbf{R}^*$ is a real form of $\mathbf{SL}_{2n}(\mathbf{C})$ and thus all irreducible representations of $\mathbf{Sp}(1) \times \mathbf{GL}_n(\mathbf{H})$ occur in tensor products of π_H, π_E, π_E^* and $(\det \pi_E)^s$ with $s \in \mathbf{R}$. The irreducible representations occuring in a tensor product $(\det \pi_E)^s \otimes \pi_H^{\otimes k} \otimes \pi_E^{\otimes a} \otimes \pi_E^{*\otimes b}$ with $k + a + b$ even descend to $\mathbf{Sp}(1) \cdot \mathbf{GL}_n(\mathbf{H})$, in particular all irreducible representations of $\mathbf{Sp}(1) \cdot \mathbf{GL}_n(\mathbf{H})$ carry real structures and so do all vector bundles associated to the bundle of quaternionic frames e. g. the complexified tangent bundle $TM \otimes_{\mathbf{R}} \mathbf{C}$ is associated to the representation $\pi_H \otimes \pi_E$. We will write $TM \otimes_{\mathbf{R}} \mathbf{C} \cong H \otimes E$ although this notation has to be taken with care as neither E nor H are globally defined vector bundles in general. The trivializable line bundles associated to the representations $(\det \pi_E)^s$ will be denoted by L^s.

The invariant symplectic form on π_H defines a real, positive section σ_H of the vector bundle $\Lambda^2 H$, which is parallel for every adapted torsion free connection. Choosing similarly a real, positive section σ_E of $\Lambda^2 E^*$ amounts to choosing a Riemannian metric on M compatible with the quaternionic structure. In particular quaternionic Kähler manifolds are quaternionic manifolds M with a fixed real, positive section σ_E of $\Lambda^2 E$, which is parallel for an adapted torsion free connection, necessarily equal to the Levi–Civita connection of the Riemannian metric $\sigma_H \otimes \sigma_E$.

The tensor product decomposition $TM \otimes_{\mathbf{R}} \mathbf{C} \cong H \otimes E$ of the complexified tangent bundle of a quaternionic manifold induces a corresponding decomposition $T^*M \otimes_{\mathbf{R}} \mathbf{C} \cong H \otimes E^*$ of its cotangent bundle and of the whole exterior algebra of forms. According to the theory of Schur functors [FH] this decomposition reads

$$\Lambda^\bullet(T^*M \otimes_{\mathbf{R}} \mathbf{C}) \quad \cong \quad \bigoplus_{\substack{2n \geq a \geq b \geq 0 \\ a+b=\bullet}} \mathrm{Sym}^{a-b} H \otimes \Lambda^{a,b} E^*\,, \tag{1}$$

where $\Lambda^{a,b}E^* \subset \Lambda^a E^* \otimes \Lambda^b E^*$ is the kernel of a $\mathbf{GL}_n(\mathbf{H})$–equivariant map

$$0 \;\longrightarrow\; \Lambda^{a,b}E^* \;\stackrel{\subset}{\longrightarrow}\; \Lambda^a E^* \otimes \Lambda^b E^* \;\stackrel{\mathrm{Pl}}{\longrightarrow}\; \Lambda^{a+1}E^* \otimes \Lambda^{b-1}E^* \;\longrightarrow\; 0\,,$$

whose precise definition is immaterial for the arguments below. Wedging with a 1–form in $H \otimes E^*$ maps $\mathrm{Sym}^{a-b} H \otimes \Lambda^{a,b} E^*$ to the sum $\mathrm{Sym}^{a-b+1} H \otimes \Lambda^{a+1,b} E^* \oplus \mathrm{Sym}^{a-b-1} H \otimes \Lambda^{a,b+1} E^*$ of course and a little more elaboration provides us with an explicit isomorphism (1) such that

$$\begin{aligned}(h \otimes \eta)\wedge \;\; &= \;\; \frac{1}{a-b+1}\, h \cdot \otimes \eta \wedge \otimes \mathrm{id} \; + \; (-1)^{a-b} h^\sharp \lrcorner \otimes \mathrm{id} \otimes \eta \wedge \\ &\quad - \; \frac{(-1)^{a-b}}{a-b+1} (\mathrm{id} \otimes \mathrm{Pl}^*) \circ (h^\sharp \lrcorner \otimes \eta \wedge \otimes \mathrm{id})\end{aligned} \tag{2}$$

with some linear map Pl^* twin to Pl above. Now for a quaternionic manifold the decomposition of the exterior algebra is respected by some torsion free connection and consequently the de Rham complex of M gives rise both to a quotient complex and a subcomplex of the form:

$$0 \to \quad \mathbf{C} \quad \xrightarrow{\;d\;} \quad H \otimes \Lambda^{1,0} E^* \quad \xrightarrow{\;d\;} \ldots \xrightarrow{\;d\;} \mathrm{Sym}^{2n} H \otimes \Lambda^{2n,0} E^* \to 0$$

$$0 \leftarrow \Lambda^{2n,2n} E^* \xleftarrow{\;\delta\;} H \otimes \Lambda^{2n,2n-1} E^* \xleftarrow{\;\delta\;} \ldots \xleftarrow{\;\delta\;} \mathrm{Sym}^{2n} H \otimes \Lambda^{2n,0} E^* \leftarrow 0.$$

We note that $\Lambda^{q,0} E^* \cong \Lambda^q E^*$ are canonically isomorphic whereas the choice of an isomorphism $\Lambda^{2n,2n-q} E^* \cong \Lambda^q E \cong \Lambda^q E^*$ amounts to choosing a volume form and a metric respectively on M. Somewhat more general than the two complexes arising from the de Rham complexes are complexes of first order differential operators d and δ first defined by Salamon

$$d: \quad L^{-s} \otimes \mathrm{Sym}^{k+q} H \otimes \Lambda^{q,0} E^* \quad \longrightarrow L^{-s} \otimes \mathrm{Sym}^{k+q+1} H \otimes \Lambda^{q+1,0} E^* \tag{3}$$

$$\delta: \quad L^{s} \otimes \mathrm{Sym}^{k+q} H \otimes \Lambda^{2n,2n-q} E^* \longrightarrow L^{s} \otimes \mathrm{Sym}^{k+q-1} H \otimes \Lambda^{2n,2n-q+1} E^* \tag{4}$$

for all even $k \geq 0$ with $s := \frac{k}{2n+2}$, for odd k the bundles involved are in general ill–defined. The twist with the auxiliary line bundles L^s and L^{-s} is inserted to make the definition of the operators d and δ independent of the choice of a torsion free connection and can be ignored for any other purpose. On the quaternionic projective space $\mathbf{H}P^n$ the two complexes above arise from the Bernstein–Gelfand–Gelfand resolution of the irreducible representation $\mathrm{Sym}^k(H \oplus E)$ of $\mathbf{PGL}_{n+1}(\mathbf{H})$. In this sense the two complexes above are curved analogues of the Bernstein–Gelfand–Gelfand resolution [BS].

Leung and Yi studied the case $k = 0$ arising from the de Rham complex and proposed to choose a Riemannian metric adapted to the quaternionic structure in order to construct an isomorphism γ between these two complexes:

$$\begin{array}{ccccccc}
\mathbf{C} & \xrightarrow{\;d\;} & H \otimes \Lambda^{1,0} E^* & \xrightarrow{\;d\;} & \ldots & \xrightarrow{\;d\;} & \mathrm{Sym}^{2n} H \otimes \Lambda^{2n,0} E^* \\
{\scriptstyle\gamma}\big\downarrow & & {\scriptstyle\gamma}\big\downarrow & & {\scriptstyle\gamma}\big\downarrow & & {\scriptstyle\gamma}\big\downarrow \\
\Lambda^{2n,2n} E^* & \xleftarrow{\;\delta\;} & H \otimes \Lambda^{2n,2n-1} E^* & \xleftarrow{\;\delta\;} & \ldots & \xleftarrow{\;\delta\;} & \mathrm{Sym}^{2n} H \otimes \Lambda^{2n,0} E^*
\end{array}$$

Using this isomorphism they first defined the elliptic second order differential operator $\Delta := (d + \gamma^{-1}\delta\gamma)^2$ and then quaternionic analytic torsion as the torsion associated to this Laplacian. Note that $\gamma^{-1}\delta\gamma$ will never be the formal adjoint of d unless the isomorphism γ is parallel. However even if the isomorphism γ can be chosen to be parallel there remains the delicate problem as to its proper choice and the naive choice is certainly not the optimal one.

