Topology of the Compactified Jacobians of Singular Curves

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Abstract

We compute the Euler number of the compactified Jacobian of a curve whose minimal unibranched normalization has only plane irreducible singularities with characteristic Puiseux exponents (p, q), $(4, 2q, s)$, $(6, 8, s)$, or $(6, 10, s)$. Further, we derive a combinatorial method to compute the Betti numbers of the compactified Jacobian of an unibranched rational curve with singularities like above. Some of the Betti numbers can be stated explicitly.

Let C be an irreducible and reduced projective curve and $\Sigma \subset C$ its singularities. The generalized Jacobian JC of C consists of the locally free sheaves of rank 1 and degree 0 on C. It is an extention of the Jacobian of the normalization $\tilde{C}$ of C by an affine commutative subgroup of dimension $\delta := \sum_{p\in\Sigma} \delta_{(C,p)}$, thus its dimension equals the arithmetic genus $g_a(C)$ of C. Unfortunately, JC is never compact except when C is smooth, but it is an open subspace of the compactified Jacobian $\bar{J}C$, which consists of all rank one torsion free sheaves $\mathcal{F}$ of degree zero, i.e., $\chi(\mathcal{F}) = 1 - g_a(C)$. The compactified Jacobian is irreducible if and only if all singularities of C are planar [AIK, R]. Only in this case JC is dense in $\bar{J}C$, and $\bar{J}C$ is in fact a compactification of JC. The Euler number $e(\bar{J}C)$ of $\bar{J}C$ is of particular interest because of the following two applications:

Inspired by the work of Yau and Zaslow, Beauville showed that while counting the rational curves in a complete linear system on a K3–surface the Euler number of $\bar{J}C$ is the multiplicity every curve has to be counted with [YZ, B]. Beauville also showed that the Euler number of the compactified Jacobian of a rational curve C equals the Euler number of the compactified Jacobian of the minimal unibranched partial normalization $\check{C}$ of C. Further, for a rational unibranched curve C its compactified Jacobian is homeomorphic to the direct product of compact spaces, the Jacobi factors $J_{(C,p)}$, $p \in \Sigma$, which depend only on the analytic type of the singularities of C. The Jacobi factors can be defined to be the compactified Jacobian of any rational curve with (C, p) as its unique singularity. Hence, it remains to compute the Euler numbers of the Jacobi factors for the unibranched plane singularities.

Fantechi, Götsche, and van Straten proved that the Euler number of the Jacobi factor of a plane singularity (C, p) equals the multiplicity of the δ–constant strata in the base of the semi–universal deformation of the singularity [FGS].

Unfortunately, this surprising result did not help to compute the Euler numbers $e(J_{(C,p)})$. So far the only known Euler numbers of the Jacobi factors are those of the plane singularities with $\mathbb{C}^*$–action, $V(x^p - y^q)$ with $\gcd(p,q) = 1$, whose Euler numbers, $\frac{1}{p+q}\binom{p+q}{p}$, were computed by Beauville. Here, we will use a natural decomposition of the Jacobi factors to compute further examples:

Main Theorem. *The following table assigns to an unibranched plane singularity with characteristic Puiseux exponents which occur in the left column the Euler number of its Jacobian factor:*

Puiseux exponents	Euler number
(p,q) with $\gcd(p,q) = 1$	$\dfrac{1}{p+q}\dbinom{p+q}{p}$
$(4, 2q, s)$ with $\gcd(qs, 2) = 1$	$\dfrac{(q+1)(q^2+5q+3)}{12} + \dfrac{(q+1)^2}{8}s$
$(6, 8, s)$ with $\gcd(s, 2) = 1$	$\dfrac{229}{2} + \dfrac{25}{2}s$
$(6, 10, s)$ with $\gcd(s, 2) = 1$	$\dfrac{511}{2} + \dfrac{49}{2}s$

The reason for the restriction to the above Puiseux exponents is that in these cases a natural decomposition of the Jacobian factor is a cell decomposition into complex cells, $J_X = \bigcup_{i=1}^{e(J_X)} \mathbb{C}^{n_i}$. We show by several examples that this is not the case for the more complicated cases. From the cell decomposition the Betti numbers of the Jacobi factors can be computed by purely combinatorial means. Explicit formulas are harder to derive, we will prove in Section 5 the following:

Theorem. *Let X be a unibranched plane singularity with characteristic Puiseux exponents (p,q) and J_X its $\delta_X = (p-1)(q-1)/2$ dimensional Jacobi factor. Then the odd homology groups of J_X all vanish. The even homology groups are free abelian groups. The ranks of $H_0(J_X), H_2(J_X), \ldots, H_{2(q-\lceil \frac{q}{p} \rceil)}(J_X)$ are the same as the first $q - \lceil \frac{q}{p} \rceil + 1$ coefficients of the power series $\prod_{i=1}^{p-1}(1-t^i)^{-1}$. The ranks of $H_{2\delta_X}(J_X), H_{2\delta_X - 2}(J_X), \ldots, H_{2\delta_X - 2\lfloor \frac{q}{p} \rfloor}(J_X)$ are the same as the first $\lceil \frac{q}{p} \rceil$ coefficients of the power series $(1-t)^{1-p}$.*

This proves in particular the conjectures of Warmt about the odd homology groups and $H_2(J_X), H_4(J_X)$ [W2, 5.8.4]. An analogous theorem is shown for singularities with characteristic Puiseux exponents $(4, 2q, s)$. In this case one can also describe all Betti numbers conjecturally.

As singularities with the same characteristic Puiseux exponents are topologically equivalent, these theorems provide evidence for the general conjecture that the topology of the compactified Jacobian depends only on the topology of the curve.

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1 Jacobi factors and their cell decomposition

The definition of the Jacobi factor of a singularity X as the compactified Jacobian of any rational curve with X as its unique singularity is unsuitable for us, because the definition is not purely local. We will use Rego's definition, which we will explain in a moment after fixing some notation [R, GP]. We always assume that X is a unibranched plane singularity given by the equation $f \in \mathbb{C}[[x,y]]$. The complete local ring $R = \mathbb{C}[[x,y]]/(f)$ of the singularity has $\tilde{R} = \mathbb{C}[[t]]$ as its normalization. By Puiseux's Theorem there exist $t^n, \varphi \in \mathbb{C}[[t]]$ such that R is embedded as $R \cong \mathbb{C}[[t^n, \varphi]]$ into $\tilde{R} = \mathbb{C}[[t]]$. The conductor of R is $C := \mathrm{Ann}_R(\tilde{R}/R)$. Since X is planar, we have $\delta_X := \delta_R := \dim \tilde{R}/R = \dim R/C$ and $C = (t^{2\delta_R})$ [JP, 5.2.4].

Let M be any torsion free R–module of rank 1. Such a module M can be embedded into $\tilde{R}$. In this situation we define the conductor $C(M)$ of M to be $C(M) := \mathrm{Ann}_R(\tilde{R}/M)$. Because it is an ideal in $\tilde{R}$ as well, we identified it with the natural number $c = c(M)$ such that $C = (t^c) \subset \tilde{R}$. The embedding of M into $\tilde{R}$ can be chosen such that $C \subset M \subset \tilde{R}$ and $\dim \tilde{R}/M = \dim M/C = \delta_R$; we will call such an embedding δ_R–normalized. A δ_R–normalized module M can be considered as a point of the Grassmannian $\mathrm{G}(\tilde{R}/C, \delta_R)$, which consists of the δ_R–dimensional subspaces of $\tilde{R}/C$. The Jacobi factor J_X or J_R of the singularity is the set of points of $\mathrm{G}(\tilde{R}/C, \delta_R)$, which are R–modules. Therefore, $M/C \in \mathrm{G}(\tilde{R}/C, \delta_R)$ lies in J_R if $RM \subseteq M$. This turns out to be a linear condition on $\mathrm{G}(\tilde{R}/C, \delta_R)$ when one considers $\mathrm{G}(\tilde{R}/C, \delta_R)$ to be embedded by the Plücker embedding [GP, 1.4]. Different points of J_R may correspond to isomorphic R–modules. In fact, one has

Theorem 1 *The subsets of J_R consisting of isomorphic modules are biregular to affine spaces.*

Proof. Two R–submodules $M_1, M_2 \subset \tilde{R}$ are isomorphic if there is an $x \in Q(\tilde{R}) = \mathbb{C}((t))$ such that $M_2 = xM_1$. If M_1 and M_2 are δ_R–normalized, the order of x must be zero, i.e., $x \in \tilde{R}^*$. Therefore, the subsets of isomorphic modules are the orbits of the action of $(\tilde{R}/C)^*$ on J_R. Since $\mathbb{C}^* \subset (\tilde{R}/C)^*$ acts trivially and the representation of $(\tilde{R}/C)^*/\mathbb{C}^*$ on J_R is unipotent, the orbits are affine spaces by the Theorem of Chevalley–Rosenlicht [CG, 3.14]. □

Unfortunately, there are infinitely many isomorphism classes of torsion free modules of rank 1 if the singularity is not an A_{2k}, E_6, or E_8 singularity by a theorem of Greuel and Knörrer [GK]. To get a finite cell decomposition, we use the natural valuation $v : \tilde{R} = \mathbb{C}[[t]] \to \mathbb{N}$ and decompose J_R according to the images of the modules under the map v. To prove the Main Theorem, we will show that this decomposition is a cell decomposition into affine complex spaces in the cases of the Main Theorem, then we will count the nonempty ones. This will require some work, because the Theorem of Chevalley–Rosenlicht cannot be applied anymore. We start by translating parts of the problem to a combinatorial problem with the help of the valuation v.

We have $v(\tilde{R}) = \mathbb{N}$ and the image of R under v is a semi–group Γ. The above properties of the conductor translate into $\#(\mathbb{N} - \Gamma) = \delta_R$ and $\min\{x \in \mathbb{N} | x + \mathbb{N} \subset \Gamma\} = 2\delta_R$. For a module $M \subset \tilde{R}$, we get an associated Γ–semi–module $\Delta :=$

$v(M)$, i.e., $\Gamma+\Delta \subseteq \Delta$. If M is δ_R–normalized, then $\#(\mathbb{N}-\Delta) = \delta_R$. We will call a semi–module Δ with this property δ_R–normalized, too. Two Γ–modules are isomorphic if one is the shift of the other by an integer. Corresponding to the definition of the conductor of a module $M \subset \tilde{R}$, we define the conductor $c(\Delta)$ of the semi–module $\Delta \subset \mathbb{N}$ to be the smallest natural number c with $c+\mathbb{N} \subseteq \Delta$.

We call the subset of modules of J_X with associated semi–module Δ simply the Δ–subset of J_X. This decomposition of J_X into Δ–subsets corresponds to the Schubert cell decomposition of the Grassmannian. More precisely, consider the flag in $\tilde{R}/C = \mathbb{C}[[t]]/(t^{2\delta_X})$ given by the ideals (t^i), $i = 1, \ldots, 2\delta_X$, and the Schubert cell decomposition corresponding to it. Then the valuation map $v : \mathbb{C}[[t]] \to \mathbb{N}$ induces a map

$$\begin{array}{ccc} G(\tilde{R}/C, \delta_X) & \longrightarrow & \{S \subset \{0, \ldots, 2\delta_X - 1\} \mid \#S = \delta_X\} \\ \Lambda + C & \longmapsto & v(\Lambda + C) \cap \{0, \ldots, 2\delta_X - 1\}, \end{array}$$

and its fibers are precisely the Schubert cells. Recalling that J_X is the intersection of $G(\tilde{R}/C, \delta_X)$ and a linear subspace L, we see that the Δ–subsets are linear sections of these Schubert cells. We will show in Section 3 that these Δ–subsets are again complex cells in the cases of the Main Theorem. To prove that they form a CW–complex like the Schubert cell decomposition seems incredible tedious mainly because the dimension of the Schubert cells does not drop uniformly during the intersection process. Luckily, this is not necessary to compute the homology groups by [F, 19.1.11]. In particular, we obtain that all odd homology groups are zero, and the even ones are free abelian groups whose rang equals the number of Δ–subsets of the corresponding dimension.

Before attacking the problem of proving that the Δ–subsets are affine in the cases of the Main Theorem, we discuss the Γ–semi–modules. In particular, we need to count them. Later on we need "syzygies" of the generators of a semi–module. However, such a notion seems cumbersome to define. Therefore, we pass over to the graded semi–group algebra $\mathbb{C}[\Gamma] = \mathrm{span}\,\{t^\gamma | \gamma \in \Gamma\}$ and correspondingly to the graded $\mathbb{C}[\Gamma]$–module $\mathbb{C}[\Delta] = \mathrm{span}\,\{t^\delta | \delta \in \Delta\}$, where we can use the conventional definition of syzygies. The connection of these objects with an R–module M with $v(M) = \Delta$ is as follows:

Define the initial term of a power series $f = \sum_{i=k}^{\infty} \lambda_i t^i$, $\lambda_k \neq 0$, to be $\mathrm{in}(f) := \lambda_k t^k$ and set $\mathrm{in}(0) := 0$. Then the graded semi–group algebra $\mathbb{C}[\Gamma]$ equals the initial ring $\mathrm{in}(R) := \mathrm{span}\,\{\mathrm{in}(f) | f \in R\} \subseteq \mathbb{C}[[t]]$. Analogously, for any maximal CM–module M the graded semi–module module $\mathbb{C}[\Delta]$ equals the initial module $\mathrm{in}(M) := \mathrm{span}\,\{\mathrm{in}(f) | f \in M\} \subseteq \mathbb{C}[[t]]$.

The study of the $\mathbb{C}[\Gamma]$–semi–modules is done in the next section; the proof that the Δ–subsets are complex cells in the following section. Everything concerning the Puiseux exponents $(6, 8, s)$ and $(6, 10, s)$ was moved to Section 4 which is combinatorically more complicated and included only for completeness and the most interested reader. In the final Section 5 the Betti numbers of the Jacobi factors are discussed.

2 The number and the syzygies of the $\mathbb{C}[\Gamma]$–modules

During this section we will always assume that any Γ–semi–module Δ is 0–normalized, i.e., $\min \Delta = 0$, to obtain unique representatives in the isomorphism classes of the semi–modules. In particular, one has $\Gamma \subseteq \Delta$.

For a singularity with only two characteristic Puiseux exponents (p, q), $p < q$, the semi–group Γ is generated by p and q, $\Gamma = \langle p, q \rangle$. To study the Γ–semi–modules, we introduce the notion of a basis for them, modeled after the Apéry–basis for semi–groups (see [JP, A, H] for the semi–group case).

