Géométrie différentielle/*Differential geometry*

Holomorphic torsion on Hermitian symmetric spaces

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Abstract – We calculate explicitly the equivariant holomorphic Ray-Singer torsion for all equivariant Hermitian vector bundles over Hermitian symmetric spaces G/K with respect to any isometry $g \in G$. In particular, we obtain the value of the usual non-equivariant torsion. The result is shown to provide very strong support for Bismut's conjecture of an equivariant arithmetic Grothendieck-Riemann-Roch theorem.

Torsion holomorphe pour des espaces hermitiens symétriques

Résumé – On calcule explicitement la torsion équivariante holomorphe de Ray-Singer pour tous les fibrés vectoriels hermitiens équivariants sur les espaces hermitiens symétriques compactes G/K relativement à chaque isométrie $g \in G$. En particulier on obtient la valeur de la torsion non-équivariante. Le resultat va dans le sens de la conjecture de Bismut d'un théorème de Grothendieck-Riemann-Roch arithmétique equivariant.

Version française abrégée – Soit E un fibré holomorphe hermitien sur une variété complexe compacte M. Soit $\square_q := (\bar\partial + \bar\partial^*)^2$ l'opérateur de Laplace-Kodaira agissant sur $\Gamma(\Lambda^q T^{*0,1}M \otimes E)$. Soit g une isométrie holomorphe de M et supposons que le fibré hermitien soit invariant par l'action de g. Soit $\tau_g(M,E)$ la torsion équivariante comme définie dans [8]. Elle est donnée par la dérivée en zéro d'une certaine fonction zêta associée au spectre de $\square$ et à l'action de g sur les espaces propres de $\square$. La torsion joue un rôle crucial dans la définition d'une image directe dans la K-théorie arithmétique de Gillet-Soulé.

Considérons un groupe de Lie compact et semi-simple G. Soit G/K un espace hermitien symétrique equipé d'une métrique G-invariante $\langle\cdot,\cdot\rangle_\diamond$. Soit $T \subseteq K$ un tore maximal et Ψ un système de racines d'une structure complexe invariante de G/K dans le sens de [4]. Soit ρ_G la demi-somme des racines positives de G. On pose $(\alpha, \rho_G) := 2\langle\alpha,\rho_G\rangle_\diamond/\|\alpha\|_\diamond^2$ pour chaque poids α. Soit χ_α le caractère virtuel associé à α.

Choisissons une représentation irréductible V de K du poids maximal Λ. Soit $E := (G\times V)/K$ le fibré vectoriel associé. Pour exprimer la torsion de E, il faut d'abord établir quelques notations. Soit $P : \mathbf{Z} \to \mathbf{C}$ une fonction du type $P(k) = \sum_{j=0}^m c_j k^{n_j} e^{ik\phi_j}$ où $m, n_j \in \mathbf{N}_0$, $c_j \in \mathbf{C}$, $\phi_j \in \mathbf{R}$. Soit ζ_L la fonction zêta de Lerch. On pose $P^{\rm odd}(k) := (P(k) - P(-k))/2$ et

$$\boldsymbol{\zeta} P := \sum_{j=0}^m c_j \zeta_L(-n_j, \phi_j), \qquad \boldsymbol{\zeta'} P := \sum_{j=0}^m c_j \zeta_L'(-n_j, \phi_j)$$

$$\text{et} \quad P^*(p) := -\sum_{\substack{j=0 \\ \phi_j \equiv 0 \mod 2\pi}}^m c_j \frac{p^{n_j+1}}{4(n_j+1)} \sum_{\ell=1}^{n_j} \frac{1}{\ell}.$$

Alors on obtient le résultat suivant par des méthodes similaires à [8]

Theorem 1 ([9]) *Le logarithme de la torsion équivariante de E sur G/K est donné par*

$$\log \tau_g = \boldsymbol{\zeta}' \sum_{\alpha\in\Psi} \chi^{\mathrm{odd}}_{\rho_G+\Lambda+k\alpha}(g) - \sum_{\alpha\in\Psi} \chi_{\rho_G+\Lambda-k\alpha}(g)^* \left((\alpha, \rho_G+\Lambda)\right)$$
$$+\frac{1}{2}\sum_{\alpha\in\Psi}\sum_{k=1}^{(\alpha,\rho_G+\Lambda)} \chi_{\rho_G+\Lambda-k\alpha}(g)\log k + \frac{1}{2}\sum_{\alpha\in\Psi}\left(\frac{1}{2} - \boldsymbol{\zeta}\chi^{\mathrm{odd}}_{\rho_G+\Lambda+k\alpha}(g)\right)\log\frac{\|\alpha\|^2_\diamond}{2}$$

pour $g \in G$.