In order to analyze this problem we will restrict attention to quaternionic Kähler manifolds or in dimension 4 to half conformally flat Einstein manifolds. Recall that choosing a quaternionic Kähler metric is equivalent to choosing a positive, real section σ_E parallel for some torsion free connection compatible with the quaternionic structure. Evidently its highest power $\frac{1}{n!}\sigma_E^n$ defines a parallel trivialization of all the bundles $L^s,\ s \in \mathbf{R}$. Any natural choice for γ is parallel, too, and for appropriate choices of the Hermitian metrics on the bundles involved the operator $\gamma^{-1}\delta\gamma$ will be the formal adjoint of d as expected. Recall that the operators d and δ for $k = 0$ arise as quotient or subcomplexes of the de Rham complex on forms. In particular both d and δ are determined by their symbols $\sigma_d[\alpha\otimes\eta] : \ \mathrm{Sym}^{\,q}H\otimes\Lambda^{\,q}E^* \longrightarrow \mathrm{Sym}^{\,q+1}H\otimes\Lambda^{\,q+1}E^*$ and $\sigma_\delta[\alpha\otimes\eta] : \ \mathrm{Sym}^{\,q}H\otimes\Lambda^{\,2n,\,2n-q}E^* \longrightarrow \mathrm{Sym}^{\,q-1}H\otimes\Lambda^{\,2n,\,2n-q+1}E^*$ respectively, which are given by

$$\sigma_d[\alpha \otimes \eta] \ := \ \tfrac{1}{q+1}\alpha^\flat \cdot \otimes \eta\,\wedge \qquad\qquad \sigma_\delta[\alpha \otimes \eta] \ := \ (-1)^q \alpha\lrcorner \otimes \mathrm{id} \otimes \eta\wedge \qquad (5)$$

according to formula (2). In fact they are the composition of their symbol with the covariant derivate with respect to some adapted torsion free connection. Similarly the operators d and δ are defined simply by specifying their symbols $\sigma_d[\alpha\otimes\eta] : \ L^{-s}\otimes\mathrm{Sym}^{\,k+q}H\otimes\Lambda^{\,q}E^* \longrightarrow L^{-s}\otimes\mathrm{Sym}^{\,k+q+1}H\otimes\Lambda^{\,q+1}E^*$ and $\sigma_\delta[\alpha\otimes\eta] : L^s\otimes\mathrm{Sym}^{\,k+q}H\otimes\Lambda^{\,2n,2n-q}E^* \longrightarrow L^s\otimes\mathrm{Sym}^{\,k+q-1}H\otimes\Lambda^{\,2n,2n-q+1}E^*$ generalizing (5):

$$\sigma_d[\alpha\otimes\eta] \ := \ \tfrac{1}{k+q+1}\mathrm{id}\otimes\alpha^\flat\cdot\otimes\eta\wedge \qquad \sigma_\delta[\alpha\otimes\eta] \ := \ (-1)^q\mathrm{id}\otimes\alpha\lrcorner\otimes\mathrm{id}\otimes\eta\wedge\,. \quad (6)$$

Given now a Riemannian metric on M adapted to the quaternionic structure or equivalently a real, positive section σ_E of $\Lambda^{\,2}E^*$ we may naively choose γ to be the musical isomorphism $\flat : \ \Lambda^{\,q}E^* \longrightarrow \Lambda^{\,q}E \cong \Lambda^{\,2n,2n-q}E^*$. However the associated formal Laplacian $(d + \flat^{-1}\delta\flat)^2$ fails to have the right symbol to be properly called a Laplacian even for $k = 0$, in fact its symbol at an isotropic covector $\alpha \otimes \eta \in (H \otimes E)^*$ does not act trivially on $\mathrm{Sym}^{\,q}H \otimes \Lambda^{\,q}E^*$, $q > 0$:

$$\begin{aligned}\sigma_{(d+\flat^{-1}\delta\flat)^2}[\alpha \otimes \eta] \ &= \ \{\, \sigma_d[\alpha \otimes \eta],\, \sigma_{\flat^{-1}\delta\flat}[\alpha \otimes \eta]\,\} \\ &= \ -\,\tfrac{1}{q(q+1)}\,\alpha^\flat \cdot \alpha\lrcorner \otimes \eta^\flat \wedge \eta\lrcorner\end{aligned}$$

One way to understand this problem is to observe that the vector bundle $\Lambda^q E \cong \Lambda^q E^*$ involved is no longer irreducible under the holonomy group $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n) \subset \mathbf{Sp}(1) \cdot \mathbf{GL}_n(\mathbf{H})$ of the Levi–Civita connection of a quaternionic Kähler manifold but decomposes into parallel subbundles according to

$$\Lambda^q E^* \quad = \quad \bigoplus_{\substack{r=0 \\ r \equiv q\,(2)}}^{q \wedge (2n-q)} \Lambda^r_\circ E^* \,,$$

where the trace free exterior power $\Lambda^r_\circ E^*$ is the kernel of the contraction with the dual of the symplectic form. Consider now the spinor representation of $\mathbf{Sp}(1) \times \mathbf{Sp}(n)$ (cf. [BS], [W]):

$$\pi_{\not{S}} \quad = \quad \bigoplus_{r=0}^{n} \pi_{\not{S}_r} \quad := \quad \bigoplus_{r=0}^{n} \mathrm{Sym}^{\,n-r} \pi_H \otimes \Lambda^r_\circ \pi^*_E \,.$$

As noted in [S] there is a $\mathbf{Z}_2$–graded isomorphism of vector bundles

$$\begin{aligned} \bigoplus_{q=0}^{2n} \mathrm{Sym}^{\,k+q} H \otimes \Lambda^q E^* \;&\cong\; \bigoplus_{q=0}^{2n} \bigoplus_{\substack{r=0 \\ r \equiv q\,(2)}}^{q \wedge (2n-q)} \mathrm{Sym}^{\,k+q} H \otimes \Lambda^r_\circ E^* \\ &\cong\; \bigoplus_{r=0}^{n} \bigoplus_{\substack{q=r \\ q \equiv r\,(2)}}^{2n-r} \mathrm{Sym}^{\,k+q} H \otimes \Lambda^r_\circ E^* \;\cong\; \not{S} \otimes \mathrm{Sym}^{\,k+n} H \end{aligned}$$

and it is natural to ask whether the isomorphism γ we are looking for can be chosen in such a way that the operator $d + \gamma^{-1} \delta \gamma$ and the twisted Dirac operator on $\not{S} \otimes \mathrm{Sym}^{\,k+n} H$ are intertwined. A complete answer to that question involves the following technical lemma:

Lemma 2.1 *Consider the subspace* $\mathrm{Sym}^{\,k+q} H \otimes \Lambda^r_\circ E^*$ *of* $\mathrm{Sym}^{\,k+q} H \otimes \Lambda^q E^*$ *and set* $l := \frac{q-r}{2}$ *for convenience. The Clifford module structure of the twisted spinor bundle* $\bigoplus_q \mathrm{Sym}^{\,k+q} H \otimes \Lambda^q E^* \;\cong\; \not{S} \otimes \mathrm{Sym}^{\,k+n} H$ *gives rise to the following Clifford multiplication on this subspace:*

$$\begin{aligned} \tfrac{1}{\sqrt{2}} (h \otimes e) \bullet \quad = \quad & \tfrac{1}{k+q+1}\, h \cdot \otimes e^\sharp \wedge_\circ \quad - \quad \tfrac{1}{k+q+1}\, \tfrac{l+1}{n-r+1}\, h \cdot \otimes e \lrcorner \\ & + \tfrac{k+n+l+1}{k+q+1}\, h^\sharp \lrcorner \otimes e^\sharp \wedge_\circ \;+\; \tfrac{k+q-l}{k+q+1}\, \tfrac{n-r-l+1}{n-r+1}\, h^\sharp \lrcorner \otimes e \lrcorner \,. \end{aligned} \tag{7}$$

Naturally the Clifford multiplication is defined only up to conjugation by an $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$–equivariant isomorphism and this freedom allows to chose the first two constants more or less arbitrarily as long as a simple compatibility condition is met; the other two constants are fixed uniquely by this choice. For the time being we state the formula as it is with constants convenient to relate the operator $d + \gamma^{-1}\delta\gamma$ to a twisted Dirac. Of course it is only natural to be curious about a satisfactory explanation for the constants appearing in this formula (7).