Definition 2 *Let $\Gamma = \langle p, q \rangle$. A p–*basis *of a Γ–semi–module Δ is the unique p–tuple $(a_0, \dots, a_{p-1})$ such that*

$$\Delta = \bigcup_{i=0}^{p-1} (a_i + p\mathbb{N}) \qquad \textit{and} \qquad a_i \equiv iq \mod p.$$

In particular, the $\{a_i\}$ generate Δ as a Γ–semi–module and $c(\Delta) = \max\{a_i\} - p + 1$.

By the definition of the p–basis and $\mathbb{N}q \subset \Gamma \subseteq \Delta$, there exist $\alpha_1, \dots, \alpha_{p-1} \in \mathbb{N}$ such that

$$a_0 = 0, \quad a_1 = q - \alpha_1 p, \quad a_2 = 2q - \alpha_2 p, \quad \dots, a_{p-1} = (p-1)q - \alpha_{p-1} p.$$

To simplify the notation, we define $\alpha_0 = 0$. The condition that $\Delta = \bigcup (a_i + p\mathbb{N})$ is a Γ–semi–module is equivalent to $a_i + q \in \Delta$ and — with a cyclic notation of the indices — to $a_i + q \geq a_{i+1}$. The latter is the same as $0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{p-1} \leq q$. Due to the 0–normalization we have $a_i \geq 0$, i.e., $\alpha_i \leq iq/p$.

Proposition 3 *For the semi–group $\Gamma = \langle p, q \rangle$, $\gcd(p, q) = 1$, the number of isomorphism classes of Γ–semi–modules is $\frac{1}{p+q}\binom{p+q}{p}$.*

Proof. Beauville proves this result with the help of generating functions [B, 4.3]. Fantechi, Götsche, and van Straten derive this from a local computation in a moduli space for rational curves [FGS, G1]. We give a third, shorter proof using the p–bases. For a moment we normalize our Γ–modules only by $\min(\Delta \cap p\mathbb{N}) = 0$, i.e., $a_0 = 0$. Then by the above arguments all such modules can be obtained by choosing $0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{p-1} \leq q$; hence, there are $\binom{p+q-1}{p-1} = \frac{p}{p+q}\binom{p+q}{p}$ of them. If we shift $\Delta = \bigcup (a_i + p\mathbb{N})$ by $-a_j$, $j = 0, \dots, p-1$, we obtain an isomorphic semi–module Δ' with $\min(\Delta' \cap p\mathbb{N}) = 0$ and these are also the only shifts of Δ that satisfy the additional condition. Therefore, to get the number of isomorphism classes of Γ–semi–modules, we have to divide the above number by p. $\square$

For the purpose of the next section we need to compute the syzygies of the graded $\mathbb{C}[\Gamma]$–semi–modules. We start with a very general lemma.

Lemma 4 *Let Γ be any semi–group $\Gamma \subset \mathbb{N}$ with $\#(\mathbb{N} \setminus \Gamma) < \infty$ and Δ a 0–normalized semi–module. Let $A = (t^{a_1}, \dots, t^{a_k})$ be a graded generating set of a $\mathbb{C}[\Gamma]$–module $\mathbb{C}[\Delta]$. There is a minimal generating set C of syzygies of A consisting of bivectors, i.e., vectors $v = (0, \dots, 0, t^{\gamma_i}, 0, \dots, 0, -t^{\gamma_j}, 0, \dots, 0) \in \mathbb{C}[\Gamma]^k$ with $A \cdot v = 0$.*

Proof. Clearly, any relation between the generators can be splitted into the sum of graded ones. Next, we show that any graded relation between the generators can be splitted into a linear combination of bivectors that are relations as well. Assume we have a graded vector $w = (w_1 t^{\gamma_1}, \ldots, w_n t^{\gamma_n})$ with $A \cdot w = 0$, i.e., $\sum w_i t^{\gamma_i + a_i} = 0$ and $\gamma_i + a_i = const$ for all i with $w_i \neq 0$. Therefore $\sum w_i = 0$. Choose j with $w_j \neq 0$ and set $v_i = (0, \ldots, 0, t^{\gamma_i}, 0, \ldots, 0, -t^{\gamma_j}, 0, \ldots, 0)$ for $i \neq j$ with $w_i \neq 0$ where the nonzero entries are at the positions i and j. Then $\sum w_i v_i = w$ using $\sum w_i = 0$. Finally, we can choose a minimal generating set among all these bivectors using Nakayama's lemma. $\square$

The degree $\deg(v)$ of the above bivector syzygy v is by definition $a_i + \gamma_i$. The bivector syzygy is — up to an unimportant choice of sign — determined by the exponents; hence, we will sometimes use the shorter additive notation

$$a_i + \gamma_i = a_j + \gamma_j.$$

The syzygies of a $\mathbb{C}[\Gamma]$–module for $\Gamma = \langle p, q \rangle$ are nearly obvious.

Proposition 5 *Let $\Gamma = \langle p, q \rangle$ and $\Delta = \bigcup_{i=0}^{p-1} (a_i + p\mathbb{N})$ be a Γ–semi–module like above. Then the $\mathbb{C}[\Gamma]$–module $\mathbb{C}[\Delta]$ is generated by $A = (t^{a_0}, t^{a_1}, \ldots, t^{a_{p-1}})$ and the syzygies of A are generated minimally by the following p–bivectors:*

$$\begin{array}{ll} v_0 := (t^q, -t^{\alpha_1 p}, 0, \ldots, 0), & v_1 := (0, t^q, -t^{(\alpha_2 - \alpha_1)p}, 0, \ldots, 0), \\ v_2 := (0, 0, t^q, -t^{(\alpha_3 - \alpha_2)p}, 0, \ldots, 0), & \ldots \\ v_{p-2} := (0, \ldots, 0, t^q, -t^{(\alpha_{p-1} - \alpha_{p-2})p}), & v_{p-1} := (-t^{(q - \alpha_{p-1})p}, 0, \ldots, 0, t^q). \end{array}$$

In particular, the degree of one of these syzygies is greater than $c(\Delta)$.

Proof. Because $\{a_0, \ldots, a_{p-1}\}$ generate the Γ–semi–module Δ, A generates the $\mathbb{C}[\Gamma]$–module $\mathbb{C}[\Delta]$. Clearly, all of the above bivectors are syzygies and none of them is a linear combination of the others. Therefore, it remains to show that the above bivectors form a generating set. By Lemma 4 all syzygies are generated by bivectors and finding a bivector $(0, \ldots, 0, t^{\gamma_i}, 0, \ldots, 0, -t^{\gamma_j}, 0, \ldots, 0)$ relating t^{a_i} and t^{a_j} is the same as finding γ_i and γ_j with $a_i + \gamma_i = a_j + \gamma_j$. W.l.o.g. assume $i < j$. Because we are looking for a minimal generating set, we may assume that neither $(\gamma_i - q, \gamma_j - q) \in \Gamma^2$ nor $(\gamma_i - p, \gamma_j - p) \in \Gamma^2$, thus either $\gamma_i \in q\mathbb{N}$ and $\gamma_j \in p\mathbb{N}$ or the other way around. Recalling that $a_i \equiv iq \bmod p$ and that q generates the group $\mathbb{Z}/p\mathbb{Z}$, we get that either $\gamma_i = (j - i)q$ and $\gamma_j = (\alpha_j - \alpha_i)p$ or $\gamma_i = (q + \alpha_i - \alpha_j)p$ and $\gamma_j = (p + i - j)q$. Thus we found only the two bivectors

$$\begin{array}{l} (0, \ldots, 0, \quad t^{(j-i)q} \quad, 0, \ldots, 0, -t^{(\alpha_j - \alpha_i)p}, 0, \ldots, 0) \quad \text{and} \\ (0, \ldots, 0, -t^{(q + \alpha_i - \alpha_j)p}, 0, \ldots, 0, \quad t^{(p+i-j)q}, 0, \ldots, 0), \end{array}$$

which are the linear combinations of the elementary bivectors $v_0, \ldots, v_{p-1}$. Namely, the first is

$$\sum_{l=i}^{j-1} t^{(j-l-1)q + (\alpha_l - \alpha_i)p} v_l$$

and the second is

$$\sum_{l=j}^{p-1} t^{(p+i-l-1)q+(\alpha_l-\alpha_j)p} v_l + \sum_{l=0}^{i-1} t^{(i-l-1)q+(q+\alpha_l-\alpha_j)p} v_l.$$

This shows that the v_k generate the syzygies. Since $c(\Delta) = \max\{a_k - p + 1\}$, at least one of the degrees of the v_k, $\deg v_k = a_k + q$, is greater than $c(\Delta)$. $\square$

Now we turn to the singularities with the three characteristic Puiseux exponents $(2p, 2q, s)$ with $\gcd(p, q) = 1$, $\gcd(s, 2) = 1$, and $2p < 2q < s$. We will give here the general definitions and then restrict ourselves to $(4, 2q, s)$, leaving the $(6, 2q, s)$ case for Section 4. The semi–group Γ is generated by $\gamma_0 := 2p$, $\gamma_1 := 2q$, and $\gamma_2 := (p-1)\gamma_1 + s$ [A, H]. Note that these generators are related by $p\gamma_1 = q\gamma_0$ and $2\gamma_2 = \beta\gamma_1 + \eta\gamma_0$ for suitable $\beta \in \{0, \ldots, p-1\}$ and $\eta \in \mathbb{N}$. Any $\gamma \in \Gamma$ can be written uniquely as

$$\gamma = \mu_2\gamma_2 + \mu_1\gamma_1 + \mu_0\gamma_0 \quad \text{with } \mu_2 \in \{0, 1\},\ \mu_1 \in \{0, \ldots, p-1\},\ \mu_0 \in \mathbb{N}.$$

The same holds for $\gamma \in \mathbb{Z}$ if one allows $\mu_0 \in \mathbb{Z}$. We use this to define a special basis for any Γ–semi–module:

Definition 6 *Let $\Gamma = \langle \gamma_0 = 2p,\ \gamma_1 = 2q,\ \gamma_2 = 2(p-1)q + s\rangle$. A $2 \times p$–basis of a Γ–semi–module Δ is the unique $2 \times p$–matrix $\left(\begin{smallmatrix} a_{00} & a_{01} & \cdots & a_{0,p-1} \\ a_{10} & a_{11} & \cdots & a_{1,p-1} \end{smallmatrix}\right)$ such that*

$$\Delta = \bigcup_{\substack{i=0,1 \\ j=0,\ldots,p-1}} (a_{ij} + \gamma_0\mathbb{N}) \quad \textit{and} \quad a_{ij} \equiv i\gamma_2 + j\gamma_1 \bmod \gamma_0.$$

Here, the a_{0J} are even and the a_{1J} are odd numbers. Again, there exist $\alpha_{ij} \in \mathbb{N}$ ($\alpha_{00} := 0$) with

$$a_{ij} = i\gamma_2 + j\gamma_1 - \alpha_{ij}\gamma_0 \qquad \text{for } i \in \{0, 1\},\ j \in \{0, \ldots, p-1\}.$$

The fact that Δ is a Γ–semi–module is equivalent to $a_{ij} + \gamma_1, a_{ij} + \gamma_2 \in \Delta$. With a cyclic notation of the indices, the first is equivalent to $a_{ij} + \gamma_1 \geq a_{i,j+1}$ and the second to $a_{0j} + \gamma_2 \geq a_{1j}$ and $a_{1j} + \gamma_2 \geq a_{0,j+\beta}$, using the above relation $2\gamma_2 = \beta\gamma_1 + \eta\gamma_0$. Expressed in terms of the α_{ij}, this means

$$\begin{array}{ccccccccccc}
0 & \leq & \alpha_{01} & \leq\cdots\leq & \alpha_{0,p-1-\beta} & \leq & \alpha_{0,p-\beta} & \leq\cdots\leq & \alpha_{0,p-1} & \leq & q \\
\wedge\!| & & \wedge\!| & & \wedge\!| & & \wedge\!| & & \wedge\!| & & \\
\alpha_{10} & \leq & \alpha_{11} & \leq\cdots\leq & \alpha_{1,p-1-\beta} & \leq & \alpha_{1,p-\beta} & \leq\cdots\leq & \alpha_{1,p-1} & \leq & q+\alpha_{10} \\
\wedge\!| & & \wedge\!| & & \wedge\!| & & \wedge\!| & & \wedge\!| & & \\
\eta+\alpha_{0\beta} & & \eta+\alpha_{0,\beta+1} & & \eta+\alpha_{0,p-1} & & \eta+q+\alpha_{00} & & \eta+q+\alpha_{0,\beta-1} & &
\end{array}$$

The 0–normalization of Δ is equivalent to $a_{ij} \geq 0$ or to $\alpha_{00} = 0$, $\alpha_{0j} < jq/p$ and $\alpha_{1j} < (\gamma_2 + j\gamma_1)/\gamma_0$. In particular, we have $\alpha_{0,p-1} < q - q/p$ sharpening $\alpha_{0,p-1} \leq q$. For $j \leq p-1-\beta$ we get

$$\alpha_{1j} < \frac{2\gamma_2 - \gamma_2 + j\gamma_1}{\gamma_0} = \frac{\eta\gamma_0 + (\beta+j)\gamma_1 - \gamma_2}{\gamma_0} \leq \eta + \frac{(p-1)\gamma_1 - \gamma_2}{\gamma_0} < \eta$$

and for any $j \leq p-1$ we obtain similarly

$$\alpha_{1j} < \frac{\eta\gamma_0 + (\beta+j)\gamma_1 - \gamma_2}{\gamma_0} \leq \eta + \frac{(p-1)\gamma_1}{\gamma_0} + \frac{(p-1)\gamma_1 - \gamma_2}{\gamma_0} < \eta + q.$$

Therefore, the 0–normalization of the Γ–semi–module already implies the last row of the inequalities between the α_{ij} in the above diagram. Now, we are ready to compute the number of Γ–semi–modules for $p = 2$.