Ce resultat correspond très bien à la conjecture d'un théorème de Grothendieck-Riemann-Roch arithmétique équivariante de Bismut [2].

The Ray-Singer analytic torsion is a positive real number associated to the spectrum of the Kodaira-Laplacian on Hermitian vector bundles over compact Hermitian manifolds [10]. It was shown by Quillen, Bismut, Gillet and Soulé that the torsion provides a metric with very beautiful properties on the determinant line bundle of direct images in K-theory over Kähler manifolds.

The main application of this construction is related to arithmetic geometry. Extending ideas of Arakelov, Gillet and Soulé constructed for arithmetic varieties $\mathcal{X}$ (i.e. flat regular quasi-projective schemes over Spec $\mathbf{Z}$ with projectiv fibre $\mathcal{X}_{\mathbf{Q}}$ over the generic point) a Chow intersection ring and a K-theory by using differential geometric objects on the Kähler manifold $X := \mathcal{X} \otimes \mathbf{C}$ [11]. In particular, the K-theory consists of arithmetic vector bundles on $\mathcal{X}$ with Hermitian metric over X and certain classes of differential forms. Using the torsion as part of a direct image, Bismut, Lebeau, Gillet and Soulé were able to prove an arithmetic Grothendieck-Riemann-Roch theorem relating the determinant of the direct image in the K-theory to the direct image in the arithmetic Chow ring. For a generalization of these concepts to higher degrees, see Bismut-Köhler [3] and Faltings [5].

One important step in the proof of the theorem was its explicit verification for the canonical projection of the projective spaces to Spec $\mathbf{Z}$ by Gillet, Soulé and Zagier [6]. In particular, the Gillet-Soulé R-genus, a rather complicated characteristic class occuring in the theorem was determined this way. The discovery of the same genus in a completely different calculation of secondary characteristic classes associated to short exact sequences by Bismut gave further evidence for the theorem.

In [8], an equivariant version of the analytic torsion was introduced and calculated for rotations with isolated fixed points of complex projective spaces. The result led Bismut to conjecture an equivariant arithmetic Grothendieck-Riemann-Roch formula [2]. Redoing his calculations concerning short exact sequences, he found an equivariant characteristic class R which equals the Gillet-Soulé R-genus in the non-equivariant case and the function R^{rot} in the case of isolated fixed points. In [1], he was able to show the compatibility of his conjecture with immersions.

In this note, we give the equivariant torsion for all compact Hermitian symmetric spaces G/K with respect to the action of any $g \in G$ as calculated in [9]. The result is of interest also in the non-equivariant case: The torsion was known only for very few manifolds; the projective spaces, the elliptic curves and the tori of dimension > 2 (for which it is zero for elementary reasons). Also, Wirsching [12] found a complicated algorithm for the determination of the torsion of complex Grassmannians $G(p,n)$, which allowed him to calculate it for $G(2,4)$, $G(2,5)$ and $G(2,6)$. Thus, our results extend largely the known examples for the torsion.

Let M be a compact n-dimensional Kähler manifold with holomorphic tangent bundle TM. Consider a hermitian holomorphic vector bundle E on M and let $\bar\partial$ and $\bar\partial^*$ denote the associated Dolbeault operator and its formal adjoint. Let $\Box_q := (\bar\partial + \bar\partial^*)^2$ be the Kodaira-Laplacian acting on $\Gamma(\Lambda^q T^{*0,1}M \otimes E)$. We denote by $\mathrm{Eig}_\lambda(\Box_q)$ the eigenspace of $\Box_q$ corresponding to an eigenvalue λ. Consider a holomorphic isometry g of M which induces a holomorphic isometry g^* of E. Then the equivariant analytic torsion is defined via the zeta function