Checking the Clifford relation for the Clifford multiplication (7) directly seems prohibitively difficult. However it is much easier to see that the anticommutator $\{(h \otimes e)\bullet, (\tilde{h} \otimes \tilde{e})\bullet\}$ maps $\mathrm{Sym}^{k+q}H \otimes \Lambda^r_\circ E$ to itself. We will give a brief sketch of this calculation before we proceed to the actual proof of Lemma 2.1 to vindicate formula (7) and to convince the reader that the constants above are much less arbitrary as they may seem at first glance.

Consider the components of the anticommutator $\{(h \otimes e)\bullet, (\tilde{h} \otimes \tilde{e})\bullet\}$ mapping $\mathrm{Sym}^{k+q}H \otimes \Lambda^r_\circ E^*$ to the various summands of $\not{S} \otimes \mathrm{Sym}^{k+n}H$. By definition $e^\sharp\wedge_\circ$ is the composition of $e\wedge$ with the projection to the trace free subspace $\Lambda^{r+1}_\circ E^*$ of $\Lambda^{r+1}E^*$, in particular $e^\sharp\wedge_\circ$ and $\tilde{e}^\sharp\wedge_\circ$ anticommute as do $e^\sharp\wedge$ and $\tilde{e}^\sharp\wedge$. Hence the components of the anticommutator mapping to $\mathrm{Sym}^{k+q\pm 2}H \otimes \Lambda^{r\pm 2}_\circ E^*$ certainly vanish. Moreover there is a fundamental identity on two dimensional symplectic vector spaces like H, namely $\sigma_H(h,a)\tilde{h} - \sigma_H(\tilde{h},a)h = \sigma_H(h,\tilde{h})a$ for all $h, \tilde{h}$ and $a \in H$, which implies the identity $h \cdot \tilde{h}^\sharp\lrcorner - \tilde{h} \cdot h^\sharp\lrcorner = (k+q)\sigma_H(h,\tilde{h})$ on $\mathrm{Sym}^{k+q}H$ or:

$$(k+q+2)\, h \cdot \tilde{h}^\sharp\lrcorner \;+\; (k+q)\, h^\sharp\lrcorner\tilde{h}\cdot \;\;=\;\; (k+q+1)\,\Big(\; h \cdot \tilde{h}^\sharp\lrcorner \;+\; \tilde{h} \cdot h^\sharp\lrcorner \;\Big)\;.$$

Using this identity the component of the composition $(h \otimes e) \bullet (\tilde{h} \otimes \tilde{e})\bullet$ mapping $\mathrm{Sym}^{k+q}H \otimes \Lambda^r_\circ E^*$ to $\mathrm{Sym}^{k+q}H \otimes \Lambda^{r+2}_\circ E^*$ can be written

$$\begin{aligned} &\tfrac{k+n+l+1}{k+q+1}\,\Big(\; \tfrac{1}{k+q}\, h \cdot \tilde{h}^\sharp\lrcorner \otimes e^\sharp \wedge_\circ \tilde{e}^\sharp \wedge_\circ \;+\; \tfrac{1}{k+q+2}\, h^\sharp\lrcorner\tilde{h} \cdot \otimes e^\sharp \wedge_\circ \tilde{e}^\sharp\wedge_\circ \;\Big) \\ &\qquad\qquad = \;\; \tfrac{k+n+l+1}{(k+q)(k+q+2)}\,\Big(\; h \cdot \tilde{h}^\sharp\lrcorner \;+\; \tilde{h} \cdot h^\sharp\lrcorner \;\Big) \otimes e^\sharp \wedge_\circ \tilde{e}^\sharp\wedge_\circ\,, \end{aligned}$$

which is skew in $h \otimes e$ and $\tilde{h} \otimes \tilde{e}$ and hence vanishes upon symmetrization. The same argument with a different leading constant shows that the anticommutator does not map $\mathrm{Sym}^{k+q}H \otimes \Lambda^r_\circ E^*$ to $\mathrm{Sym}^{k+q}H \otimes \Lambda^{r-2}_\circ E^*$ either. Completely analogous arguments replacing the fundamental identity of two dimensional symplectic vector spaces by $e\lrcorner\tilde{e}^\sharp \wedge_\circ + \tilde{e}^\sharp \wedge_\circ e\lrcorner = \sigma_E(e,\tilde{e}) + \frac{1}{n-r+1} e^\sharp \wedge_\circ \tilde{e}\lrcorner$ on $\Lambda^r E^*$ (cf. [KSW]), more usefully written as

$$\begin{aligned} &(n-r+1)\, e\lrcorner\tilde{e}^\sharp \wedge_\circ \;+\; (n-r)\, e^\sharp \wedge_\circ \tilde{e}\lrcorner \\ &\qquad = \;\; (n-r+1)\;\Big(\; \sigma_E(e,\tilde{e}) \;+\; e^\sharp \wedge_\circ \tilde{e}\lrcorner \;-\; \tilde{e}^\sharp \wedge_\circ e\lrcorner \;\Big) \end{aligned}$$

show that the components of the anticommutator $\{(h\otimes e)\bullet, (\tilde{h}\otimes\tilde{e})\bullet\}$ mapping to $\mathrm{Sym}^{k+q\pm 2}H \otimes \Lambda^r_\circ E^*$ vanish, too, consequently the anticommutator maps $\mathrm{Sym}^{k+q}H \otimes \Lambda^r_\circ E^*$ to itself as claimed.

Proof: Let us choose embeddings $\iota_{q,r} \;:\; \mathrm{Sym}^{k+q}H \longrightarrow \mathrm{Sym}^{n-r}H \otimes \mathrm{Sym}^{k+n}\mathrm{H}$ for all $r \leq q \leq 2n-r$ with $q \equiv r\ (2)$, which piece together to an isomorphism:

$$\iota \;:=\; \oplus(\iota_{q,r}\otimes \mathrm{id}) : \quad \bigoplus_{\substack{q\equiv r\,(2)\\ r\leq q\leq 2n-r}} \mathrm{Sym}^{k+q}H \otimes \Lambda^r_\circ E^* \quad\longrightarrow\quad \$ \otimes \mathrm{Sym}^{k+n}H\,.$$

The diagonal multiplication $\sigma\cdot : \mathrm{Sym}^s H\otimes \mathrm{Sym}^t H \longrightarrow \mathrm{Sym}^{s+1}H\otimes\mathrm{Sym}^{t+1}H$ with σ and the Plücker map $\mathrm{Pl} : \mathrm{Sym}^s H\otimes\mathrm{Sym}^t H \longrightarrow \mathrm{Sym}^{s+1}H\otimes\mathrm{Sym}^{t-1}H$ give in fact rise to an embedding

$$\iota_{q,r} \quad:=\quad \frac{1}{l!}\mathrm{Pl}^l\,\frac{1}{(n-r-l)!}\,(\sigma\cdot)^{n-r-l}$$

with $l := \frac{q-r}{2}$ and $\mathrm{Sym}^{k+q}H \;\cong\; \mathbf{C}\otimes\mathrm{Sym}^{k+q}H$. To make sense out of this expression we need to choose a pair $\{h_\mu\}$, $\{h^\vee_\nu\}$ of dual bases for H and H^* to fix $\sigma\cdot := \sum(h^\vee_\nu)^\flat\cdot\otimes h_\nu\cdot$ and $\mathrm{Pl} := \sum h_\nu\cdot\otimes h^\vee_\nu\lrcorner$ explicitly. It is not difficult to check that in terms of the embeddings $\iota_{q,r}$ the symmetric product with $h\in H$ or the contraction with $\alpha\in H^*$ in the first factor of $\mathrm{Sym}^{n-r}H\otimes\mathrm{Sym}^{k+n}H$ is expressed by the following formulas:

$$\begin{aligned}(h\cdot\otimes 1)\,(\iota_{q,r}\omega) &= \tfrac{l+1}{k+q+1}\,\iota_{q+1,r-1}(h\cdot\omega) \;+\; \tfrac{n-r-l+1}{k+q+1}\,\iota_{q-1,r-1}(h^\sharp\lrcorner\omega)\\ (\alpha\lrcorner\otimes 1)\,(\iota_{q,r}\omega) &= -\,\tfrac{k+q+1-l}{k+q+1}\,\iota_{q+1,r+1}(\alpha^\flat\cdot\omega) \;+\; \tfrac{k+n+l+1}{k+q+1}\,\iota_{q-1,r+1}(\alpha\lrcorner\omega)\,.\end{aligned}$$