Proposition 7 *The number of* 0*–normalized* Γ*–semi–modules for the semi–group* $\Gamma = \langle 4, 2q, 2q+s\rangle$ *with* $\gcd(qs, 2) = 1$ *is*

$$\frac{(q+1)(2q^2+4q+3)}{12} + s\frac{(q+1)^2}{8}.$$

Proof. We have to count the triples $\alpha = (\alpha_{01}, \alpha_{10}, \alpha_{11})$ with the restrictions

$$\begin{array}{ccccc} 0 & \le & \alpha_{01} & \le & q/2 \\ \wedge\!| & & \wedge\!| & & \\ \alpha_{10} & \le & \alpha_{11} & \le & q+\alpha_{10}\,; \\ \wedge\!| & & \wedge\!| & & \\ \frac{2q+s}{4} & & \frac{4q+s}{4} & & \end{array}$$

thus we have to count the elements of

$$A := \left\{\alpha \in [0, \tfrac{q}{2}] \times [0, \tfrac{2q+s}{4}] \times \mathbb{N} \mid \max\{\alpha_{01}, \alpha_{10}\} \le \alpha_{11} \le \min\{q+\alpha_{10}, q+\tfrac{s}{4}\}\right\}.$$

We set

$$\begin{aligned} A_0 &= \left\{\alpha \in [0, \tfrac{q}{2}] \times [0, \tfrac{2q+s}{4}] \times \mathbb{N} \mid \alpha_{10} \le \alpha_{11} \le q+\alpha_{10}\right\} \\ A_1 &= \left\{\alpha \in [0, \tfrac{q}{2}] \times [0, \tfrac{2q+s}{4}] \times \mathbb{N} \mid \alpha_{10} \le \alpha_{11} < \alpha_{01}\right\} \\ A_2 &= \left\{\alpha \in [0, \tfrac{q}{2}] \times [0, \tfrac{2q+s}{4}] \times \mathbb{N} \mid q+\tfrac{s}{4} < \alpha_{11} \le q+\alpha_{10}\right\}. \end{aligned}$$

Due to $\alpha_{01} \le q/2 \le q+\alpha_{10}$ and $\alpha_{10} \le q/2 + s/4$, we have $A_1, A_2 \subseteq A_0$. Obviously, also $A = A_0 \setminus (A_1 \cup A_2)$ and $A_1 \cap A_2 = \emptyset$ holds. Therefore, the number of elements of A is $\#A_0 - \#A_1 - \#A_2$ or

$$\left\lceil\frac{q}{2}\right\rceil \cdot \left\lceil\frac{2q+s}{4}\right\rceil \cdot (q+1) - \sum_{\alpha_{01}=1}^{\lfloor \frac{q}{2}\rfloor} \sum_{\alpha_{10}=0}^{\alpha_{01}} (\alpha_{01}-\alpha_{10}) - \left\lceil\frac{q}{2}\right\rceil \sum_{\alpha_{10}=\lceil \frac{s}{4}\rceil}^{\lfloor \frac{2q+s}{4}\rfloor} \left(q+\alpha_{10} - \left\lfloor q+\frac{s}{4}\right\rfloor\right).$$

Using the substitution $q = 2\bar{q}+1$ and $s = 4\bar{s}+1$ resp. $s = 4\bar{s}+3$ for the intermediate steps, the above sum can be evaluated easily to obtain the number in the statement of the proposition. $\square$

Later we will show that some of the Γ–semi–modules cannot occur as the 0–normalization of an associated semi–module of a maximal CM–module over the local ring of the singularity. We call the ones that occur admissible. Their combinatorial definition is as follows:

Definition 8 *Let* $\Gamma = \langle \gamma_0 = 2p,\ \gamma_1 = 2q,\ \gamma_2 = (p-1)\gamma_1 + s\rangle$ *with* $p = 2$ *or* $(p, q) \in \{(3, 4), (3, 5)\}$*. A* 0*–normalized* Γ*–semi–module* Δ *is* admissible *iff*

$$\Delta \cap \{a_{0j} + s \mid j = 0, \ldots, p-1\} \neq \emptyset.$$

Proposition 9 *Let $\Gamma = \langle 4, 2q, 2q+s\rangle$ be the above semi–group for $p=2$. The number of admissible Γ–semi–modules is*

$$\frac{(q+1)(q^2+5q+3)}{12} + \frac{(q+1)^2}{8}s.$$

Proof. We are going to count the nonadmissible semi–modules, i.e., the 0–normalized semi–modules with $s, a_{01}+s \notin \Delta$. We have

$$\begin{aligned} s = \gamma_2 - \gamma_1 = \gamma_2 + \gamma_1 - q\gamma_0 &\equiv a_{11} \bmod \gamma_0 \\ a_{01} + s = \qquad \gamma_2 - \alpha_{01}\gamma_0 \quad &\equiv a_{10} \bmod \gamma_0. \end{aligned}$$

Hence, $s, a_{01}+s \notin \Delta$ is equivalent to $s < a_{11}$ and $a_{01}+s < a_{10}$, i.e., $\alpha_{11} < q$ and $\alpha_{10} < \alpha_{01}$. Together with the conditions for Δ being a 0–normalized semi–module,

$$\alpha_{01} < \tfrac{q}{2}, \quad \alpha_{10} < \tfrac{2q+s}{4}, \quad \max\{\alpha_{01}, \alpha_{10}\} \le \alpha_{11} \le \min\{q+\alpha_{10}, q+\tfrac{s}{4}\},$$

the nonadmissible semi–modules correspond to the triples $(\alpha_{01}, \alpha_{10}, \alpha_{11}) \in \mathbb{N}^3$ with

$$\alpha_{01} < \tfrac{q}{2}, \quad \alpha_{10} < \alpha_{01}, \quad \alpha_{01} \le \alpha_{11} < q.$$

Clearly, the numbers of these is

$$\sum_{\alpha_{01}=0}^{\lfloor \frac{q}{2} \rfloor} \alpha_{01}(q-\alpha_{01}) = \frac{1}{2}\binom{q+1}{3}.$$

We obtain the number of admissible semi–modules as the difference of the number of all semi–modules (Proposition 7) and this term. □

We already know from Lemma 4 that the syzygies of a graded generating set of a $\mathbb{C}[\Gamma]$–module $\mathbb{C}[\Delta]$ are generated by bivectors. We are going to select a small subset of these bisectors, which generate the syzygies of degree less than the conductor $c(\Delta)$. Later on, only these syzygies will be of interest to us.

Proposition 10 *Let $\Gamma = \langle \gamma_0 = 4,\ \gamma_1 = 2q,\ \gamma_2 = \gamma_1 + s\rangle$, and $\Delta = \langle 0, a_{01}; a_{10}, a_{11}\rangle$ like above. The $\mathbb{C}[\Gamma]$–module $\mathbb{C}[\Delta]$ is generated by $A = (1, t^{a_{01}}, t^{a_{10}}, t^{a_{11}})$, and the syzygies of A of degree less than $c(\Delta)$ are generated by*

$$(t^{\gamma_1}, -t^{\alpha_{01}\gamma_0}, 0, 0), \quad (-t^{(q-\alpha_{01})\gamma_0}, t^{\gamma_1}, 0, 0), \quad \textit{and} \;\; (t^{\gamma_2}, 0, -t^{\alpha_{10}\gamma_0}, 0).$$

Proof. By Lemma 4 there is a generating set of syzygies consisting of bivectors. Any bivector syzygy of degree d may be written additively as

$$a_{ij} + (\xi_2\gamma_2 + \xi_1\gamma_1 + \xi_0\gamma_0) = a_{kl} + (\zeta_2\gamma_2 + \zeta_1\gamma_1 + \zeta_0\gamma_0) = d$$

with $\xi_1, \xi_2, \zeta_1, \zeta_2 \in \{0,1\}$ and $\xi_0, \zeta_0 \in \mathbb{N}$. We may assume that the bivector syzygy is not the multiple of another; hence, for all $r \in \{0,1,2\}$ one of the ξ_r, ζ_r is zero.

Recall that $c(\Delta) = \max\{a_{ij}\} - \gamma_0 + 1 \le c(\Gamma)$. We have $a_{00} = 0$, $a_{01} \le \gamma_1$, $a_{10} + \gamma_1 \ge a_{11}$, and $a_{11} + \gamma_1 \ge a_{10}$, thus $a_{1j} + \gamma_2 > a_{1j} + \gamma_1 > c(\Delta)$. Consequently, of all possible bivector syzygies only the following four may be of degree less than $c(\Delta)$:

$$\begin{array}{ll} a_{00} + \gamma_1 = a_{01} + \alpha_{01}\gamma_0 & a_{01} + \gamma_1 = a_{00} + (q - \alpha_{01})\gamma_0 \\ a_{00} + \gamma_2 = a_{10} + \alpha_{10}\gamma_0 & a_{01} + \gamma_2 = a_{11} + (\alpha_{11} - \alpha_{01})\gamma_0. \end{array}$$

However, the degree of the last relation is also greater than $c(\Delta)$ due to $a_{00}, a_{10} \leq \gamma_2$. □

3 The cell decomposition

We return to our singularity $(X, 0)$ with local ring R. Its Jacobi factor $J_R \subset G(\tilde{R}/C, \delta_R)$ consists of the torsion free modules of rank 1 and was decomposed according to their associated semi–modules. We will show that these subsets are biregular to an affine space $\mathbb{C}^N$. Given a Γ–semi–module Δ we are going to explicitly construct all R–modules $M \subseteq \tilde{R} = \mathbb{C}[[t]]$ with associated semi–module Δ. There is a small technical problem: The associated semi–module Δ_M of a module M is δ_R–normalized, while we were using 0–normalized semi–modules in the last section for notational convenience. However, if $d := \min \Delta_M$ then $-d + \Delta_M$ is 0–normalized and the module $t^{-d}M \subseteq \tilde{R}$ has this semi–module as associated semi–module. With the help of this obvious bijection we may continue to assume that the occurring Γ—semi–modules are 0–normalized.

We start our proof with several remarks about the elements of $M \subset \mathbb{C}[[t]]$. Any element $x = \sum_{k \in \mathbb{N}} \lambda_k t^k \in M$ can be normalized as follows: To get the coefficient of the initial term $\lambda_{v(x)} t^{v(x)}$ equal to one, we multiply by $1/\lambda_{v(x)}$. Then one removes the terms $\lambda_\delta t^\delta$, $\delta \in \Delta$, $\delta > v(x)$, in increasing order by subtracting suitable multiples of elements $y \in M$ with $v(y) = \delta$; thus as the normal form of x we obtain a polynomial of type

$$t^{v(x)} + \sum_{k \in]v(x), \infty[\setminus \Delta} \lambda_k t^k.$$

There is only one normalized x of a fixed order $v(x)$, because the difference of two such lies in M and has no powers of t which can occur as an initial term and must therefore vanish.

The same ideas lead to a reduction algorithm for an element $x \in \mathbb{C}[[t]]$ with respect to a set $\{m_0, \ldots, m_n\} \subset \mathbb{C}[[t]]$: Let Δ be the Γ–semi–group generated by $\{v(m_0), \ldots, v(m_n)\}$. Set $x_0 = x \in \mathbb{C}[[t]]$. Starting with $i = 0$ we do for increasing $i \in \mathbb{N}$ the following: If $i \notin \Delta$, set $s_i = 0$ and $x_{i+1} = x_i$. If $i \in \Delta$, then locate the t^i–term, $\tilde{\lambda}_i t^i$, in x_i, find $s_i \in R$, $j_i \in \{0, \ldots, n\}$ with $\tilde{\lambda}_i t^i = s_i m_{j_i}$ and set $x_{i+1} = x_i - s_i m_{j_i}$. The x_i converge to an

$$x_\infty = \sum_{k \notin \Delta} \mu_k t^k.$$

Unfortunately, x_∞ depends in general on the choices made. However, this does not make the reduction process useless. Its main application is the following: If the $m_0, \ldots, m_n$ generate the R–module M and $v(M) = \Delta$, then $x \in M$ iff $x_\infty = 0$. Namely, on the one hand if $x_\infty = 0$ then the algorithm yields

$x = \sum s_i m_{j_i} \in M$, on the other hand if $x_\infty \neq 0$ then $v(x_\infty) \notin \Delta$ and this implies $x_\infty \notin M$ and $x = x_\infty + \sum s_i m_{j_i} \notin M$.

Often one starts with a module M and then picks normalized generators $m_0, \ldots, m_n$ such that $\Delta := v(M)$ is generated as a Γ–semi–module by $v(m_0), \ldots, v(m_n)$. We will call such a set a Δ–generating set of M. We write the generators as

$$\begin{aligned} m_0 &= \quad 1 + \sum_{k \in]0,\infty[\setminus \Delta} \lambda_k^0 t^k &&= \quad 1 + \sum_{k \notin \Delta} \lambda_k^0 \, t^k \\ m_1 &= t^{a_1} + \sum_{k \in]a_1,\infty[\setminus \Delta} \lambda_{k-a_1}^1 t^k &&= t^{a_1} + \sum_{a_1+k \notin \Delta} \lambda_k^1 \, t^{a_1+k} \\ &\vdots \\ m_n &= t^{a_n} + \sum_{k \in]a_n,\infty[\setminus \Delta} \lambda_{k-a_n}^n t^k &&= t^{a_n} + \sum_{a_n+k \notin \Delta} \lambda_k^n \, t^{a_n+k}. \end{aligned}$$

The special choice of the lower indices of the λ will be crucial later on.

Now we start the other way around. Given Δ with generators $\{a_0 = 0, \ldots, a_n\}$, let $m_0, \ldots, m_n$ be like above and M the module generated by them. Clearly, $\Delta \subseteq v(M)$. We want to see which conditions the λ must satisfy such that $v(M) = \Delta$; we consider them now as variables. To count the number of the λ, we introduce the *gap counting function* g_Δ by $g_\Delta(k) = \#([k, \infty[\setminus \Delta)$ for $k \in \mathbb{N}$, i.e., g_Δ counts the gaps of Δ greater than or equal to k. With this notation, there are $\sum_{j=0}^{n} g_\Delta(a_j)$ of the λ variables.

We want to consider syzygies between the m_j as well as between their initial terms. For a graded vector $r = (r_j) \in \bigoplus_j R(-a_j)$ we define the *initial vector* as follows: Let $\delta := \min\{v(r_j) + a_j\}$. Then $\mathrm{in}(r) = (s_j)$ with $s_j = \mathrm{in}(r_j)$ if $v(r_j) + a_j = \delta$ and 0 otherwise. The important consequence is that if r is a syzygy of the generators (m_j) of M then $\mathrm{in}(r)$ is a syzygy of $(\mathrm{in}(m_j)) = (t^{a_j})$.

Our leading idea for the following is

Lemma 11 *With the above notation let M be the R–module generated by $\{m_0, \ldots, m_n\}$. Further, let $V \subset \bigoplus_j R(-a_j)$ such that the initial vectors $\{\mathrm{in}(r) | r \in V\}$ of V generate the syzygies of the generating set $A = (t^{a_j})$ of $\mathbb{C}[\Delta]$. Then $v(M) = \Delta$ if and only if for each $r = (r_j) \in V$ the following holds:*

Let $\delta := \min\{v(r_j m_j)\}$. Then the initial terms of $\sum r_j m_j$ cancel, i.e., $v(\sum r_j m_j) > \delta$, and there exist $s_j \in R$ with $v(s_j m_j) > \delta$ and $\sum r_j m_j = \sum s_j m_j$.