$$Z_g(s) := \sum_{q>0} (-1)^q q \sum_{\substack{\lambda\in\mathrm{Spec}\Box_q\\ \lambda\neq 0}} \lambda^{-s}\,\mathrm{Tr}\, g^*_{|\mathrm{Eig}_\lambda(\Box_q)}$$

for $\mathfrak{Re}\, s \gg 0$. Classically, this zeta function has a meromorphic continuation to the complex plane which is holomorphic at zero. The equivariant analytic torsion is defined as $\tau_g := \exp(-Z_g'(0)/2)$. This gives for $g = \mathrm{Id}_M$ the ordinary analytic torsion τ of Ray and Singer [10].

Let G/K be a compact hermitian symmetric space, equipped with any G-invariant metric $\langle\cdot,\cdot\rangle_\diamond$. We may assume G to be compact and semi-simple. Let $T \subseteq K$ denote a fixed maximal torus. Let Θ be a system of positive roots of K (with respect to some ordering) and let Ψ denote the set of roots of an invariant complex structure in the sense of [4]. Then $\Theta \cup \Psi =: \Delta^+$ is a system of positive roots of G for a suitable ordering, which we fix [4, 13.7].

Let ρ_G denote the half sum of the positive roots of G and let W_G be its Weyl group. As usual, we define $(\alpha, \rho_G) := 2\langle\alpha,\rho_G\rangle_\diamond/\|\alpha\|^2_\diamond$ for any weight α. For any weight b, the (virtual) character χ_b evaluated at $t \in T$ is given via the Weyl character formula by

$$\chi_b(t) = \frac{\sum_{w\in W_G} \det(w)\, e^{2\pi i wb(t)}}{\sum_{w\in W_G} \det(w)\, e^{2\pi i w\rho_G(t)}}.$$

This extends to all of G by setting χ_b to be invariant under the adjoint action. Let V be an irreducible K-representation with highest weight Λ and let $E := (G\times V)/K$ denote the associated G-invariant holomorphic vector bundle on G/K. The metric $\langle\cdot,\cdot\rangle_\diamond$ on $\mathfrak{g}$ induces a hermitian metric on E. Using similar methods as in [8], one may reduce the problem of determining $Z_g(s)$ to a problem in finite-dimensional representation theory. This way one gets our key result

Theorem 2 *The zeta function Z associated to the vector bundle E over G/K is given by*

$$Z(s) = -2^s \sum_{\substack{\alpha\in\Psi\\ k>0}} \langle k\alpha, k\alpha + 2\rho_G + 2\Lambda\rangle_\diamond^{-s} \chi_{\rho_G+\Lambda+k\alpha}.$$

Let for $\phi \in \mathbf{R}$ and $s > 2$

$$\zeta_L(s,\phi) = \sum_{k>0} \frac{e^{ik\phi}}{k^s}$$

denote the Lerch zeta function. Let $P : \mathbf{Z} \to \mathbf{C}$ be a function of the form

$$P(k) = \sum_{j=0}^{m} c_j k^{n_j} e^{ik\phi_j}$$

with $m \in \mathbf{N}_0$, $n_j \in \mathbf{N}_0$, $c_j \in \mathbf{C}$, $\phi_j \in \mathbf{R}$ for all j. Set $P^{\mathrm{odd}}(k) := (P(k) - P(-k))/2$. We define analogously to [6, 2.3.4]

$$\boldsymbol{\zeta} P := \sum_{j=0}^{m} c_j \zeta_L(-n_j, \phi_j), \qquad \boldsymbol{\zeta'} P := \sum_{j=0}^{m} c_j \zeta_L'(-n_j, \phi_j)$$

$$\overline{\boldsymbol{\zeta}} P := \sum_{j=0}^{m} c_j \zeta_L(-n_j, \phi_j) \sum_{\ell=1}^{n_j} \frac{1}{\ell} \quad \text{and} \quad P^*(p) := -\sum_{\substack{j=0 \\ \phi_j \equiv 0 \mod 2\pi}}^{m} c_j \frac{p^{n_j+1}}{4(n_j+1)} \sum_{\ell=1}^{n_j} \frac{1}{\ell}.$$