According to the formula for the Clifford multiplication in the untwisted case $\$ \cong \bigoplus_r \mathrm{Sym}^{n-r}H\otimes\Lambda^r_\circ E^*$ given in [KSW] the twisted Clifford multiplication on $\mathrm{Sym}^{k+q}H\otimes\Lambda^r_\circ E^* \;\subset\; \$\otimes\mathrm{Sym}^{k+n}H$ becomes

$$\begin{aligned}\tfrac{1}{\sqrt{2}}(h\otimes e)\bullet &= \iota^{-1}\circ\big(h\cdot\otimes e\lrcorner\otimes\mathrm{id} \;-\; \tfrac{1}{n-r}h^\sharp\lrcorner\otimes e^\sharp\wedge_\circ\otimes\mathrm{id}\big)\circ\iota\\ &= \tfrac{l+1}{k+q+1}\,h\cdot\otimes e\lrcorner \;+\; \tfrac{n-r-l+1}{k+q+1}\,h^\sharp\lrcorner\otimes e\lrcorner\\ &\qquad +\,\tfrac{1}{n-r}\,\tfrac{k+q+1-l}{k+q+1}\,h\cdot\otimes e^\sharp\wedge_\circ \;-\; \tfrac{1}{n-r}\,\tfrac{k+n+l+1}{k+q+1}\,h^\sharp\lrcorner\otimes e^\sharp\wedge_\circ\end{aligned}$$

under the isomorphism ι, the change of sign in the first line is due to the fact that we are working with $\Lambda^r_\circ E^*$ instead of $\Lambda^r_\circ E$. Evidently this Clifford multiplication is conjugated to the Clifford multiplication stated in (7) under the $\mathbf{Sp}(1)\cdot\mathbf{Sp}(n)$–equivariant isomorphism of $\bigoplus\mathrm{Sym}^{k+q}H\otimes\Lambda^r_\circ E^*$, which is $(-1)^l(k+q-l)!(n-r)!$ on $\mathrm{Sym}^{k+q}H\otimes\Lambda^r_\circ E^*$. **Q.E.D.**

The straightforward embeddings $\Lambda^r_\circ E^* \longrightarrow \Lambda^q E^*$ and $\Lambda^r_\circ E^* \longrightarrow \Lambda^{2n,2n-q} E^*$ sending $\psi \in \Lambda^r_\circ E^*$ to $\frac{1}{l!}(\sigma_E\wedge)^l\psi$ and $\frac{1}{n!}\sigma_E^n \otimes \frac{1}{(n-r-l)!}(\sigma_E\wedge)^{n-r-l}\psi$ respectively translate the symbols of the operators d and δ given explicitly in (6) into the following maps on the subspace $\mathrm{Sym}^{k+q} H \otimes \Lambda^r_\circ E^*$ of $\mathrm{Sym}^{k+q} H \otimes \Lambda^q E^*$ and $\mathrm{Sym}^{k+q} H \otimes \Lambda^{2n,2n-q} E^*$:

$$\begin{array}{rcl} \sigma_d[h^\sharp \otimes \eta] & = & \frac{1}{k+q+1}\, h \cdot \otimes \eta \wedge_\circ \; - \; \frac{1}{k+q+1}\,\frac{l+1}{n-r+1}\, h \cdot \otimes \eta^\flat \lrcorner \\ (-1)^q \sigma_\delta[h^\sharp \otimes \eta] & = & h^\sharp \lrcorner \otimes \eta \wedge_\circ \; - \; \frac{n-r-l+1}{n-r+1}\, h^\sharp \lrcorner \otimes \eta^\flat \lrcorner \end{array}$$

Comparing this with the formula (7) for the Clifford multiplication we immediately deduce the following proposition which is the main result of this section:

Proposition 2.2 *Identify the spaces* $\mathrm{Sym}^{k+q} H \otimes \Lambda^r_\circ E^*$, $r \leq q \leq 2n-r$ *with subspaces of both* $\mathrm{Sym}^{k+q} H \otimes \Lambda^q E^*$ *and* $\mathrm{Sym}^{k+q} H \otimes \Lambda^{2n,2n-q} E^*$ *as above and consider the* $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$*–equivariant isomorphism*

$$\gamma : \quad \mathrm{Sym}^{k+q} H \otimes \Lambda^q E^* \;\longrightarrow\; \mathrm{Sym}^{k+q} H \otimes \Lambda^{2n,2n-q} E^*\,,$$

which is $(-1)^l \frac{(k+q-l)!(k+n+l+1)!}{(k+q+1)!}$ *on these subspaces. The differential operator*

$$D_{\mathrm{Sym}^{k+n} H} \quad := \quad \sqrt{2}\,(d + \gamma^{-1} \circ \delta \circ \gamma)$$

is the twisted Dirac operator on $\not{S} \otimes \mathrm{Sym}^{k+n} H \;\cong\; \bigoplus_q \mathrm{Sym}^{k+q} H \otimes \Lambda^q E^*$.

Moreover γ is uniquely characterized by this property up to an overall constant with respect to the present choice of the Clifford multiplication (7) and the symbols (6) of the operators d and δ. Other conventions simply conjugate d, δ and $D_{\mathrm{Sym}^{k+n} H}$ by $\mathbf{Sp}(1) \cdot \mathbf{GL}_n(\mathbf{H})$– and $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$–equivariant isomorphisms respectively leading essentially to the same conclusion but with appropriately conjugated γ. It is important however to note that the operator $D^2_{\mathrm{Sym}^{k+n} H}$ respects the decomposition of $\not{S} \otimes \mathrm{Sym}^{k+n} H$ into $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$–irreducible subspaces and is hence genuinely defined independent of all choices.

Since the operator $d + \gamma^{-1}\delta\gamma$ is a twisted Dirac operator on a quaternionic Kähler manifold the cohomology of the complexes (3) and (4) can be presented by harmonic twisted spinors. Quite a lot is known about the existence of harmonic spinors in this situation and consequently about the cohomology of these complexes ([NN], [SW]). In particular the d–complex is acyclic for all even $k \geq 0$ except in degree $q = 0$, if the scalar curvature $\kappa > 0$ is positive.

Moreover it is assumed that its cohomology in degree zero governs the classification of quaternionic Kähler manifolds with $\kappa > 0$. If the scalar curvature $\kappa < 0$ is negative, then the d–complex is acyclic except in degree $q = 2n$ (sic!) for all even $k > 0$, but for $k = 0$ it has trivial cohomology $\mathbf{C}$ in degree $q = 0$ and it may have exceptional cohomology in degrees $q = n, \ldots, 2n$. In the hyperkähler case $\kappa = 0$ the cohomology of the d–complex can be represented by holomorphic forms and thus faithfully reflects the decomposition of the manifold into irreducible factors.

For our calculations we are also interested in twisted versions of the complexes introduced above. However extra curvature terms arising from a twisting bundle $\mathcal{W}$ will spoil $d^2 = 0$ unless the curvature of $\mathcal{W}$ will be an antiselfdual two form, i. e. a section of $\mathrm{Sym}^2 E \otimes \mathrm{End}\, \mathcal{W} \subset \Lambda^2(TM \otimes_{\mathbf{R}} \mathbf{C}) \otimes \mathrm{End}\, \mathcal{W}$. Consequently we restrict ourselves to Hermitian vector bundles $\mathcal{W}$ with an antiselfdual Hermitian connection. Associated to such an antiselfdual bundle $\mathcal{W}$ and all even $k \geq 0$ are twisted versions

$$d_{\mathcal{W}} : \ \mathrm{Sym}^{k+q} H \otimes \Lambda^{q,0} E^* \otimes \mathcal{W} \ \longrightarrow \ \mathrm{Sym}^{k+q+1} H \otimes \Lambda^{q+1,0} E^* \otimes \mathcal{W}$$

and

$$\delta_{\mathcal{W}} : \ \mathrm{Sym}^{k+q} H \otimes \Lambda^{2n,2n-q} E^* \otimes \mathcal{W} \longrightarrow \mathrm{Sym}^{k+q-1} H \otimes \Lambda^{2n,2n-q+1} E^* \otimes \mathcal{W}$$

of the elliptic complexes considered above. The cohomology of the d–complex defines the quaternionic cohomology $H^{*,k}(M, \mathcal{W})$ of $\mathcal{W}$. Let $\pentagon_{q,k}$ denote the operator $\pentagon_k := (d_{\mathcal{W}} + \gamma^{-1} \delta_{\mathcal{W}} \gamma)^2$ restricted to $\mathrm{Sym}^{k+q} H \otimes \Lambda^{q,0} E^* \otimes \mathcal{W}$ with spectrum $\sigma(\pentagon_{q,k})$. The usual arguments of Hodge theory imply that quaternionic cohomology can be represented by harmonic sections $H^{q,k}(M, \mathcal{W}) \cong \ker \pentagon_{q,k}$. In the quaternionic Kähler case the twisted complexes are related to the Dolbeault complex of suitable holomorphic vector bundles on the twistor space via the Penrose transform, in particular the main motivation for studying these complexes arise from complex geometry ([NN],[MS]).