We will call $\sum s_j m_j$ a *higher order expression* of $\sum r_j m_j$. Note that a higher order expression can be obtained trivially if $\delta \geq c(\Delta)$ because $t^{c(\Delta)+k} \in M$ for $k \in \mathbb{N}$. A higher order expression for a term $T = \sum r_j m_j$ may be found by reducing it with the above algorithm. If T reduces to zero, then the algorithm produces an expression with $T = \sum s_i m_{j_i}$ with $v(s_i m_{j_i}) > \delta$ which we can reorder to get the higher order expression. If T reduces to $T_\infty \neq 0$, then $\mathrm{in}(T_\infty) \in v(M) \setminus \Delta$ showing $v(M) \neq \Delta$.

Proof. Clearly, we always have $v(M) \supseteq \Delta$. We have equality iff there is no element $x = \sum r_j m_j \in M$ with $v(x) \notin \Delta$. We claim that this is the case if and only if the following holds:

$(*)$ For any $\delta \in \Delta$ and $r = (r_j) \in \bigoplus_j R(-a_j)$ with $v(r_j) + a_j = \delta$ or $r_j = 0$ such that the initial terms of $\sum r_j m_j$ cancel, i.e., $v(\sum r_j m_j) > \delta$, $\sum r_j m_j$ can be expressed as $\sum s_j m_j$ with $v(s_j m_j) > \delta$.

To prove this claim, assume that $(*)$ holds, and there is an $x = \sum r_j m_j \in M$ with $v(x) \notin \Delta$. Further, we assume that the linear combination is chosen such that $\delta := \min\{v(r_j m_j)\} \in \Delta$ is maximal. Since $\delta \in \Delta$ and $v(x) \notin \Delta$, the initial terms of $\sum r_j m_j$ cancel and by $(*)$ there exist s_j with $x = \sum r_j m_j = \sum s_j m_j$ and $v(s_j m_j) > \delta$. But this contradicts the maximality assumption on $\min\{v(r_j m_j)\}$.

The other way around, assume we have r_j such that $(*)$ fails. Set $x_0 := \sum r_j m_j$ and reduce it with the above algorithm to x_∞. x_0 cannot be reduced to zero because $\sum r_j m_j = x_0 = \sum_{i=0}^{\infty} s_i m_{j_i}$ would show that $(*)$ holds for r_j. Therefore, $x_\infty \neq 0$ and $v(x_\infty) \notin \Delta$ shows $v(M) = \Delta$.

Cancellation of the initial terms in $(*)$ means precisely that $\mathrm{in}(r)$ is a syzygy of the generating set $A = (\mathrm{in}(m_j)) = (t^{a_j})$ of $\mathbb{C}[\Delta]$. Note that if $(*)$ holds for r then it holds for all r' with $\mathrm{in}(r') = \mathrm{in}(r)$, because $\sum r'_j m_j = \sum r_j m_j + \sum (r'_j - r_j) m_j = \sum (s_j + r_j - r'_j) m_j$ and $v(s_j), v(r_j - r'_j) > v(r'_j)$. Therefore, it is enough to check $(*)$ for a set of vectors which generate the syzygies of $\mathbb{C}[\Delta]$. □

We now study the different cases of characteristic Puiseux exponents separately. We start again with a unibranched plane singularity that has the characteristic Puiseux exponents (p, q). Then by Puiseux's Theorem the local ring R of the singularity is isomorphic to a ring $\mathbb{C}[[t^p, \varphi]] \subset \mathbb{C}[[t]] \cong \tilde{R}$ where $\varphi = t^q +$ higher order terms. Further $\Gamma = \langle p, q \rangle$. Let Δ be any 0–normalized Γ–semi–module. As a generating set for Δ we choose the unique p–basis $(a_0, \dots, a_{p-1})$ and define m_j as above. By Proposition 5 and Lemma 11 the module M has $v(M) = \Delta$ iff higher order expressions for the following p terms can be found:

$$\begin{aligned} T^j &:= \varphi m_j - t^{(\alpha_{j+1} - \alpha_j)p} m_{j+1} =: \sum_{k=1}^{\infty} c_k^j t^{a_j + q + k} \qquad j = 0, \dots, p-2 \\ T^{p-1} &:= \varphi m_{p-1} - t^{(q - \alpha_{p-1})p} m_0 \ \ =: \sum_{k=1}^{\infty} c_k^{p-1} t^{a_{p-1} + q + k}. \end{aligned}$$

We study the coefficients c_k^j more closely. For $k \in \mathbb{N}$ define the *gap function* $\tilde{g}_\Delta$ by $\tilde{g}_\Delta(k) = 1$ if $k \notin \Delta$ and 0 otherwise. Then with cyclic index notation

$$c_k^j = \tilde{g}_\Delta(a_j + k)\lambda_k^j - \tilde{g}_\Delta(a_{j+1} + k)\lambda_k^{j+1} + \text{polynomial in } \lambda_l^j \text{ with } l < k.$$

To find the higher order expressions for the T^j, we reduce the T^j by the above algorithm. We denote the resulting terms by $\tilde{T}^j$. These terms must vanish, otherwise $v(M) \neq \Delta$. The terms $\tilde{T}^j$ have only powers of t whose exponents do not lie in Δ. Let us study the coefficients of these t–powers more closely. There are two important observations: The first is that during this process a coefficient

c_k^j is only modified by the addition of polynomials in the λ_l^i with $l < k$, except it is made to vanish. The second concerns the occurrence of the λ_k^j and λ_k^{j+1} in the final coefficients $\tilde{c}_k^j$. If $\tilde{g}_\Delta(a_j + k) = 0$ or $\tilde{g}_\Delta(a_{j+1} + k) = 0$, then $a_j + k \in \Delta$ or $a_{j+1} + k \in \Delta$ and further $a_j + k + q = a_{j+1} + k + (\alpha_{j+1} - \alpha_j)p \in \Delta$ for $j < p - 1$ resp. $a_{p-1} + k + q = a_0 + k + (q + \alpha_0 - \alpha_{p-1})p \in \Delta$ for $j = p - 1$, showing that the t–power t^{a_j+q+k} will be eliminated in the process. Therefore, all c_k^j which do not have a $\lambda_k^j - \lambda_k^{j+1}$ term vanish during this process. Thus in the end the remaining coefficients $\tilde{c}_k^j$ with $a_j + q + k \notin \Delta$ are of the form

$$\tilde{c}_k^j = \lambda_k^j - \lambda_k^{j+1} + \text{polynomial in } \lambda_l^* \text{ with } l < k,$$

and there are $\sum_{j=0}^{p-1} g_\Delta(a_j + q)$ of these. The vanishing of these coefficients is equivalent to M being an R–module with associate semi–module Δ. For fixed k we may view $\tilde{c}_k^j = 0$ as an inhomogeneous linear equation system in the variables λ_k^j. Because of Proposition 5 there is a $J \in \{0, \ldots, p-1\}$ with $c(T_J) \geq c(\Delta)$ and consequently $\tilde{T}_j = 0$, therefore there are at most $p-1$ nonzero equations and the linear system is in row echelon form. Hence, we can easily obtain a dependency of some of the λ_k^* on the other λ_k^* and the λ_l^* with $l < k$. Finally, we substitute successively the solutions for the λ with lower index less than k into the solutions for the λ with lower index k; thereby obtaining an explicit form of the equations $\tilde{c}_k^j = 0$, expressing some λ–variables as polynomial functions of the other.

Summarizing, we have shown that all possible coefficients for the m_i such that $v(M) = \Delta$ can be obtained as the graph of a polynomial function in $\sum g_\Delta(a_j) - \sum g_\Delta(a_j + q)$ variables. Setting $d = \delta_R - g_\Delta(0)$, we note that different values for the remaining free λ–variables lead to different modules $t^d M$ and modulo C to different points of $G(\tilde{R}/C, \delta_R)$, because of the normalized form of the m_j. Thus we have proved

Theorem 12 *Let R be the local ring of a unibranched plane singularity with characteristic Puiseux exponents (p, q) and Δ be a δ_R–normalized $\langle p, q\rangle$–semi–module, whose 0–normalization Δ_0 has the p–basis $(a_0, \ldots, a_{p-1})$. Then the subset of modules of J_R with associated semi–module Δ is biregular to an affine space $\mathbb{C}^N$ with*

$$N = \sum_{j=0}^{p-1} \left(g_{\Delta_0}(a_j) - g_{\Delta_0}(a_j + q) \right),$$

where for a $k \in \mathbb{N}$ the number $g_{\Delta_0}(k) := \#([k, \infty[\setminus \Delta_0)$ is the number of gaps in Δ_0 equal to or greater than k.

Since the number of $\langle p, q\rangle$–semi–modules is $\frac{1}{p+q}\binom{p+q}{p}$ by Proposition 3, the Jacobi factor J_R has a cell decomposition into the same number of complex cells. In particular, its Euler number is also $\frac{1}{p+q}\binom{p+q}{p}$, proving the main theorem in this case.

Now we treat the case of a singularity with characteristic Puiseux exponents $(4, 2q, s)$ using the notation of the preceeding section. The local ring R of such a singularity is isomorphic to $\mathbb{C}[[t^4, \varphi]] \subset \mathbb{C}[[t]]$, where $\varphi = t^{2q} + t^s +$ higher order terms [Z, p. 784]. Let $\psi \in R$ be the normalized element with $v(\psi) = \gamma_2 = 2q + s$.

A 0–normalized Γ–semi–module Δ has a 2×2–basis $(a_{00}=0, a_{01}; a_{10}, a_{11})$, thus we have the ansatz

$$m_{00} = \quad 1 + \sum_{k\in]0,\infty[\setminus\Delta} \lambda^{00}_k t^k \qquad\qquad m_{01} = t^{a_{01}} + \sum_{k\in]a_{01},\infty[\setminus\Delta} \lambda^{01}_{k-a_{01}} t^k$$

$$m_{10} = t^{a_{10}} + \sum_{k\in]a_{10},\infty[\setminus\Delta} \lambda^{10}_{k-a_{10}} t^k \qquad\qquad m_{11} = t^{a_{11}} + \sum_{k\in]a_{11},\infty[\setminus\Delta} \lambda^{11}_{k-a_{11}} t^k$$

for the generators of an R–module M with associated semi–module Δ. By Proposition 10 and Lemma 11 we have $v(M)=\Delta$ iff we can find higher order expressions for the three terms

$$T^1 := \varphi m_{00} - t^{4\alpha_{01}} m_{01} \qquad =: \sum_{k=1}^{\infty} c^1_k t^{\gamma_1+k}$$

$$T^2 := t^{4(q-\alpha_{01})} m_{00} - \varphi m_{01} =: \sum_{k=1}^{\infty} c^2_k t^{a_{01}+\gamma_1+k}$$

$$T^3 := \psi m_{00} - t^{4\alpha_{10}} m_{10} \qquad =: \sum_{k=1}^{\infty} c^3_k t^{\gamma_2+k}.$$

We follow the same strategy as before: reduce T^1, T^2, T^3 with respect to $\{m_{ij}\}$ and solve the equations given by the remaining coefficients. However, the resulting equations are not so easy to solve, and we have to take more care in the reduction process of T_1 and T_2, which we think of being processed at the same time with increasing index of the λ variables. First, we note that

$$\begin{aligned}
c^1_k &= c^2_k = \tilde g_\Delta(k)\lambda^{00}_k - \tilde g_\Delta(a_{01}+k)\lambda^{01}_k \quad \text{for } k=1,\dots,s-\gamma_1-1\\
c^1_{s-\gamma_1} &= \tilde g_\Delta(s-\gamma_1)\lambda^{00}_{s-\gamma_1} - \tilde g_\Delta(a_{01}+s-\gamma_1)\lambda^{01}_{s-\gamma_1} + 1\\
c^2_{s-\gamma_1} &= \tilde g_\Delta(s-\gamma_1)\lambda^{00}_{s-\gamma_1} - \tilde g_\Delta(a_{01}+s-\gamma_1)\lambda^{01}_{s-\gamma_1} - 1
\end{aligned}$$

by the special form of φ. We want to organize the reduction process in such a way that in the intermediate stages the coefficients $\tilde c^1_k, \tilde c^2_k$ satisfy $\tilde c^1_k = \tilde c^2_k$ for $k=1,\dots,s-\gamma_1-1$ and $\tilde c^1_{s-\gamma_1} - \tilde c^2_{s-\gamma_1} = 2$ as long as possible. Because $v(T^1), v(T^2) \geq \gamma_1$ and $\gamma_1 + 2\mathbb{N} \subset \Gamma$, the even powers of t, $\tilde c^1_{2k} t^{\gamma_1+2k}$ in T^1 and $\tilde c^2_{2k} t^{a_{01}+\gamma_1+2k}$ in T^2, can be eliminated by subtracting elements of the form $\tilde c^1_{2k} t^{4l}\varphi^i m_{00}$, $i\in\{0,1\}$. Whereas one has to use different pairs of (l,i) for T^1 and T^2, the coefficient $\tilde c^1_{2k}$ is equal to $\tilde c^2_{2k}$ for $0<2k<s-\gamma_1$. Therefore, the coefficients of the resulting terms differ only in and after the $(s-\gamma_1)$–th t–power term; in particular, the differences $\tilde c^1_j - \tilde c^2_j$ for $j=1,\dots,s-\gamma_1$ are the same before and after the subtraction. The lower odd powers of t in T^1 and T^2 we do not eliminate at all while (a) the degree is less than s resp. $a_{01}+s$ and (b) there has not been an odd degree γ_1+2k+1 resp. $a_{01}+\gamma_1+2k+1$ where both degrees lie in Δ. After that we eliminate as many powers of t as possible in the usual way. By Lemma 11 $v(M)=\Delta$ holds iff the remaining coefficients $\tilde c^1_k, \tilde c^2_k, \tilde c^3_k$ of the reduced terms $\tilde T^1, \tilde T^2, \tilde T^3$ vanish. Our special treatment of the lower odd powers of t does not influence this, because for each of them $\tilde c^1_{2k+1} t^{\gamma_1+2k+1}$ resp. $\tilde c^2_{2k+1} t^{a_{01}+\gamma_1+2k+1}$ which we might have removed by subtraction there was a nonremovable $\tilde c^2_{2k+1} t^{a_{01}+\gamma_1+2k+1}$ resp. $\tilde c^1_{2k+1} t^{\gamma_1+2k+1}$ term, which forces $\tilde c^2_{2k+1}$

resp. $\tilde{c}^1_{2k+1}$ to vanish and with it the other one due to $\tilde{c}^1_{2k+1} = \tilde{c}^2_{2k+1}$. The advantage of this process is that we keep the difference $\tilde{c}^1_k - \tilde{c}^2_k$ fixed as long as possible. Let us exploit this.