Then theorem 2 implies by some calculus on zeta functions

Theorem 3 *The logarithm of the equivariant torsion of E on G/K is given by*

$$\begin{aligned} -\frac{1}{2} Z'(0) &= \boldsymbol{\zeta'} \sum_{\alpha \in \Psi} \chi^{\mathrm{odd}}_{\rho_G + \Lambda + k\alpha} - \sum_{\alpha \in \Psi} \chi^*_{\rho_G + \Lambda - k\alpha}\left((\alpha, \rho_G + \Lambda)\right) \\ &+ \frac{1}{2} \sum_{\alpha \in \Psi} \sum_{k=1}^{(\alpha, \rho_G + \Lambda)} \chi_{\rho_G + \Lambda - k\alpha} \log k + \frac{1}{2} \sum_{\alpha \in \Psi} \left(\frac{1}{2} - \boldsymbol{\zeta} \chi^{\mathrm{odd}}_{\rho_G + \Lambda + k\alpha} \right) \log \frac{\|\alpha\|^2_\diamond}{2}. \end{aligned}$$

One can show that the polynomial degree in k of $\sum_\Psi \chi_{\rho_G + \Lambda + k\alpha}(g)$ for any $g \in G$ is at most the dimension of the fixed point set of the action of g on G/K. In particular, it is less or equal $\#\Psi$. The torsion behaves additively under direct sum of vector bundles, thus this result gives the torsion for any homogeneous vector bundle.

Remark: If the decomposition of the space G/K in its irreducible components does not contain one of the spaces $\mathbf{SO}(p+2)/\mathbf{SO}(p) \times \mathbf{SO}(2)$ $(p \geq 3)$ or $\mathbf{Sp}(n)/\mathbf{U}(n)$ $(n \geq 2)$, one may choose the metric $\langle \cdot, \cdot \rangle_\diamond$ in such a way that $\log \|\alpha\|^2_\diamond / 2 = 0$ for all $\alpha \in \Psi$. Thus the corresponding term in theorem 3 vanishes.

We shall now compare the result with Bismut's conjecture of an equivariant Riemann-Roch formula. Consider again a compact Kähler manifold M and a vector bundle E acted on by g and let M_g denote the fixed point set. Let N be the normal bundle of the imbedding $M_g \hookrightarrow M$. Let $\gamma^N_{|x}$ (resp. $\gamma^E_{|x}$) denote the infinitesimal action of g at $x \in M_g$. Let Ω^{TM_g}, Ω^N and Ω^E denote the curvatures of the corresponding bundles with respect to the hermitian holomorphic connection. Define the function Td on square matrices A as Td(A):=det A/(1-exp(-A)).

Definition 1 *Let $\mathrm{Td}_g(TM)$ and $\mathrm{ch}_g(TM)$ denote the following differential forms on M_g:*

$$\mathrm{Td}_g(TM) := \mathrm{Td}\left(\frac{-\Omega^{TM_g}}{2\pi i}\right) \det\left(1 - (\gamma^N)^{-1} \exp \frac{\Omega^N}{2\pi i}\right)^{-1}$$

and

$$\mathrm{ch}_g(TM) := \mathrm{Tr}\,\gamma^E \exp \frac{-\Omega^E_{|M_g}}{2\pi i}.$$

Assume now for simplicity that E is the trivial line bundle. In [2], Bismut introduced the equivariant R-genus $R_g(TM)$. Using this genus we may reformulate theorem 3 as follows:

Theorem 4 *The logarithm of the torsion is given by the equation*

$$2\log\tau_g(G/K) - \log\mathrm{vol}_\diamond(\mathrm{G/K}) + \sum_\Psi \big(\frac{1}{2} + \boldsymbol{\zeta}\chi^{\mathrm{odd}}_{\rho_G+\mathrm{k}\alpha}(\mathrm{g})\big)\log\frac{\|\alpha\|^2_\diamond}{2}$$
$$= \int_{(G/K)_g} \mathrm{Td}_g(T(G/K))R_g(T(G/K)) - \overline{\boldsymbol{\zeta}}\sum_\Psi \chi^{\mathrm{odd}}_{\rho_G+k\alpha}(g) - 2\sum_\Psi \chi_{\rho_G-k\alpha}(g)^*\left((\alpha,\rho_G)\right).$$