Consider an isometry g of the quaternionic Kähler manifold M, preserving the quaternionic structure (e. g. the identity). Assume furthermore an isometry of vector bundles $g^{\mathcal{W}} : \mathcal{W} \to g^* \mathcal{W}$. Then the quaternionic torsion is defined via the zeta function

$$Z_g(s) := \sum_{q=0}^{2n} (-1)^{q+1} q \sum_{\substack{\lambda \in \sigma(\pentagon_{q,k}) \\ \lambda \neq 0}} \lambda^{-s} \mathrm{Tr}\, g^*_{|\mathrm{Eig}_\lambda(\pentagon_{q,k})}$$

for $\mathrm{Re}\ s \gg 0$. This zeta function has a meromorphic continuation to the complex plane which is holomorphic at $s = 0$ by a general result by Donnelly ([Do]).

Definition 2.3 *The equivariant quaternionic analytic torsion is defined as*

$$T_g^k(M, \mathcal{W}) := Z_g'(0) \ .$$

Similarly, one can define an equivariant Quillen metric on the equivariant determinant of the quaternionic cohomology. Let g be an isometry of an hermitian vector space E. Let Θ denote the set of eigenvalues ζ of g with associated eigenspaces E_ζ. The g-**equivariant determinant** of E is defined as

$$\det_g E := \bigoplus_{\zeta \in \Theta} \det E_\zeta .$$

The g-**equivariant metric** associated to the metric on E is the map

$$\begin{array}{rcl} \log \| \cdot \|^2_{\det_g E} : \det_g E & \rightarrow & \mathbf{C} \\ (s_\zeta)_\zeta & \mapsto & \displaystyle\sum_{\zeta \in \Theta} \log \|s_\zeta\|^2_\zeta \cdot \zeta , \end{array}$$

where $\| \cdot \|^2_\zeta$ denotes the induced metric on $\det E_\zeta$. Now in our situation the isometry g induces an isometry g^* of the Dolbeault cohomology $H^{q,k}(M, \mathcal{W}) := \ker \pentagon_{q,k}$ equipped with the restriction of the L^2-metric.

Definition 2.4 *Set* $\lambda_g(M, \mathcal{W}) := [\det_g H^{q,k}(M, \mathcal{W})]^{-1}$. *The equivariant Quillen metric on* $\lambda_g(M, E)$ *is defined as*

$$\log \| \cdot \|^2_{Q, \lambda_g(M,E)} := \log \| \cdot \|^2_{L^2, \lambda_g(M, \mathcal{W})} - T_g^k(M, \mathcal{W}). \tag{8}$$

3 Quaternionic torsion for symmetric spaces

On a symmetric space G/K Partharasarty's formula relates the squares of twisted Dirac operators to the Casimirs of G and K. In consequence the operators $2\,(d_{\mathcal{W}} + \gamma^{-1} \delta_{\mathcal{W}} \gamma)^2$, which are squares of twisted Dirac operators on every quaternionic Kähler manifold, can be expressed in terms of the Casimirs of G and K on every quaternionic Kähler symmetric space. Recall that these two Casimirs induce an elliptic second order differential operator and a curvature operator respectively on every homogeneous vector bundle on G/K. Analogues of these two operators are defined in [SW] for all vector bundles associated to the holonomy bundle $\mathrm{Hol}(M)$ of an arbitrary Riemannian manifold M via representations of the holonomy group.

In fact the Levi–Civita connection ∇ of M defines an elliptic second order differential operator on every homogeneous vector bundle, namely the horizontal Laplacian $\nabla^*\nabla$. On a symmetric space G/K with metric induced by the Killing form B of G on $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the horizontal Laplacian is the "Casimir operator" of $\mathfrak{p}$ up to sign. It is more difficult to write down the analogue of the Casimir of K. Consider for this purpose a point p in a Riemannian manifold M with holonomy group $\mathrm{Hol}_p M \subset \mathbf{O}(T_p M)$ and holonomy algebra $\mathfrak{hol}_p M \subset \Lambda^2(T_p M)$. The completely contravariant curvature tensor R of M at p is by its very definition an element of the space $\mathrm{Sym}^2 \mathfrak{hol}_p M \subset \mathrm{Sym}^2 \Lambda^2 T_p^* M$ and thus the quantization map

$$q : \quad \mathrm{Sym}\ \mathfrak{hol}_p M \longrightarrow \mathcal{U}\ \mathfrak{hol}_p M, \quad \mathfrak{X}^l \longmapsto \mathfrak{X}^l$$

defines a curvature term $2\,q(R) \in \mathcal{U}\,\mathfrak{hol}_p M$ acting on every vector bundle on M associated to the holonomy bundle $\mathrm{Hol}(M)$. Straightforward computation shows that this curvature term reduces to the Casimir operator of K with respect to the restriction of the Killing form B to $\mathfrak{k}$ on every homogeneous vector bundle on G/K. Consequently the elliptic differential operator

$$\Delta \quad := \quad \nabla^*\nabla \; + \; 2\,q(R)$$

agrees with the Casimir operator of G on the symmetric space $M = G/K$. The operator Δ allows us to write the Bochner–Weitzenböck formula for a twisted Dirac operator $D_{\mathcal{R}}$ on a twisted spinor bundle $\not{S} \otimes \mathcal{R}$ associated to the holonomy bundle in the form:

$$D^2_{\mathcal{R}} \quad = \quad \Delta \; + \; \frac{\kappa}{8} \; - \; \mathrm{id}_{\not{S}} \otimes 2\,q(R) \tag{9}$$

where κ is the scalar curvature of M (cf. [SW]). We will employ this formula for tensor products $\mathcal{R} = \mathrm{Sym}^{k+n} H \otimes \mathcal{W}$ of $\mathrm{Sym}^{k+n} H$ with antiselfdual homogeneous vector bundles $\mathcal{W}$ on a quaternionic symmetric space G/K. Its isotropy group $K = \mathbf{Sp}(1) \cdot K' := (\mathbf{Sp}(1) \times K')/\mathbf{Z}_2$ splits almost into a direct product and by definition an antiselfdual homogeneous vector bundle $\mathcal{W}$ is associated to a representation on which $\mathbf{Sp}(1) \subset K$ acts trivially. Hence $\mathcal{W}$ is induced by a representation of $K_\circ := K'/\mathbf{Z}_2$. In this case formula (9) provides the following corollary to Proposition 2.2:

Corollary 3.1 *Let $\mathcal{W}$ be an antiselfdual homogeneous vector bundle on a quaternionic Kähler symmetric space G/K, i. e. the subgroup $\mathbf{Sp}(1) \subset K$ acts trivially on the corresponding representation of K. The square of the operator $d_{\mathcal{W}} + \gamma^{-1}\delta_{\mathcal{W}}\gamma$ for even $k \geq 0$ can be expressed as:*

$$(\,d_{\mathcal{W}} + \gamma^{-1}\,\delta_{\mathcal{W}}\,\gamma\,)^2 \quad = \quad \frac{1}{2}\left(\mathrm{Cas}_G - \frac{\kappa}{8}\,\frac{k(k+2n+2)}{n(n+2)} - \mathrm{Cas}_K^{\mathcal{W}}\right).$$