We show that the 0–normalized associated semi–module Δ of an R–module has to be admissible, i.e., $\{s, a_{01}+s\} \cap \Delta \neq \emptyset$. Assume that this is not the case. Since $\{s, a_{01}+s\} \equiv \{1,3\} \bmod 4$, there is no odd number equal or less than s in Δ and condition (b) is satisfied up to $(s, a_{01}+s)$; hence $\tilde{c}^1_{s-\gamma_1} - \tilde{c}^2_{s-\gamma_1} = 2$ even at the end of the reduction process. Therefore, not both coefficients $\tilde{c}^1_{s-\gamma_1}, \tilde{c}^2_{s-\gamma_1}$ can vanish at the same time, and we cannot find higher order expressions for T^1 and T^2 simultaneously.

Now, we will show that when Δ is admissible the final equations $\tilde{c}^i_k = 0$ are solvable. We claim that either $\tilde{c}^i_k$ is already zero or

$$\tilde{c}^1_k = \lambda^{00}_k - \lambda^{01}_k + \ldots, \quad \tilde{c}^2_k = \lambda^{00}_k - \lambda^{01}_k + \ldots, \quad \tilde{c}^3_k = \lambda^{00}_k - \lambda^{10}_k + \ldots,$$

where the dots stand for polynomials in the λ with lower index less than k. This follows as before. We discuss as an example the coefficient $\tilde{c}^1_k$. Looking at the definition of $\tilde{c}^1_k$, we see immediately that

$$\tilde{c}^1_k = \tilde{g}_\Delta(k)\lambda^{00}_k - \tilde{g}_\Delta(a_{01}+k)\lambda^{01}_k + \ldots.$$

Now $\tilde{g}_\Delta(k) = 0$ or $\tilde{g}_\Delta(a_{01}+k) = 0$ implies $k \in \Delta$ or $a_{01}+k \in \Delta$ thus $\gamma_1 + k = a_{01} + k + 4\alpha_{01} \in \Delta$ and the term $\tilde{c}^1_k t^{\gamma_1+k}$ will be eliminated.

Again, we solve the equations $\tilde{c}^i_k = 0$ first for fixed index k and then successively substitute the solutions for the index less than k into the solutions for index k. The difficulty is that $\tilde{c}^1_k$ and $\tilde{c}^2_k$ have the same term $\lambda^{00}_k - \lambda^{01}_k$, and it is therefore impossible to solve these equations for λ^{00}_k and λ^{01}_k when $\tilde{c}^1_k$ and $\tilde{c}^2_k$ are nonzero and not the same. We have to treat two cases separately.

Let us assume that the smallest odd number $n \in \Delta \cap [\gamma_1, \infty[$ is less or equal to s. We visualize which coefficients can be eliminated by the following diagram:

	0	1	2	3	$\cdots$	$n-\gamma_1$	$+1$	$+2$	$+3$	$+4$	$\cdots$
T^1	0	$=/\circ$	$=/\times$	$=/\circ$	$\cdots$	$\times$	$\times$		$\times$	$\times$	$\cdots$
T^2	0	$=/\circ$	$=/\times$	$=/\circ$	$\cdots$		$\times$	$\times$	$\times$		$\cdots$

The k–th column stands for the coefficients of t^{γ_1+k} and $t^{a_{01}+\gamma_1+k}$ in T_1 and T_2; "=" stands for equal coefficients in the these terms, "$\times$" for "coefficient can be eliminated" and "$\circ$" for "coefficient cannot be eliminated". After the $(n-\gamma_1)$–th coefficient at least one of the coefficients with the same odd index k can be eliminated, because the corresponding t–powers have the degrees $(k, a_{01}+k)$, hence they are $(1,3)$ or $(3,1)$ modulo 4 and both are greater than n, thus one of them lies in $n + 4\mathbb{N} \subset \Delta$.

Therefore, up to the index $n-\gamma_1$, $\tilde{c}^1_k = \tilde{c}^2_k$, and after that at least one of the $\tilde{c}^1_k$, $\tilde{c}^2_k$ vanishes; thus there is at most one equation for each odd index and solving it is trivial. In addition, a higher order expression for T^3 is trivially obtained because the order of each of its t–powers is greater than the conductor $c(\Delta) = \max\{a_{ij}\} - 3 \leq \max\{n, n+\gamma_1, a_{01}\} \leq \gamma_2$. Therefore, we found explicit polynomial equations for the $g_\Delta(\gamma_1)$ nontrivial equations $\tilde{c}^1_k = 0$ and the additional $g_\Delta(a_{01}+n)$ nontrivial equations $\tilde{c}^2_k = 0$ with $k \geq n-\gamma_1$. This shows that the Δ–subset of J_X is a complex cell of dimension

$$\sum g_\Delta(a_{ij}) - g_\Delta(\gamma_1) - g_\Delta(a_{01}+n).$$

The second case we have to consider is when the smallest odd number of Δ is greater than s. Because Δ is admissible, we have $a_{01}+s \in \Delta$, in a diagram

	0	1	2	3	$\cdots$	$s-\gamma_1$	$+1$	$+2$	$+3$	$+4$	$\cdots$
T^1	0	$=/\circ$	$=/\times$	$=/\circ$	$\cdots$	$\circ$	$\times$		$\times$		$\cdots$
T^2	0	$=/\circ$	$=/\times$	$=/\circ$	$\cdots$	$\times$	$\times$		$\times$	$\times$	$\cdots$

During the reduction process for the first $s-\gamma_1-1$ coefficients we use only multiples of m_{00}. To eliminate the $t^{a_{01}+s}$–term of T^2, we have to use a multiple of m_{10} due to $a_{01}+s \equiv a_{10} \bmod 4$. Because of $c^2_{s-\gamma_1} = -1 + \ldots$, we add $(1+\ldots)t^{a_{01}+s-a_{10}}m_{10}$ to T^2, in particular we add λ^{10}_{2+4l} to the coefficient $\tilde{c}^2_{s-\gamma_1+2+4l}$ for $a_{01}+s+2+4l \notin \Delta$. Tracking the variables λ^{00}_k, λ^{01}_k, λ^{10}_k with the greatest index in these coefficients we find

$$\begin{aligned}
\tilde{c}^1_{s-\gamma_1+2+4l} &= \lambda^{00}_{s-\gamma_1+2+4l} - \lambda^{01}_{s-\gamma_1+2+4l} + \ldots \\
\tilde{c}^2_{s-\gamma_1+2+4l} &= \lambda^{00}_{s-\gamma_1+2+4l} - \lambda^{01}_{s-\gamma_1+2+4l} + \lambda^{10}_{2+4l} + \ldots
\end{aligned}$$

at this point, and this will not change later during the reduction process except when $\tilde{c}^1_{s-\gamma_1+2+4l}$ is made to vanish. Hence, $\tilde{c}^1_{s-\gamma_1+2+4l} = \tilde{c}^2_{s-\gamma_1+2+4l} = 0$ can be solved for λ^{10}_{2+4l} and $\lambda^{00}_{s-\gamma_1+2+4l}$. Due to $a_{01}+s+4l \in \Delta$, we have $\tilde{c}^2_{s-\gamma_1+4l} \equiv 0$ and solving $\tilde{c}^1_{s-\gamma_1+4l} = \tilde{c}^2_{s-\gamma_1+4l} = 0$ is trivial. Plugging these solutions back into T^1 and T^2, we see that with our reduction process we have found higher order expressions for T^1 and T^2 of the form

$$\begin{aligned}
\varphi m_{00} - t^{4\alpha_{01}} m_{01} \quad &= f_{00}m_{00} + f_{01}m_{01} + f_{10}m_{10} + f_{11}m_{11} \\
t^{4(q-\alpha_{01})}m_{00} - \varphi m_{01} &= g_{00}m_{00} + g_{01}m_{01} - t^{4(\alpha_{10}-\alpha_{01})}m_{10} + g_{10}m_{10} + g_{11}m_{11}
\end{aligned}$$

$$\begin{aligned}
&\text{with } v(f_{00}m_{00}), v(f_{01}m_{01}) > \gamma_1, \qquad v(f_{10}m_{10}), v(f_{11}m_{11}) > s \\
&\text{and } v(g_{00}m_{00}), v(g_{01}m_{01}) > a_{01}+\gamma_1,\ v(g_{10}m_{10}), v(g_{11}m_{11}) > a_{01}+s.
\end{aligned}$$

The amazing fact is that from these two equations we can find a higher order expression for T^3 without imposing further restrictions on the λ–variables. We multiply the first equation by φ and the second by $t^{4\alpha_{01}}$, then all products on the left hand sides are of order $2\gamma_1$. We subtract the second from the first equation and move the terms from the left hand side to the right hand side to obtain

$$0 = h_{00}m_{00} + h_{01}m_{01} - t^{4\alpha_{10}}m_{10} + h_{10}m_{10} + h_{11}m_{11} \qquad (+)$$

$$\text{with } v(h_{00}m_{00}), v(h_{01}m_{01}) > 2\gamma_1,\ v(h_{10}m_{10}), v(h_{11}m_{11}) > s+\gamma_1 = \gamma_2.$$

Assume that $v(h_{00}m_{00}), v(h_{01}m_{01}) < \gamma_2$, then cancellation of the initial terms takes place in $h_{00}m_{00} + h_{01}m_{01}$, i.e., $(\mathrm{in}(h_{00}), \mathrm{in}(h_{01}))$ is a syzygy of

$(\mathrm{in}(m_{00}), \mathrm{in}(m_{01}))$. As the syzygies of $(\mathrm{in}(m_{00}), \mathrm{in}(m_{01}))$ are generated by $(\mathrm{in}(\varphi), -t^{4\alpha_{01}})$ and $(t^{4(q-\alpha_{01})}, -\mathrm{in}(\varphi))$, we can find $r_1, r_2 \in R$ with

$$(\mathrm{in}(h_{00}), \mathrm{in}(h_{01})) = \mathrm{in}(r_1)(\mathrm{in}(\varphi), -t^{4\alpha_{01}}) + \mathrm{in}(r_2)(t^{4(q-\alpha_{01})}, -\mathrm{in}(\varphi));$$

thus in

$$T := r_1 T^1 + r_2 T^2 = (r_1\varphi + r_2 t^{4(q-\alpha_{01})})m_{00} + (-r_1 t^{4\alpha_{01}} - r_2\varphi)m_{01}$$

the coefficients of m_{00} and m_{01} have also the same initial terms, $(\mathrm{in}(h_{00}), \mathrm{in}(h_{01}))$. From the higher order expressions of T^1 and T^2 we obtain one for T, $\sum s_{ij}m_{ij}$. We subtract $T - \sum s_{ij}m_{ij} = 0$ from (+) to get rid of the initial terms of $h_{00}m_{00}$ and $h_{01}m_{01}$ in (+) without changing any of the extra conditions. Continuing this way we arrive at the stage where we may assume that $v(h_{00}m_{00}), v(h_{01}m_{01}) \geq \gamma_2$.

As γ_2 is the smallest odd number in Γ and $v(m_{01}) > 0$ is even, we conclude that $v(h_{01}m_{01}) = v(h_{01}) + v(m_{01}) > \gamma_2$. Therefore, the cancellation of the initial terms in (+) takes places between $h_{00}m_{00}$ and $t^{4\alpha_{10}}m_{10}$ with $v(h_{00}) = \gamma_2$, providing a higher order expression for the term $h_{00}m_{00} + t^{4\alpha_{10}}m_{10}$, which is essentially T^3 and may be used instead of it. Namely, because the syzygy $(\mathrm{in}(h_{00}), -t^{4\alpha_{10}})$ between $\mathrm{in}(m_{00})$ and $\mathrm{in}(m_{10})$ together with the above two syzygies between $\mathrm{in}(m_{00})$ and $\mathrm{in}(m_{01})$ generate all the syzygies of $\mathbb{C}[\Delta]$ below the degree of the conductor $c(\Delta)$, the conditions of the Lemma 11 are satisfied. As we solved the $g_\Delta(\gamma_1)$ nontrivial equations $\tilde{c}^1_k = 0$ and the additional $g_\Delta(a_{01} + s)$ nontrivial equations $\tilde{c}^2_k = 0$ by polynomial functions we have shown that the Δ–subset of the Jacobi factor J_R is a complex cell of dimension

$$\sum g_\Delta(a_{ij}) - g_\Delta(\gamma_1) - g_\Delta(a_{01} + s).$$

Summarizing we proved

Theorem 13 *Let R be the local ring of a unibranched plane singularity with characteristic Puiseux exponents $(4, 2q, s)$ and Δ be a δ_R–normalized $\langle 4, \gamma_1 = 2q, \gamma_2 = 2q + s\rangle$–semi–module, whose 0–normalization Δ_0 has a 2×2–basis $(a_{00} = 0, a_{01}; a_{10}, a_{11})$. Then the subset of modules of J_R with associated semi–module Δ is nonempty if Δ_0 is admissible, i.e., $\{s, a_{01} + s\} \cap \Delta_0 \neq \emptyset$. In this case it is biregular to an affine space $\mathbb{C}^N$ with*

$$N = \sum g_{\Delta_0}(a_{ij}) - g_{\Delta_0}(\gamma_1) - g_{\Delta_0}(a_{01} + n)$$

where n is the smallest odd number in $(\Delta_0 \cup \{s\}) \cap [\gamma_1, \infty[$.

As a consequence the number of admissible semi–modules (Proposition 9) is the Euler number of J_R, as stated in the main theorem.

In the next section we are going to prove an analogous theorem for the characteristic Puiseux exponents $(6, 8, s)$ and $(6, 10, s)$. It seems that there are no further Puiseux exponents where such a theorem holds — with the probable exception of $(6, 14, 15)$. The rows of the following table consist of a ring R and its associated semi–group Γ together with the 0–normalization of a Γ–semi–module Δ such that the Δ–subset of J_R is not affine, but $\mathbb{C}^N \times \mathbb{C}^*$, a union of two affine spaces, or worse.