Using the R-genus, Bismut formulated a conjectural equivariant arithmetic Grothendieck-Riemann-Roch theorem [2]. Suppose that M is given by $\mathcal{M}\otimes\mathbf{C}$ for a flat regular scheme $\pi : \mathcal{M} \to \mathrm{Spec}\,\mathbf{Z}$ and that E stems from an algebraic vector bundle $\mathcal{E}$ over $\mathcal{M}$. Let $\sum(-1)^q R^q\pi_*\mathcal{E}$ denote the direct image of $\mathcal{E}$ under π. We equip the associated complex vector space with a hermitian metric via Hodge theory. Bismut's conjecture implies that the equivariant torsion verifies the equation

$$2\log\tau_g(M,E) + \hat{c}^1_g\left(\sum_{q\geq 0}(-1)^q R^q\pi_*\mathcal{E}\right) = \pi_*\left(\widehat{\mathrm{Td}}_g(T\mathcal{M})\widehat{\mathrm{ch}}_g(\mathcal{E})\right)^{(1)}$$
$$+\int_{(G/K)_g}\mathrm{Td}_g(T(G/K))R_g(T(G/K))\mathrm{ch}_g(E) \qquad (1)$$

(We identify the first arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathrm{Spec}\,\mathbf{Z})$ with $\mathbf{R}$). Here $\hat{c}^1_g$, $\widehat{\mathrm{Td}}_g$ and $\widehat{\mathrm{ch}}_g$ denote certain equivariant arithmetic characteristic classes which are only defined in a non-equivariant situation up to now (see [11]). Bismut [1] has proven that this formula is compatible with the behaviour of the equivariant torsion under immersions and changes of the occuring metrics. In the non-equivariant case, equation (1) has been proven by Gillet, Soulé, Bismut and Lebeau [7]. In our case, the $\hat{c}^1_g$ term in (1) should be independent of g. By the definition of $\hat{c}^1$, it should equal minus the logarithm of the norm of the element $1 \in H^0(G/K)$, thus $-\log\mathrm{vol}_\diamond(\mathrm{G/K})$. Hence, theorem 4 fits very well with Bismut's conjecture.

We consider now the case $g = \mathrm{Id}$. For this action, the equivariant torsion equals the original Ray-Singer torsion. The values of the characters $\chi_{\rho_G+k\alpha}$ at zero are given by the Weyl dimension formula

$$\chi_{\rho_G+k\alpha}(0) = \dim V_{\rho_G+k\alpha} = \prod_{\beta\in\Delta^+}\left(1 + k\frac{\langle\beta,\alpha\rangle}{\langle\beta,\rho_G\rangle}\right).$$

In particular, the first term in theorem 3 is given by $\boldsymbol{\zeta'}$ applied to the odd part of the polynomial

$$\sum_{\alpha\in\Psi}\chi_{\rho_G+k\alpha}(0) = \sum_{\alpha\in\Psi}\prod_{\beta\in\Delta^+}\left(1 + k\frac{\langle\beta,\alpha\rangle}{\langle\beta,\rho_G\rangle}\right).$$

At a first sight, this looks like a polynomial of degree $\#\Delta^+$, but it has in fact degree $\leq \#\Psi$, thus all higher degree terms cancel. By combining theorem 4 with the arithmetic Riemann-Roch theorem, we get the following formula:

Theorem 5 *The direct image of the arithmetic Todd class is given by*

$$\begin{aligned}\Big(\pi_* \widehat{\mathrm{Td}}(T\mathcal{M})\Big)^{(1)} &= \sum_{\Psi} \big(\frac{1}{2} + \boldsymbol{\zeta}\,(\dim V_{\rho_G+k\alpha})^{\mathrm{odd}}\big) \log \frac{\|\alpha\|_\diamond^2}{2} \\ &+ \overline{\boldsymbol{\zeta}} \sum_{\Psi} (\dim V_{\rho_G+k\alpha})^{\mathrm{odd}} + 2 \sum_{\Psi} (\dim V_{\rho_G+k\alpha})^* \left((\alpha, \rho_G)\right).\end{aligned}$$

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