Proof: The algebraic relation between $d + \gamma^{-1}\,\delta\,\gamma$ and $D_{\mathrm{Sym}^{k+n}H}$ proved in Proposition 2.2 remains valid under arbitrary twists. In particular we may use the identification of $\not{S} \otimes \mathrm{Sym}^{k+n}H \otimes \mathcal{W}$ with $\bigoplus_q \mathrm{Sym}^{k+q}H \otimes \Lambda^q E^* \otimes \mathcal{W}$ to write the operator $(d_{\mathcal{W}} + \gamma^{-1}\,\delta_{\mathcal{W}}\gamma)$ as a twisted Dirac operator:

$$\sqrt{2}\,(\,d_{\mathcal{W}} \,+\, \gamma^{-1}\,\delta_{\mathcal{W}}\,\gamma\,) \quad = \quad D_{\mathrm{Sym}^{k+n}\otimes\mathcal{W}}$$

Equation (9) relates the operator $D^2_{\mathrm{Sym}^{k+n}H\otimes\mathcal{W}}$ to the Casimirs of G and K:

$$2\,(\,d_{\mathcal{W}} + \gamma^{-1}\,\delta_{\mathcal{W}}\,\gamma\,)^2 \quad = \quad D^2_{\mathrm{Sym}^{k+n}H\otimes\mathcal{W}} \quad = \quad \mathrm{Cas}_G + \frac{\kappa}{8} - \mathrm{Cas}_K^{\mathrm{Sym}^{k+n}H\otimes\mathcal{W}}$$

However the Lie algebra of K splits into commuting subalgebras $\mathfrak{k} = \mathfrak{sp}(1) \oplus \mathfrak{k}_\circ$ and $\mathfrak{sp}(1)$ acts trivially on the representation corresponding to $\mathcal{W}$ by assumption whereas $\mathfrak{k}_\circ$ acts trivially on the representation corresponding to $\mathrm{Sym}^{k+n}H$. Hence the Casimir of K on $\mathrm{Sym}^{k+n}H \otimes \mathcal{W}$ is the sum:

$$\mathrm{Cas}_K^{\mathrm{Sym}^{k+n}H\otimes\mathcal{W}} \quad = \quad \frac{\kappa}{8}\,\frac{(k+n)(k+n+2)}{n(n+2)} \;+\; 1 \otimes \mathrm{Cas}_K^{\mathcal{W}}\,.$$

In fact the Casimir of $\mathrm{Sym}^{k+n}H$ is proportional to $(k+n)(k+n+2)$ and necessarily equals $\frac{\kappa}{8}$ for $k = 0$, because $\not{S} \otimes \mathrm{Sym}^n H$ occurs in the forms. **Q.E.D.**

Fix a maximal torus T of $K = \mathbf{Sp}(1) \cdot K'$ containing a maximal torus of $\mathbf{Sp}(1)$. T is automatically a maximal torus of G. As T contains a maximal torus of $\mathbf{Sp}(1)$ the subalgebra $\mathfrak{sp}(1)$ is invariant under T and we denote its weights by -2α, 0, 2α. Under the action of $\mathfrak{sp}(1)$ the Lie algebra of $\mathfrak{g}$ splits into $\mathfrak{g}^{-2\alpha} \oplus \mathfrak{g}^{-\alpha} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^\alpha \oplus \mathfrak{g}^{2\alpha}$. We choose an ordering of the roots of G such that 2α is the highest root and the weights of $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{2\alpha}$ are positive. In particular α is the positive weight of H. We denote the set of positive roots by Σ^+.

Set $\mathfrak{t}_{\mathrm{reg}} := \{X \in \mathfrak{t} | \beta(X) \notin \mathbf{Z}\ \forall \beta \in \Sigma\}$. For $X \in \mathfrak{t}$ let $e^X \in T$ denote the associated group element. Let ρ denote half the sum of the positive weights of G and define similarly ρ_K etc. Let W_G, W_K etc. denote the Weyl groups. Set for $b \in \mathfrak{t}^*$

$$\mathrm{Alt}_G\{b\} := \sum_{w \in W_G} \mathrm{sign}(w) e^{2\pi i w b}\,.$$

We denote the G-representation with highest weight λ by $V^G_{\rho+\lambda}$ and its character is denoted by $\chi_{\rho+\lambda}$. In general, for a weight λ and $X \in \mathfrak{t}_{\mathrm{reg}}$ we define $\chi_{\rho+\lambda}$ by the Weyl character formula

$$\chi_{\rho+\lambda}(e^X) := \frac{\mathrm{Alt}_G\{\rho+\lambda\}(X)}{\mathrm{Alt}_G\{\rho\}(X)}$$

with $\mathrm{Alt}_G\{\rho\}(X) = \prod_{\beta\in\Sigma^+} 2i\sin\pi\beta(X)$. For an irreducible representation π, we shall denote the sum of ρ and the highest weight by b_π. Thus the Casimir acting on V_π is given by $\|b_\pi\|^2 - \|\rho\|^2$.

An irreducible K-representation $V^K_{\rho_K+\lambda}$ induces a G-invariant vector bundle $\mathcal{W}^K_{\rho_K+\lambda}$ on M. As $V^K_{\rho_K+\lambda}$ carries a K-invariant Hermitian metric which is unique up to a factor, we get corresponding G-invariant metrics on $\mathcal{W}^K_{\rho_K+\lambda}$. Consider a $K_\circ$-representation $V^{K_\circ}_{\rho_{K_\circ}+\lambda_\circ}$ of highest weight $\lambda_\circ$ and the induced equivariant bundle $\mathcal{W}$ on the quaternionic Kähler symmetric space G/K. Set $\lambda := \lambda_\circ + k\alpha$. By Corollary 3.1, the zeta function defining the torsion $T^k(M,\mathcal{W})$ of $\mathcal{W}$ equals

$$\begin{aligned} Z(s) &= \sum_{q=1}^{2n}(-1)^{q+1}q\sum_{\pi \text{ irr.}}\left(\frac{2}{\|b_\pi\|^2-\|\rho+\lambda\|^2}\right)^s \\ &\quad \cdot\chi_{b_\pi}\dim\mathrm{Hom}_K(V_\pi, \Lambda^q E\otimes \mathrm{Sym}^{\,k+q}H\otimes V^{K_\circ}_{\rho_{K_\circ}+\lambda_\circ})\ . \end{aligned}$$

Let Θ^E denote the representation of K on E and let Ψ_0 denote its weights. Analogously to [K2, Lemma4] and [K3, Lemma7] we show

Lemma 3.2 *Let G/K be a n-dimensional quaternionic Kähler symmetric space. For any irreducible G-representation (V_π,π) the sum*

$$\sum_{q=1}^{2n}(-1)^q q\dim\mathrm{Hom}_K(V_\pi,\Lambda^q E\otimes\mathrm{Sym}^{\,k+q}H\otimes V^{K_\circ}_{\rho_{K_\circ}+\lambda_\circ})$$

equals the sum of $-\chi_{\rho+\lambda+\ell(\alpha+\beta)}$ over those $\ell\in\mathbf{N}$, $\beta\in\Psi_0$ such that b_π is in the W_G-orbit of $\rho+\lambda+\ell(\alpha+\beta)$.