R	Γ	Δ_0
$\mathbb{C}[t^6, t^{14}+t^{17}]$	$\langle 6,14,45\rangle$	$\langle 0,8,16,23,31,39\rangle$
$\mathbb{C}[t^6, t^9+t^{10}]$	$\langle 6,9,19\rangle$	$\langle 0,3,7,10,17,20\rangle$
$\mathbb{C}[t^6, t^9+t^{13}]$	$\langle 6,9,22\rangle$	$\langle 0,3,7,10,17,20\rangle$
$\mathbb{C}[t^9, t^{12}+t^{14}]$	$\langle 9,12,38\rangle$	$\langle 0,3,13,28,32,35\rangle$
$\mathbb{C}[t^{10}, t^{14}+t^{17}]$	$\langle 10,14,73\rangle$	$\langle 0,4,16,31,37\rangle$
$\mathbb{C}[t^8, t^{12}+t^{14}+t^{15}]$	$\langle 8,12,26,53\rangle$	$\langle 0,4,13,17,19,22,23\rangle$

4 The Puiseux exponents $(6,8,s)$ and $(6,10,s)$

In this section we deal with the cases when the characteristic Puiseux exponents of the singularity are $(6,8,s)$ and $(6,10,s)$. The above examples suggest that these two cases are the last ones where the natural cell decomposition of the Jacobi factor is affine. The basic ideas for the proof of this are the same as in the $(4,2q,s)$ case, but the arguments have to be sharpened. In particular, the combinatorics of the Γ–semi–modules is more complicated. As most of this section is very technical, we recommend it only for the most interested reader.

We follow the proof for the $(4,2q,s)$ case. Recall that Γ is generated by $\gamma_0=6$, $\gamma_1=2q$, and $\gamma_2=2\gamma_1+s$ and that a 0–normalized Γ–semi–module has a 2×3–basis, see Definition 6. We compute the number of Γ–semi–modules.

Proposition 14 *The number of* 0*–normalized* Γ*–semi–modules for the semi–group* $\Gamma=\langle 6,2q,4q+s\rangle$ *with* $\gcd(q,3)=\gcd(s,2)=1$ *is*

$$\frac{(q+1)(q+2)(7q^3+24q^2+29q+15)}{180}+s\frac{(q+1)^2(q+2)^2}{72}.$$

Proof. This time we have to count the number of 5–tuples $\alpha=(\alpha_{01},\alpha_{02},\alpha_{10},\alpha_{11},\alpha_{12})$ which satisfy

$$\begin{array}{ccccccc}
 & & q/3 & & 2q/3 & & \\
 & & \vee & & \vee & & \\
0 & \leq & \alpha_{01} & \leq & \alpha_{02} & & \\
\wedge\!\mid & & \wedge\!\mid & & \wedge\!\mid & & \\
\alpha_{10} & \leq & \alpha_{11} & \leq & \alpha_{12} & \leq q+\alpha_{10}. & \\
\wedge & & \wedge & & \wedge & & \\
\frac{4q+s}{6} & & \frac{6q+s}{6} & & \frac{8q+s}{6} & &
\end{array}$$

Let A be the set of these. We may view this set as $A=\bar{A}\setminus(A_3\cup A_4)$ with

$$\bar{A}=\Big\{\alpha\in\mathbb{N}^5 \mid \alpha_{01}<\tfrac{q}{3}, \alpha_{01}\leq\alpha_{02}<\tfrac{2q}{3}; \alpha_{1j}<\tfrac{(4+2j)q+s}{6}, \\ \alpha_{10}\leq\alpha_{11}\leq\alpha_{12}\leq q+\alpha_{10}\Big\}$$

$$\begin{aligned}
A_3 &= \bar{A}\cap\{\alpha\in\mathbb{N}^5 \mid \alpha_{02}>\alpha_{12}\} \\
&= \{\alpha\in\mathbb{N}^5 \mid \alpha_{01}<\tfrac{q}{3}, \alpha_{01}\leq\alpha_{02}<\tfrac{2q}{3}; \alpha_{10}\leq\alpha_{11}\leq\alpha_{12}<\alpha_{02}\} \\
A_4 &= \bar{A}\cap\{\alpha\in\mathbb{N}^5 \mid \alpha_{02}\leq\alpha_{12}, \alpha_{01}>\alpha_{11}\} \\
&= \{\alpha\in\mathbb{N}^5 \mid \alpha_{01}<\tfrac{q}{3}, \alpha_{01}\leq\alpha_{02}<\tfrac{2q}{3}; \alpha_{10}\leq\alpha_{11}<\alpha_{01}, \alpha_{02}\leq\alpha_{12}\leq q+\alpha_{10}\}.
\end{aligned}$$

We split $\bar{A}$ again as $\bar{A} = A_0 \setminus (A_1 \cup A_2)$ with

$$\begin{aligned}
A_0 &= \left\{\alpha \in \mathbb{N}^5 \mid \alpha_{01} < \tfrac{q}{3}, \alpha_{01} \le \alpha_{02} < \tfrac{2q}{3}; \alpha_{10} < \tfrac{4q+s}{6}, \alpha_{10} \le \alpha_{11} \le \alpha_{12} \le q + \alpha_{10}\right\} \\
A_1 &= A_0 \cap \left\{\alpha \in \mathbb{N}^5 \mid \alpha_{11} > \tfrac{6q+s}{6}\right\} \\
&= \Big\{\alpha \in \mathbb{N}^5 \mid \alpha_{01} < \tfrac{q}{3}, \alpha_{01} \le \alpha_{02} < \tfrac{2q}{3}; \tfrac{s}{6} < \alpha_{10} < \tfrac{4q+s}{6}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \tfrac{6q+s}{6} < \alpha_{11} \le \alpha_{12} \le q + \alpha_{10}\Big\} \\
A_2 &= \bar{A} \cap \left\{\alpha \in \mathbb{N}^5 \mid \alpha_{11} < \tfrac{6q+s}{6}, \alpha_{21} > \tfrac{8q+s}{6}\right\} \\
&= \Big\{\alpha \in \mathbb{N}^5 \mid \alpha_{01} < \tfrac{q}{3}, \alpha_{01} \le \alpha_{02} < \tfrac{2q}{3}; \tfrac{2q+s}{6} < \alpha_{10} < \tfrac{4q+s}{6}, \alpha_{10} \le \alpha_{11} < \tfrac{6q+s}{6}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \tfrac{8q+s}{6} < \alpha_{12} \le q + \alpha_{10}\Big\}\,.
\end{aligned}$$

By definition the $A_1, \ldots, A_4 \subset A_0$ are pairwise disjoint, thus $\#A = \#A_0 - \sum_{i=1}^{4} \#A_i$. The sets A_i are written down in such a way that when one reads the inequalities from the left to the right there are only restrictions on the newly appearing variables; hence, they can be counted, for example

$$\#A_1 = \sum_{\alpha_{01}=0}^{\lfloor \frac{q}{3} \rfloor} \sum_{\alpha_{02}=\alpha_{01}}^{\lfloor \frac{2q}{3} \rfloor} \sum_{\alpha_{10}=\lceil \frac{s}{6} \rceil}^{\lfloor \frac{4q+s}{6} \rfloor} \sum_{\alpha_{11}=\lceil \frac{6q+s}{6} \rceil}^{q+\alpha_{10}} (q + 1 + \alpha_{10} - \alpha_{11}).$$

It is possible to evaluate these sums and obtain for $\#A$ the number in the statement. □

Next we count the Γ–semi–modules of which we show later that they are not the 0–normalization of an associated semi–module of a torsion free module over the local ring of the singularity.

Proposition 15 *Let* $\Gamma = \langle 6, 2q, 4q + s\rangle$ *be the above semi–group with* $q \in \{4, 5\}$. *The number of admissible* Γ*–semi–modules is*

$$\frac{(q+1)(q+2)(7q^3 + 24q^2 + 29q + 15)}{180} - \frac{2(4q+7)}{15}\binom{q+2}{4} + s\frac{(q+1)^2(q+2)^2}{72}.$$

Proof. A proof of this Proposition can be obtained by mixing the ideas of the proofs of Proposition 9 and Proposition 14. □

Evaluating this formula for $q = 4, 5$, one obtains the numbers given in the statement of the Main Theorem.

It remains to compute the syzygies of the canonical generators of the $\mathbb{C}[\Gamma]$–module $\mathbb{C}[\Delta]$. We continue to use the notation of Section 2

Proposition 16 *Let* $\Gamma = \langle \gamma_0 = 2p,\ \gamma_1 = 2q,\ \gamma_2 = 2(p-1)q + s\rangle$, *choose* $\beta \in \{0, \ldots, p-1\}$ *and* $\eta \in \mathbb{N}$ *such that* $2\gamma_2 = \beta\gamma_1 + \eta\gamma_0$. *Further, let* Δ *be a* 0*–normalized* Γ*–semi–module with* 2×3*–basis* (a_{ij}). *Then the* $\mathbb{C}[\Gamma]$*–module* $\mathbb{C}[\Delta]$ *is generated by* $A = (t^{a_{ij}})$, *and the syzygies of this generating set are generated by the following additively written bivector syzygies:*

- $a_{ij} + \gamma_1 = a_{i,j+1} + *\gamma_0$

- $a_{0j} + \gamma_2 = a_{1,j-\mu_j} + \mu_j\gamma_1 + *\gamma_0$ *where* μ_j *is chosen maximal under the condition* $a_{0j} + \gamma_2 \geq a_{1,j-\mu_j} + \mu_j\gamma_1$.
- $a_{1j} + \gamma_2 = a_{0,j+\beta-\nu_j} + \nu_j\gamma_1 + *\gamma_0$ *where* ν_j *is chosen maximal under the condition* $a_{1j} + \gamma_2 \geq a_{0,j+\beta-\nu_j} + \nu_j\gamma_1$.

Here, we use cyclic index notation and $*$ *stands for an easily computed unique natural number.*

Proof. By Lemma 4 there is a generating set of syzygies consisting of bivectors. Any bivector syzygy of degree d may be written additively as

$$a_{ij} + \xi_2\gamma_2 + \xi_1\gamma_1 + \xi_0\gamma_0 = a_{lk} + \zeta_2\gamma_2 + \zeta_1\gamma_1 + \zeta_0\gamma_0 = d$$

with $\xi_2, \zeta_2 \in \{0, 1\}$, $\xi_1, \zeta_1 \in \{0, \ldots, p-1\}$, and $\xi_0, \zeta_0 \in \mathbb{N}$. We may assume that the bivector syzygy is not the multiple of another; hence, for all $r \in \{0, 1, 2\}$ one of the $\xi_r, \zeta_r = 0$ is zero. Considering the above relations modulo γ_0 and using $2\gamma_2 = \beta\gamma_1 + \eta\gamma_0$ one sees

$$i + \xi_2 \equiv l + \zeta_2 \mod 2 \quad \text{and} \quad j + \xi_1 + \beta\delta^2_{i+\xi_2} \equiv k + \zeta_1 + \beta\delta^2_{l+\zeta_2} \mod p;$$

here δ^2_n is the Kronecker δ–symbol, i.e., $\delta^2_n = 0$ except for $n = 2$ where $\delta^2_2 = 1$. In particular, a minimal syzygy between the $\{a_{ij}\}$ for fixed i does not involve a γ_2. Therefore, any such bivector syzygy is of the type $a_{ij} + k\gamma_1 = a_{i,j+k} + *\gamma_0$, and thus a combination of the $a_{il} + \gamma_1 = a_{i,l+1} + *\gamma_0$ for $l = j, \ldots, j+k-1$. Next, we note that a syzygy $a_{0j} + \gamma_2 + k\gamma_1 = a_{1,j+k} + *\gamma_0$ is a combination of $a_{0j}+\gamma_2 = a_{1,j}+*\gamma_0$ and $a_{1j}+k\gamma_1 = a_{1,j+k}+*\gamma_0$, and similarly for $a_{1j}+\gamma_2+k\gamma_1 = a_{0,j+\beta+k}+*\gamma_0$. Also, if there exists a relation of the type $a_{0j}+\gamma_2+*\gamma_0 = a_{1,j-k}+k\gamma_1$, it can be obtained from $a_{0j}+\gamma_2 = a_{1,j}+*_1\gamma_0$ and $a_{1,j-k}+k\gamma_1 = a_{1,j}+*_2\gamma_0$ using the assumed relation to see that $*_2 \geq *_1$. Again, an analogous statement holds for $a_{1j} + \gamma_2 + *\gamma_0 = a_{0,j+\beta-k} + k\gamma_1$.

Thus it remains to show that all relations of the type $a_{ij} + \gamma_2 = \ldots$ can be obtained from the ones in the statement of the theorem. Since the ones with the most γ_1's on the right hand side for each fixed a_{0j} or a_{1j} on the left hand side are the ones in the statement, the other can be obtained from them by replacing $a_{ij} + k\gamma_1$ on the right hand side by the corresponding $a_{i,j+k} + *\gamma_0$. □

Our main interest are the syzygies whose degree is less than the conductor of the module. Let us isolate these for $p = 3$.

Corollary 17 *Let* $\Gamma = \langle\gamma_0 = 6,\ \gamma_1 = 2q,\ \gamma_2 = 4q + s\rangle$, *and* $\Delta = \bigcup(a_{ij} + \gamma_0\mathbb{N})$ *like above. The* $\mathbb{C}[\Gamma]$*–module* $\mathbb{C}[\Delta]$ *is generated by* $A = (1, t^{a_{01}}, t^{a_{02}}, t^{a_{10}}, t^{a_{11}}, t^{a_{12}})$, *and the syzygies of* A *of degree less than* $c(\Delta)$ *are generated by the following additively written bivector syzygies:*

— *If* $\max\{a_{ij}\} = a_{0J}$ *for a suitable* $J \in \{1, 2\}$ *use the following relations:*

- $a_{02} + \gamma_1 = a_{00} + *\gamma_0$ *if* $J = 1$ *or* $a_{00} + \gamma_1 = a_{01} + *\gamma_0$ *if* $J = 2$.
- $a_{1j} + \gamma_1 = a_{1,j+1} + *\gamma_0$ *for* $j \in \{0, 1, 2\}$

— *If* $\max\{a_{ij}\} = a_{1J}$ *for a suitable* $J \in \{0, 1, 2\}$, *choose* $K, L \in \{0, 1, 2\} \setminus \{I\}$ *with* $K + 1 \equiv L \bmod 3$ *and use the following relations:*

- $a_{0j} + \gamma_1 = a_{0,j+1} + *\gamma_0$ *for* $j \in \{0,1,2\}$
- $a_{1K} + \gamma_1 = a_{1L} + *\gamma_0$
- $a_{0K} + \gamma_2 = a_{1K} + *\gamma_0$
- $a_{0L} + \gamma_2 = a_{1K} + \gamma_1 + *\gamma_0$ *if* $a_{0L} + \gamma_2 \geq a_{1K} + \gamma_1$
  $a_{0L} + \gamma_2 = a_{1L} + *\gamma_0$ *else.*

Proof. Obviously, any relation involving the maximal a_{ij} has degree greater than $c(\Delta)$. Therefore, for the first case it is enough to remark that because of $\max\{a_{ij}\} = a_{0J} \leq 2\gamma_1 < \gamma_2$ any relation involving a γ_2 has also degree greater than $c(\Delta)$. For the more general second case we need only to argue for $a_{1j}+\gamma_2 > c(\Delta)$. By the definition of the a_{ij} we have $a_{0j} \leq 2\gamma_1$, $a_{1,j+1} \leq a_{1j}+\gamma_1$, and $a_{1,j+2} \leq a_{1j} + 2\gamma_1$; hence, $c(\Delta) < \max\{a_{lk}\} \leq a_{1j} + 2\gamma_1 < a_{1j} + \gamma_2$. □

Finally, it remains to prove that the Δ–subsets of the Jacobi factor J_R are affine. As shown by examples at the end of the last section this is probably only possible for the characteristic Puiseux exponents $(6, 8, s)$ and $(6, 10, s)$.