Proof: Let χ^K denote the virtual K-character

$$\chi^K := \sum_{q=1}^{2n}(-1)^q q\chi(\Lambda^q E\otimes\mathrm{Sym}^{\,k+q}H)\ .$$

Notice that

$$\chi(\mathrm{Sym}^{\,k+q}H) = \frac{e^{2\pi i(k+q+1)\alpha}-e^{-2\pi i(k+q+1)\alpha}}{e^{2\pi i\alpha}-e^{-2\pi i\alpha}}$$

and, for $s\in\mathbf{R}$,

$$\sum_{q=0}^{2n}(-s)^q\chi(\Lambda^q E) = \det(1-s\Theta^E)\ .$$

Hence

$$\begin{aligned}
\chi_s &:= \sum_{q=0}^{2n} (-s)^q \chi(\Lambda^q E \otimes \mathrm{Sym}^{k+q} H) \\
&= \frac{1}{e^{2\pi i\alpha} - e^{-2\pi i\alpha}} \left[e^{2\pi i(k+1)\alpha} \det(1 - s e^{2\pi i\alpha}\Theta^E) - e^{-2\pi i(k+1)\alpha} \det(1 - s e^{-2\pi i\alpha}\Theta^E) \right]
\end{aligned}$$

and

$$\begin{aligned}
\chi^K &= \frac{\partial}{\partial s}_{|s=1} \chi_s = \frac{e^{2\pi i(k+1)\alpha}}{e^{2\pi i\alpha} - e^{-2\pi i\alpha}} \det(1 - e^{2\pi i\alpha}\Theta^E) \mathrm{Tr}\, \left[(1 - e^{-2\pi i\alpha}(\Theta^E)^{-1})^{-1} \right] \\
&\quad - \frac{e^{-2\pi i(k+1)\alpha}}{e^{2\pi i\alpha} - e^{-2\pi i\alpha}} \det(1 - e^{-2\pi i\alpha}\Theta^E) \mathrm{Tr}\, \left[(1 - e^{2\pi i\alpha}(\Theta^E)^{-1})^{-1} \right] \\
&= \frac{1}{e^{2\pi i\alpha} - e^{-2\pi i\alpha}} \prod_{\beta\in\Psi_0, \beta>0} \left(e^{\pi i(\alpha+\beta)} - e^{-\pi i(\alpha+\beta)} \right) \left(e^{\pi i(\alpha-\beta)} - e^{-\pi i(\alpha-\beta)} \right) \\
&\quad \cdot \Big[e^{2\pi i(k+n+1)\alpha} \sum_{\beta\in\Psi_0, \beta>0} \Big(\frac{1}{1 - e^{2\pi i(-\alpha+\beta)}} + \frac{1}{1 - e^{2\pi i(-\alpha-\beta)}} \Big) \\
&\quad - e^{-2\pi i(k+n+1)\alpha} \sum_{\beta\in\Psi_0, \beta>0} \Big(\frac{1}{1 - e^{2\pi i(\alpha+\beta)}} + \frac{1}{1 - e^{2\pi i(\alpha-\beta)}} \Big) \Big] \\
&= \frac{\overline{\mathrm{Alt}_G\{\rho\}}}{\overline{\mathrm{Alt}_K\{\rho_K\}}} \cdot \frac{1}{e^{2\pi i\alpha} - e^{-2\pi i\alpha}} \\
&\quad \cdot \Big[e^{2\pi i(k+n+1)\alpha} \sum_{\beta\in\Psi_0, \beta>0} \Big(\frac{1}{1 - e^{2\pi i(-\alpha+\beta)}} + \frac{1}{1 - e^{2\pi i(-\alpha-\beta)}} \Big) \\
&\quad - e^{-2\pi i(k+n+1)\alpha} \sum_{\beta\in\Psi_0, \beta>0} \Big(\frac{1}{1 - e^{2\pi i(\alpha+\beta)}} + \frac{1}{1 - e^{2\pi i(\alpha-\beta)}} \Big) \Big] .
\end{aligned}$$

Notice that $\Sigma_G^+ \setminus \Sigma_K^+ = \{\alpha \pm \beta | \beta \in \Psi_0, \beta > 0\}$ and $\Sigma_K^+ \setminus \Sigma_{K_\circ}^+ = \{2\alpha\}$; also, $W_K = W_{K_\circ} \times W_{\mathbf{Sp}(1)}$. In particular, $\rho - \rho_{K_\circ} = (n+1)\alpha$. Thus

$$\begin{aligned}
&\overline{\mathrm{Alt}_K\{\rho_K\}} \mathrm{Alt}_K\{\rho_K\} \overline{\chi^K \cdot \chi^{K_\circ}_{\rho_{K_\circ}+\lambda_\circ}} \\
&\quad = \mathrm{Alt}_G\{\rho\} \overline{\mathrm{Alt}_{K_\circ}\{\rho_{K_\circ} + \lambda_\circ\}} \sum_{\substack{\beta\in\Psi_0 \\ w\in W_{\mathbf{Sp}(1)}}} \mathrm{sign}(w) \frac{e^{-2\pi i w(\rho - \rho_{K_\circ} + k\alpha)}}{1 - e^{2\pi i w(\alpha+\beta)}} .
\end{aligned}$$

As in [K2, eq. (29)] we obtain for large $N \in \mathbf{N}$

$$\begin{aligned}
&\dim \mathrm{Hom}_K(V_\pi, \Lambda^q E \otimes \mathrm{Sym}^{k+q} H \otimes V^{K_\circ}_{\rho_{K_\circ}+\lambda_\circ}) \\
&\quad = \frac{1}{\# W_K} \int_T \overline{\mathrm{Alt}_K\{\rho_K\}} \mathrm{Alt}_K\{\rho_K\} \overline{\chi_K \cdot \chi^{K_\circ}_{\rho_{K_\circ}+\lambda_\circ}} \chi_\pi \, \mathrm{dvol}_T
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-1}{\#W_K}\int_T \mathrm{Alt}_G\{\rho\}\overline{\mathrm{Alt}_{K_\circ}\{\rho_{K_\circ}+\lambda_\circ\}}\\
&\quad\cdot \sum_{\substack{\beta\in\Psi_0\\ w\in W_{\mathbf{Sp}(1)}}} \mathrm{sign}(w)\frac{e^{-2\pi i w(\rho-\rho_{K_\circ}+k\alpha)}\left(e^{-2\pi i w(\alpha+\beta)}-e^{-2\pi i N w(\alpha+\beta)}\right)}{1-e^{-2\pi i w(\alpha+\beta)}}\chi_\pi\,\mathrm{dvol}_T\\
&= \frac{-1}{\#W_K}\sum_{\ell=1}^{N-1}\int_T \mathrm{Alt}_G\{b_\pi\}\overline{\mathrm{Alt}_{K_\circ}\{\rho_{K_\circ}+\lambda_\circ\}}\\
&\quad\cdot \sum_{\substack{\beta\in\Psi_0\\ w\in W_{\mathbf{Sp}(1)}}} \mathrm{sign}(w)e^{-2\pi i w(\rho-\rho_{K_\circ}+k\alpha+\ell(\alpha+\beta))}\,\mathrm{dvol}_T\\
&= \frac{-1}{\#W_K}\sum_{\ell=1}^{N-1}\sum_{\beta\in\Psi_0}\int_T \mathrm{Alt}_G\{b_\pi\}\overline{\mathrm{Alt}_K\{\rho+\lambda+\ell(\alpha+\beta)\}}\,\mathrm{dvol}_T\ .
\end{aligned}
$$

This proves the Lemma the same way as in [K2, p. 100]. **Q.E.D.**

Set $\Psi_0^+ := \{\beta\in\Psi_0 | \langle(\alpha+\beta)^\vee, \rho+\lambda\rangle \geq 0\}$ and $\Psi_0^- := \{\beta\in\Psi_0 | \langle(\alpha+\beta)^\vee, \rho+\lambda\rangle < 0\}$ with $\beta^\vee = 2\beta/\|\beta\|^2$. By Lemma 3.2 and Theorem 3.1 we find that $Z(s)$ is given by the following formula:

Theorem 3.3 *For G/K quaternionic Kähler, the zeta function Z equals*

$$
\begin{aligned}
Z(s) \;=\; & -2^s \sum_{\beta\in\Psi_0^+}\sum_{\ell>\langle(\alpha+\beta)^\vee,\rho+\lambda\rangle} \frac{\chi_{\rho+\lambda+\ell(\alpha+\beta)}}{\langle 2\rho+2\lambda+\ell(\alpha+\beta), k(\alpha+\beta)\rangle^s}\\
& +2^s \sum_{\beta\in\Psi_0^-}\sum_{\ell>-\langle(\alpha+\beta)^\vee,\rho+\lambda\rangle} \frac{\chi_{\rho+\lambda+\ell(\alpha+\beta)}}{\langle 2\rho+2\lambda+\ell(\alpha+\beta), \ell(\alpha+\beta)\rangle^s}\ .
\end{aligned}
$$

This is a zeta function of the form considered in [K2, Lemma 8]. It is actually the very same formula as in [KK, Prop. 5.1] (see also [K2, Theorem 5]), the only difference being that we consider a different kind of symmetric space there. Define for $\phi\in\mathbf{R}$ and $\mathrm{Re}\, s>1$

$$\zeta_L(s,\phi) = \sum_{\ell=1}^{\infty}\frac{e^{i\ell\phi}}{k^s}\ . \qquad (10)$$