Theorem 18 *Let R be the local ring of a unibranched plane singularity with characteristic Puiseux exponents $(6, 8, s)$ or $(6, 10, s)$ and Γ its associated semi–group. Let Δ be a δ_R–normalized Γ–semi–module. Then the Δ–subset of J_R is biregular to an affine space $\mathbb{C}^N$ and nonempty iff the 0–normalization of Δ is admissible.*

The proof proceeds as before. Again, the local ring of a singularity is isomorphic to $\mathbb{C}[[t^6, \varphi]] \in \mathbb{C}[[t]]$, where $\varphi = t^{\gamma_1} + t^s + \ldots$, because by a coordinate transformation any t–power whose exponent lies in $\Gamma \setminus \{\gamma_1\}$ or in $((\gamma_1 - \gamma_0) + \Gamma) \setminus \{\gamma_1\}$ can be eliminated [Z, p. 784]. These two characteristic Puiseux exponents series are the only ones — apart from the ones already discussed — where there are no t–powers in φ between the t–powers to the second and third Puiseux exponent. We denote the normalized element of R of order $\gamma_2 = 2\gamma_1 + s$ by ψ.

Let (a_{ij}) be the 2×3–basis of the 0–normalization of Δ. We will work during this proof only with the 0–normalization and may therefore denote it by Δ as well. For an R–module M with associated semi–module Δ we have the ansatz

$$m_{ij} = t^{a_{ij}} + \sum_{k \in]a_{ij},\infty[\setminus \Delta} \lambda^{ij}_{k-a_{ij}} t^k$$

for its six generators. These generators must satisfy the condition of Lemma 11 for the syzygies of Corollary 17. The most interesting syzygies are the ones between 1, $t^{a_{01}}$, and $t^{a_{02}}$. They lead to the terms

$$\begin{aligned}
T^1 &:= \varphi m_{00} - t^{6\alpha_{01}} m_{01} &&=: \sum_{k=1}^{\infty} c^1_k t^{\gamma_1+k} \\
T^2 &:= \varphi m_{01} - t^{6(\alpha_{02}-\alpha_{01})} m_{02} &&=: \sum_{k=1}^{\infty} c^2_k t^{a_{01}+\gamma_1+k} \\
T^3 &:= \varphi m_{02} - t^{6(q-\alpha_{02})} m_{00} &&=: \sum_{k=1}^{\infty} c^3_k t^{a_{02}+\gamma_1+k}
\end{aligned}$$

for which we have to find higher order expressions. We proceed as before: reduce T^1, T^2, T^3 with respect to $\{m_{ij}\}$ in some modified way, solve the equations given by the remaining coefficients for a fixed index k and successively substitute these solutions into each other.

As before we find that

$$\begin{aligned}
c_k^j &= \tilde{g}_\Delta(a_{0,j-1}+k)\lambda_k^{0,j-1} - \tilde{g}_\Delta(a_{0j}+k)\lambda_k^{0j} \quad \text{for } k=1,\dots,s-\gamma_1-1 \quad \text{and} \\
c_{s-\gamma_1}^j &= \tilde{g}_\Delta(a_{0,j-1}+s-\gamma_1)\lambda_{s-\gamma_1}^{0,j-1} - \tilde{g}_\Delta(a_{0j}+s-\gamma_1)\lambda_{s-\gamma_1}^{0j} + 1.
\end{aligned}$$

We note that the sum $c_k^1 + c_k^2 + c_k^3$ is zero for $k = 1,\dots,s-\gamma_1-1$ and $c_{s-\gamma_1}^1 + c_{s-\gamma_1}^2 + c_{s-\gamma_1}^3 = 3$. These are the invariants that we want to keep as long as possible during our modified reduction process. First, we consider the elimination of the even t–powers. Since all even numbers greater than or equal to $2\gamma_1 - 4$ are contained in Γ, we can subtract — for fixed even k — appropriate multiples of m_{00} from all T^j to eliminate the terms $\tilde{c}_k^j t^{a_{0j}+\gamma_1+k}$ when $\gamma_1 + k \geq 2\gamma_1 - 4$. This does not change the sum conditions, because modulo the ideals (t^{s+1}), $(t^{a_{01}+s+1})$, resp. $(t^{a_{02}+s+1})$ we subtract $\tilde{c}_k^1 t^{\gamma_1+k} m_{00}$ from T^1, $\tilde{c}_k^2 t^{a_{01}+\gamma_1+k} m_{00}$ from T^2, and $\tilde{c}_k^3 t^{a_{02}+\gamma_1+k} m_{00} = -(\tilde{c}_k^1 + \tilde{c}_k^2) t^{a_{02}+\gamma_1+k} m_{00}$ from T^3. For $\gamma_1 = 8$ this leaves only the terms $c_2^j t^{a_{0j}+\gamma_1+2}$ to discuss. If $a_{01} = 2$, then $a_{00}+\gamma_1+2, a_{02}+\gamma_1+2 \in a_{01}+\Gamma$ and $a_{01}+\gamma_1+2, a_{02}+\gamma_1+2 \in a_{00}+\Gamma$; thus, we can subtract $\tilde{c}_2^1 \varphi m_{01}$ from T^1, $\tilde{c}_2^2 t^{12} m_{00}$ from T^2, and add $\tilde{c}_2^1 t^{6(4-\alpha_{02})} m_{01} + \tilde{c}_2^2 \varphi t^{6(3-\alpha_{02})} m_{00}$ to T^3. Due to $\tilde{c}_2^3 = -\tilde{c}_2^1 - \tilde{c}_2^2$, this eliminates all the $\tilde{c}_2^j$ coefficients and leaves the sum condition intact. If $a_{01} = 8$ and $a_{02} \in \{4, 10\}$, then $a_{00}+\gamma_1+2, a_{01}+\gamma_1+2 \in a_{02}+\Gamma$ and $a_{01}+\gamma_1+2, a_{02}+\gamma_1+2 \in a_{00}+\Gamma$ and an analogous subtraction and addition works. The case of $a_{01} = 8$ and $a_{02} = 16$ is trivial because here m_{01} and m_{02} must be the normalization of φm_{00} resp. $\varphi^2 m_{00}$. For $\gamma_1 = 10$ the same ideas work, because we can always find two indices j_1, j_2 such that

$$\begin{aligned}
&\#\left(\{a_{0j}+\gamma_1+2k \mid j=0,1,2\} \cap (a_{0j_\varrho}+\Gamma)\right) \geq 2 \quad \text{for} \quad \varrho = 1,2 \quad \text{and} \\
&\{a_{0j}+\gamma_1+2k \mid j=0,1,2\} \subset (a_{0j_1}+\Gamma) \cup (a_{0j_2}+\Gamma).
\end{aligned}$$

An analogous result does not hold for $\gamma_1 > 10$.

Now, we consider the elimination of the odd t–powers. If we can eliminate only one of the terms $\tilde{c}_k^2 t^{a_{01}+\gamma_1+k}$ of T^2 or $\tilde{c}_k^3 t^{a_{02}+\gamma_1+k}$ of T^3 for an odd index $k < s-\gamma_1$ and also not $\tilde{c}_k^1 t^{\gamma_1+k}$ of T^1, we do not eliminate at all. Because we later force the remaining two coefficients to be zero, the third will be zero as well due to the sum condition. Therefore, we will still find a higher order expression for T^1, T^2, T^3 by this modified reduction process. As soon as we find an odd index n with $\gamma_1+n \in \Delta$ or $a_{01}+\gamma_1+n \in \Delta$ and $a_{02}+\gamma_1+n \in \Delta$, we eliminate all possible t–powers. We claim that these conditions imply that at least one of each of the following triples of the odd exponents $(a_{0j}+\gamma_1+n+2k)_{j=0,1,2}$ lies in Δ — with the exception of the trivial case of $a_{01} = \gamma_1$ and $a_{02} = 2\gamma_1$. If $\gamma_1+n \in \Delta$ this is obvious, as $\{a_{0j}+\gamma_1+n+2k \mid j=0,1,2\} \equiv \{1,3,5\} \bmod 6$ and $a_{0j}+\gamma_1+n+2k > \gamma_1+n$ and hence $\{a_{0j}+\gamma_1+n+2k \mid j=0,1,2\} \cap (\gamma_1+n+6\mathbb{N})$ is nonempty. If $a_{01}+\gamma_1+n \in \Delta$ and $a_{02}+\gamma_1+n \in \Delta$, this statement has to be checked case by case. We do this with the help of the following diagrams that indicate which odd terms can be eliminated; the second column stands for

t–powers with the exponents $(a_{0j}+\gamma_1+n)_{j=0,1,2}$, the third for the exponents $(a_{0j}+\gamma_1+n+2)_{j=0,1,2}$ and the last one for the exponents $(a_{0j}+\gamma_1+n+4)_{j=0,1,2}$. Since $\Delta+6\mathbb{N}\subset\Delta$, it is enough to consider only the next two odd numbers. For $\gamma_1=8$ we get

	$+0$	$+2$	$+4$
$a_{00}=0$		$\times$	
$a_{01}=2$	$\times$		
$a_{02}=?$	$\times$		$\times$

	$+0$	$+2$	$+4$
$a_{00}=0$			$\times$
$a_{01}=8$	$\times$	$\times$	
$a_{02}=4$	$\times$		

	$+0$	$+2$	$+4$
$a_{00}=0$			
$a_{01}=8$	$\times$	$\times$	
$a_{02}=10$	$\times$		$\times$

For $\gamma_1=10$ we get

	$+0$	$+2$	$+4$
$a_{00}=0$			$\times$
$a_{01}=4$	$\times$		
$a_{02}=?$	$\times$	$\times$	

	$+0$	$+2$	$+4$
$a_{00}=0$		$\times$	
$a_{01}=10$	$\times$	$\times$	$\times$
$a_{02}=2$	$\times$		

	$+0$	$+2$	$+4$
$a_{00}=0$			
$a_{01}=10$	$\times$		$\times$
$a_{02}=8$	$\times$	$\times$	

	$+0$	$+2$	$+4$
$a_{00}=0$			
$a_{01}=10$	$\times$		$\times$
$a_{02}=14$	$\times$	$\times$	

We are ready to prove that for any R–modules M its 0–normalized associated semi–module Δ is admissible, i.e., it has a nonempty intersection with $\{s, a_{01}+s, a_{02}+s\}$. If it were empty, then by the above discussion we have to apply only operations during the reduction that do not change the sum condition; hence, the requirement that after the normalization process the coefficients $\tilde{c}^1_{s-\gamma_1}, \tilde{c}^2_{s-\gamma_1}, \tilde{c}^3_{s-\gamma_1}$ have to vanish contradicts the fact that their sum is three.

We need to show that in the remaining cases the equations can be solved by expressing some of the λ–variables as polynomials of the other. The coefficients c^1_k, c^2_k, c^3_k are of the form

$$c^j_k = \tilde{g}_\Delta(a_{0,j-1}+k)\lambda^{0,j-1}_k - \tilde{g}_\Delta(a_{0j}+k)\lambda^{0j}_k + \ldots$$

where the lower dots stand for polynomials in the λ with indices less than k. During the reduction process some of the c^k_1, c^k_2, c^k_3 are made to vanish, in particular those where the gap function $\tilde{g}_\Delta$ assumes the value zero by the usual arguments. In the end we are left with either zero coefficients or coefficients $\tilde{c}^j_k$ that look like

$$\tilde{c}^1_k = \lambda^{00}_k - \lambda^{01}_k + \ldots \quad \tilde{c}^2_k = \lambda^{01}_k - \lambda^{02}_k + \ldots \quad \tilde{c}^3_k = \lambda^{02}_k - \lambda^{00}_k + \ldots.$$

For fixed k we can obviously solve the equations $\tilde{c}^j_k = 0$ for $\lambda^{00}_k, \lambda^{01}_k, \lambda^{02}_k$ if their sum is zero, which is the case for $k < \min\{n, s-\gamma_1\}$, or if at least one of them is zero, which is always the case for $k \geq n$. Thus it remains to discuss the coefficients with indices in the range $]s-\gamma_1, n[$. This range is nonempty only if Δ contains no odd number less or equal to s and either $a_{01}+s\in\Delta$ or $a_{02}+s\in\Delta$, but not both. Let us start with $a_{01}+s\in\Delta$. Assume that we reduced all T^j for the t–powers with exponents less then $a_{0j}+s$. Because $a_{01}+s\equiv\gamma_1+s\equiv\gamma_2+2\gamma_1\equiv a_{12} \bmod 6$ we can subtract $\tilde{c}^2_{s-\gamma_1}t^{a_{01}+s-a_{12}}m_{12}$ from the term T^2 to eliminate the $\tilde{c}^2_{s-\gamma_1}t^{a_{01}+s}$ term. As the constant term of the original $c^2_{s-\gamma_1}$ is -1, the constant term of $\tilde{c}^2_{s-\gamma_1}$ is -1, too. Thus we are adding λ^{12}_k to $\tilde{c}^2_{s-\gamma_1+k}$ for all k with $a_{01}+s+k\notin\Delta$. Tracking again the variables $\lambda^{0j}_k,\lambda^{12}_k$ with the greatest index, we find that at this moment in the process we have for the coefficients $\tilde{c}^j_{s-\gamma_1+k}$ with $a_{0j}+s+k\notin\Delta$ — the others are made to vanish later on anyway —

$$\tilde{c}^1_{s-\gamma_1+k}=\lambda^{00}_{s-\gamma_1+k}-\lambda^{01}_{s-\gamma_1+k}+\dots\quad \tilde{c}^2_{s-\gamma_1+k}=\lambda^{01}_{s-\gamma_1+k}-\lambda^{02}_{s-\gamma_1+k}+\lambda^{12}_k+\dots$$
$$\tilde{c}^3_{s-\gamma_1+k}=\lambda^{02}_{s-\gamma_1+k}-\lambda^{00}_{s-\gamma_1+k}+\dots$$

and this will not change later in the process. Now there is no difficulty in solving these equations for $\lambda^{00}_{s-\gamma_1+k}$, $\lambda^{02}_{s-\gamma_1+k}$, and λ^{12}_k.