The function ζ_L has a meromorphic continuation to the complex plane in s which is holomorphic for $s\neq 1$. Set $\zeta_L'(s,\phi) := \partial/\partial s(\zeta_L(s,\phi))$. Let $P:\mathbf{Z}\to\mathbf{C}$ be a function of the form

$$P(\ell) = \sum_{j=0}^{m} c_j \ell^{n_j} e^{i\ell\phi_j} \qquad (11)$$

with $m \in \mathbf{N}_0$, $n_j \in \mathbf{N}_0$, $c_j \in \mathbf{C}$, $\phi_j \in \mathbf{R}$ for all j. We define $P^{\text{odd}}(\ell) := (P(\ell) - P(-\ell))/2$. Also we define as in [K2, Section 6]

$$\begin{aligned}
\boldsymbol{\zeta} P &:= \sum_{j=0}^{m} c_j \zeta_L(-n_j, \phi_j), & (12)\\
\boldsymbol{\zeta'} P &:= \sum_{j=0}^{m} c_j \zeta'_L(-n_j, \phi_j), & (13)\\
\text{and} \qquad P^*(p) &:= -\sum_{\substack{j=0\\ \phi_j \equiv 0 \mod 2\pi}}^{m} c_j \frac{p^{n_j+1}}{4(n_j+1)} \sum_{\ell=1}^{n_j} \frac{1}{\ell} & (14)
\end{aligned}$$

for $p \in \mathbf{R}$.

Then by [K2, Lemma 8] we get the same formula as in [KK, Theorem 5.2] (compare also [K2, Theorem 9]):

Theorem 3.4 *Let G/K be a quaternionic Kähler symmetric space. The equivariant analytic torsion of $\overline{\mathcal{W}}$ on G/K is given by*

$$\begin{aligned}
T^k(G/K, \overline{\mathcal{W}}) &= -2 \sum_{\beta \in \Psi_0} \boldsymbol{\zeta'} \chi^{\text{odd}}_{\rho+\lambda-\ell(\alpha+\beta)} - 2 \sum_{\beta \in \Psi_0} \chi^*_{\rho+\lambda-\ell(\alpha+\beta)}(\langle(\alpha+\beta)^\vee, \rho+\lambda\rangle)\\
&- \sum_{\beta \in \Psi_0} \boldsymbol{\zeta} \chi_{\rho+\lambda-\ell(\alpha+\beta)} \cdot \log \frac{2}{\|\alpha\|^2 + \|\beta\|^2} - \chi_{\rho+\lambda} \sum_{\beta \in \Psi_0^+} \log \frac{2}{\|\alpha\|^2 + \|\beta\|^2}\\
&- \sum_{\beta \in \Psi_0^+} \sum_{\ell=1}^{\langle(\alpha+\beta)^\vee, \rho+\lambda\rangle} \chi_{\rho+\lambda-\ell(\alpha+\beta)} \cdot \log \ell + \sum_{\beta \in \Psi_0^-} \sum_{\ell=1}^{\langle-(\alpha+\beta)^\vee, \rho+\lambda\rangle} \chi_{\rho+\lambda+\ell(\alpha+\beta)} \cdot \log \ell\ .
\end{aligned}$$

4 The hyperkähler case

Assume that M is a hyperkähler manifold. Then choosing a subordinate complex structure is equivalent to fixing isomorphisms $T^{(1,0)}M = T^{(0,1)}M = E$ and $H = \mathcal{O} \oplus \mathcal{O}$. The antiselfdual two forms on a hyperkähler manifold can be identified with the forms of bidegree $(1,1)$ for any subordinated complex structure, so that antiselfdual vector bundles are exactly the vector bundles which are holomorphic with respect all these complex structures. In this case the twisted d–complex reads

$$\cdots \xrightarrow{\bar\partial} \Lambda^q T^{*(0,1)}M \otimes \mathcal{O}^{\oplus(k+q+1)} \otimes \mathcal{W} \xrightarrow{\bar\partial} \cdots\ ,$$

i. e. it is essentially equivalent to the twisted Dolbeault complex up to the trivial factor $\mathrm{Sym}^{k+q} H \cong \mathcal{O}^{\oplus(k+q+1)}$, which is needed to make this complex independent of the choice of a complex structure.

Let $\Box_q := (\bar\partial + \bar\partial^*)^2$ denote the Kodaira-Laplace operator acting on

$$\Gamma^\infty(M, \Lambda^q T^{*(0,1)} M \otimes \mathcal{W}) \ .$$

Let $P^\perp$ be the projection of this space to the orthogonal complement of ker $\Box_q$ and define

$$\zeta_q(s) := \mathrm{Tr}\,(\Box_q^{-s} P^\perp)$$

for Re $s \gg 0$. Then the quaternionic torsion equals

$$T^k(M, \mathcal{W}) = \sum_{q=0}^{2n} (-1)^{q+1} q(q+k+1)\zeta_q'(0) = T^0(M, \mathcal{W}) + kT_{\bar\partial}(M, \mathcal{W}) \quad (15)$$

with $T_{\bar\partial}$ denoting the holomorphic torsion. Now let $K_X := \Lambda^m T^* X$ denote the canonical line bundle on an m-dimensional compact Kähler manifold X. Let $\mathcal{W}$ denote a holomorphic Hermitian vector bundle on X. In [GS5, Th. 1.4] it was shown that $\zeta_q'(0) = \zeta_{m-q}'(0)$ and thus

$$T_{\bar\partial}(X, \mathcal{W}) = (-1)^{m+1} T_{\bar\partial}(X, \mathcal{W}^* \otimes K_X) \ .$$

In particular, if X is even-dimensional and spin and $K_X^{1/2}$ denotes a chosen square root of K_X, then $T_{\bar\partial}(X, K_X^{1/2})$ vanishes. This statement takes a particularly nice form if $K_X \cong \mathcal{O}$ as holomorphic hermitian bundles, i.e. for Calabi-Yau manifolds equipped with the Kähler-Einstein metric. This has been noticed first by J.-B. Bost and J.-M. Bismut.

In our case, this implies that for any bundle $\mathcal{W}$ with $\mathcal{W} = \mathcal{W}^*$, $T^k(M, \mathcal{W})$ is independent of k. In particular this holds for $\mathcal{W} = \mathcal{O}$. Furthermore for any holomorphic Hermitian bundle $\mathcal{W}$

$$\begin{aligned} T^k(M, \mathcal{W}) &= \sum_{q=0}^{2n} (-1)^{2n-q+1}(2n-q)(2n-q+k+1)\zeta_q'(0)(M, \mathcal{W}^*) \\ &= \sum_{q=0}^{2n} (-1)^{q+1} q(q-(4n+k+1))\zeta_q'(0)(M, \mathcal{W}^*) \end{aligned}$$

which can be interpreted as "$T^k(M, \mathcal{W}) = T^{-4n-k-1}(M, \mathcal{W}^*)$" (we did not define the latter). The quadratic term in q thus does not provide vanishing results for the quaternionic torsion in contrast to the holomorphic case.

Example: Consider a $2n$-dimensional Hermitian vector space V and a lattice $\Lambda \subset V$ of maximal rank. Let $\Lambda^\vee$ denote the dual of Λ and let $M := V/\Lambda$ be the associated flat torus. Then, as holomorphic Hermitian bundles,

$$\Lambda^q T^{*(0,1)} M \cong \mathcal{O}^{\oplus \binom{2n}{q}}$$

and the zeta function defining $T^k(M, \mathcal{O})$ equals

$$\begin{aligned}
\sum_{q=0}^{2n} (-1)^q q(q+k+1) \mathrm{Tr}\,(\Box_q^{-s} P^\perp) &= \frac{\partial^2}{\partial x^2}_{|x=1} \left(\sum_{q=0}^{2n} (-1)^q x^{q+1} \binom{2n}{q} \right) \cdot \mathrm{Tr}\,(\Box_q^{-s} P^\perp) \\
&= \frac{\partial^2}{\partial x^2}_{|x=1} \left(x(1-x)^{2n} \right) \cdot \mathrm{Tr}\,(\Box_q^{-s} P^\perp) \\
&= \begin{cases} 0 & \text{if } n > 1 \\ \sum_{\mu \in \Lambda^\vee \setminus \{0\}} \|\mu\|^{-2s} & \text{if } n = 1 \,. \end{cases}
\end{aligned}$$

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