The case of $a_{02}+s\in\Delta$ is similar, one uses a multiple of m_{10} for the term T^3. In the whole we have shown so far:

The existence of higher order expressions for the terms T^1,T^2,T^3 can be expressed as a polynomial dependence of some of the λ–variables on the other λ–variables.

Now we have to find higher order expressions for the terms derived from the remaining syzygies of the canonical generating set of $\mathbb{C}[\Delta]$. The case where $\max\{a_{ij}\}=a_{0J}$ — see Corollary 17 – is nearly trivial. In fact, as there is only one interesting cancellation of initial terms between the m_{00},m_{01},m_{02} getting the condition of Lemma 11 to hold for it is trivial and the above discussion is not needed here. The three cyclic cancellations of initial terms between the m_{10},m_{11},m_{12} derived from the syzygies between $(t^{a_{10}},t^{a_{11}},t^{a_{12}})\in\mathbb{C}[\Delta]$ are

$$\begin{aligned}
T^4 &:= \varphi m_{10}-t^{6(\alpha_{11}-\alpha_{10})}m_{11} &&=: \sum_{k=1}^{\infty} c^4_k t^{a_{10}+\gamma_1+k}\\
T^5 &:= \varphi m_{11}-t^{6(\alpha_{12}-\alpha_{11})}m_{12} &&=: \sum_{k=1}^{\infty} c^5_k t^{a_{11}+\gamma_1+k}\\
T^6 &:= \varphi m_{12}-t^{6(q+\alpha_{10}-\alpha_{12})}m_{10} &&=: \sum_{k=1}^{\infty} c^6_k t^{a_{12}+\gamma_1+k}.
\end{aligned}$$

These terms are easily expressed as higher order expressions. Namely, the coefficients have again the typical form

$$c^4_k=\lambda^{10}_k-\lambda^{11}_k+\dots\quad c^5_k=\lambda^{11}_k-\lambda^{12}_k+\dots\quad c^6_k=\lambda^{12}_k-\lambda^{10}_k+\dots;$$

here we suppress the gap function in front of the λ, because the coefficients where it is relevant will be made to vanish later on. We will show that for fixed k at least one of the coefficients vanishes during the reduction process. Let $J\in\{0,1,2\}$ be such that $\min_j\{a_{1j}\}=a_{1J}$. Then we see that $\{a_{1j}+\gamma_1+k\mid j=0,1,2\}\cap 6\mathbb{N}\neq\emptyset$ for even k and $\{a_{1j}+\gamma_1+k\mid j=0,1,2\}\cap(a_{1J}+6\mathbb{N})\neq\emptyset$ for odd k by considering the numbers modulo 6; thus at least one of the t–powers

$t^{a_{1j}+\gamma_1+k}$, $j = 0, 1, 2$, can be eliminated. In the end, at most two of three equations $\tilde{c}^4_k = \tilde{c}^5_k = \tilde{c}^6_k = 0$ are nontrivial and solving them for one or two of the λ^{1j}_k is easy.

We turn to the case of the syzygies of $\mathbb{C}[\Delta]$ described in Corollary 17, where $\max\{a_{ij}\} = a_{1J}$ and $K, L \in \{0, 1, 2\}$ with $K + 1 \equiv L$ and $K + 2 \equiv J$ modulo 3. We have to find higher order expressions for the terms

$$\begin{aligned}
T^4 &:= \varphi m_{1K} - t^{6*} m_{1L} &&=: \sum_{k=1}^{\infty} c^4_k t^{a_{1K}+\gamma_1+k} \\
T^5 &:= \psi m_{0K} - t^{6(\alpha_{1K}-\alpha_{0K})} m_{1K} &&=: \sum_{k=1}^{\infty} c^5_k t^{a_{0K}+\gamma_2+k} \\
T^6 &:= \psi m_{0L} - \varphi t^{6*} m_{1K} &&=: \sum_{k=1}^{\infty} c^6_k t^{a_{0L}+\gamma_2+k} \\
T^{6\prime} &:= \psi m_{0L} - t^{6(\alpha_{1L}-\alpha_{0L})} m_{1L} &&=: \sum_{k=1}^{\infty} c^{6\prime}_k t^{a_{0L}+\gamma_2+k}
\end{aligned}$$

where one uses T^6 if $a_{0L} + \gamma_2 \geq a_{1K} + \gamma_1$ and $T^{6\prime}$ otherwise. The coefficients are

$$\begin{aligned}
c^4_k &= \lambda^{1K}_k - \lambda^{1L}_k + \ldots & \qquad c^5_k &= \lambda^{0K}_k - \lambda^{1K}_k + \ldots \\
c^6_k &= \lambda^{0L}_k - \lambda^{1K}_k + \ldots & \qquad c^{6\prime}_k &= \lambda^{0L}_k - \lambda^{1L}_k + \ldots,
\end{aligned}$$

where we suppressed the gap function again.

Now if Δ contains an odd number $n \leq s + 6$ — for example s itself — then $n + \gamma_1, n + 2\gamma_1 \in \Delta$ and $\{n, n + \gamma_1, n + 2\gamma_1\} \equiv \{1, 3, 5\} \bmod 6$, thus $c(\Delta) \leq n + 2\gamma_1 - 6 + 1 \leq \gamma_2 + 1$. Consequently $c(\Delta) \leq \gamma_2$, because $\gamma_2 \in \Gamma \subset \Delta$. Therefore, the only t–powers in the terms $T^4, T^5, T^6, T^{6\prime}$ whose exponents may be less than $c(\Delta)$ occur in the term T^4. Solving the coefficients of its reduction is trivial, even if we already used up either λ^{1K}_k or λ^{1L}_k before.

Another exceptional case in the treatment of T^1, T^2, T^3 was when there is an even number k with $a_{01} + s - k, a_{02} + s - k \in \Delta$. Choose $I \in \{0, 1, 2\}$ such that $a_{01} + s - k \equiv a_{1I} \bmod 6$, then $a_{1I} \leq a_{01} + s - k \leq \gamma_2 - \gamma_1 - 2$, $a_{1,I+1} \leq \gamma_2 - 2$, $a_{1,I+2} \leq \gamma_2 + \gamma_1 - 2$ and we get $c(\Delta) \leq \gamma_2 + \gamma_1 - 2 - 6 + 1 \leq \gamma_2 + 3$ due to $\gamma_1 \leq 10$. Now the terms T^5, T^6 resp. $T^{6\prime}$ have order greater then $a_{0K} + \gamma_2$ and $a_{0L} + \gamma_2$. One of the a_{0K}, a_{0L} is at least 2, thus the order of the corresponding term is equal to or greater than the conductor $c(\Delta)$ and a higher order expression can be found trivially. Therefore, we need to consider only one of the terms $T^5, T^6/T^{6\prime}$ besides T^4. They contain the so far unused variables $\lambda^{1K}_k, \lambda^{1L}_k$ and finding higher order expressions for them is easy.

The final exceptional case we had during the search for higher order expressions for T^1, T^2, T^3 was when $a_{01} = \gamma_1$ and $a_{02} = 2\gamma_1$. If $\max\{a_{1i}\} \neq a_{12}$, then $a_{0K} + \gamma_2$ or $a_{0L} + \gamma_2$ equals $2\gamma_1 + \gamma_2 < c(\Delta)$ and we argue as before. If $\max\{a_{1i}\} = a_{12}$, then the term T^5 is essentially T^6, because m_{01} is the normalization of φm_{00}. Therefore, we are left again with only two terms, T^4, T^5, to consider, which we can solve easily.

Now we turn to the cases of $a_{01}+s \in \Delta$ or $a_{02}+s \in \Delta$, but $s \notin \Delta$, where we had to make use of the λ–variables in m_{12} resp. m_{10} during the search for the higher order expression for T^1, T^2, T^3. In fact, we found the following

$$\begin{aligned}
\varphi m_{00} - t^{6\alpha_{01}} m_{01} &= \sum_{j=1}^{3} f_{0j} m_{0j} + \sum_{j=1}^{3} f_{1j} m_{1j} \\
\varphi m_{01} - t^{6(\alpha_{02}-\alpha_{01})} m_{02} &= \varepsilon t^{6(\alpha_{12}-\alpha_{01}-q)} m_{12} + \sum_{j=1}^{3} g_{0j} m_{0j} + \sum_{j=1}^{3} g_{1j} m_{1j} \\
\varphi m_{02} - t^{6(q-\alpha_{02})} m_{00} &= \eta t^{6(\alpha_{10}-\alpha_{02})} m_{10} + \sum_{j=1}^{3} h_{0j} m_{0j} + \sum_{j=1}^{3} h_{1j} m_{1j}
\end{aligned}$$

with

$$\begin{array}{lll}
v(f_{0j} m_{0j}) > \gamma_1 & v(g_{0j} m_{0j}) > a_{01} + \gamma_1 & v(h_{0j} m_{0j}) > a_{02} + \gamma_1 \\
v(f_{1j} m_{1j}) > s & v(g_{1j} m_{1j}) > a_{01} + s & v(h_{1j} m_{1j}) > a_{02} + s.
\end{array}$$

Here, $\varepsilon = 1$ if $a_{01}+s \in \Delta$ and $\eta = 1$ if $a_{02}+s \in \Delta$, otherwise they are 0.

Assume now that $a_{01}+s \in \Delta$. As $a_{01}+s \equiv a_{12} \mod 6$, we find $a_{12} \leq a_{01}+s \leq \gamma_1 + s = \gamma_2 - \gamma_1$. Thus we have either $c(\Delta) \leq \gamma_2$, which can be treated like above, or $\max\{a_{1j}\} = a_{11}$. In the latter case we have to consider the following syzygies of $\mathbb{C}[\Delta]$

$$\begin{aligned}
a_{12} + \gamma_1 &= a_{10} + 6(q + \alpha_{10} - \alpha_{12}) \\
a_{02} + \gamma_2 &= a_{12} + 6(\alpha_{12} - \alpha_{02}) \\
a_{00} + \gamma_2 &= a_{12} + \gamma_1 + 6(\alpha_{12} - q)
\end{aligned}$$

and the corresponding T^4, T^5, T^6 terms. The T^4 term is the only one involving the variables λ_k^{10} and a higher order expression can be found by an appropriate choice of these. Higher order expressions for T^5, T^6 can be obtained from the equations (+). Namely, multiply the first equation by φ^2, the second by $\varphi t^{6\alpha_{01}}$, the third by $t^{6\alpha_{02}}$ and add them to obtain after moving the left hand side to the right hand side:

$$0 = \varphi t^{6(\alpha_{12}-q)} m_{12} + \eta t^{6\alpha_{10}} m_{10} + \sum_{j=1}^{3} u_{0j} m_{0j} + \sum_{j=1}^{3} u_{1j} m_{1j}$$

with $v(u_{1j} m_{1j}) > \gamma_2$. As in the case with the Puiseux exponents $(4, 2q, s)$ we replace the multiples of T^1, T^2, T^3 by their higher order expressions to achieve that $v(u_{0j} m_{0j}) \geq \gamma_2$. In fact, as the first odd number in Γ is γ_2 and $v(m_{01}), v(m_{02}) \geq 2$ are even, we find $v(u_{01} m_{01}), v(u_{02} m_{02}) > \gamma_2$. Therefore, we got a higher order expression for the cancellation of the initial terms in

$$\varphi t^{6(\alpha_{12}-q)} m_{12} + \eta t^{6\alpha_{10}} m_{10} - (1+\eta)\psi m_{00}.$$

Replacing $\eta t^{6(\alpha_{12}-q)} T^4$ by its higher order expression, which was found earlier, we find the higher order expression for T^6 or a term that can take the place of T^6 in Lemma 11.

To obtain the higher order expression for T^5, we multiply the above equations with different elements, namely $\varphi t^{6(q-\alpha_{02})}$, $t^{6(q-\alpha_{02}+\alpha_{01})}$, and φ^2 before adding them and obtain this time

$$0 = t^{6(\alpha_{12}-\alpha_{02})}m_{12} + \eta\varphi^2 t^{6(\alpha_{10}-\alpha_{02})}m_{10} + \sum_{j=1}^{3} w_{0j}m_{0j} + \sum_{j=1}^{3} w_{1j}m_{1j}$$

with $v(w_{1j}m_{1j}) > a_{02} + \gamma_2$. Again, we use the higher order expressions for T^1, T^2, T^3 to get $v(w_{0j}m_{0j}) \geq a_{02} + \gamma_2$. With further use of these we can achieve that $v(w_{00}m_{00}), v(w_{01}m_{01}) > a_{02} + \gamma_2$, thus the only terms of the least order $a_{02} + \gamma_2$ are the first two terms and $w_{02}m_{02}$. Now, if $\eta = 0$ then we may view the above equation as a higher order expression for T^5. If $\eta = 1$ then the order of T^5 is greater than $a_{10} + 2\gamma_1 + 6(\alpha_{10} - \alpha_{02}) > a_{10} + \gamma_1 \geq a_{11} \geq c(\Delta)$ and a higher order expression is obtained trivially.

The remaining regular case is when $a_{02}+s \in \Delta$. As $a_{02}+s \equiv a_{10} \equiv \gamma_2 \bmod 6$ this is only a weak restriction on a_{10}. Let us assume that $s, a_{01} + s \notin \Delta$, otherwise we are in one of the above cases. In addition, we assume $c(\Delta) > \gamma_2$, i.e., $c(\Delta) \geq \gamma_2 + 2$, because otherwise the same arguments as in the special cases apply. We claim that $\max\{a_{ij}\} = a_{12}$. If we had $\max\{a_{ij}\} = a_{11}$, then $\gamma_2 + 2 \leq c(\Delta) = a_{11} - 5 = \gamma_2 + \gamma_1 - 6\alpha_{11} - 5$; hence $\alpha_{11} = 0$ and $a_{11} = \gamma_2 + \gamma_1$. This implies $a_{01} = \gamma_1$ and $a_{10} = \gamma_2$ and from $a_{02} + s \geq a_{10}$ we get $a_{02} = 2\gamma_1$, but this was a special case discussed above.

Because of $\max\{a_{ij}\} = a_{12}$ the syzygies of $\mathbb{C}[\Delta]$ of degree below $c(\Delta)$ are generated by:

$$\begin{aligned}
a_{10} + \gamma_1 &= a_{11} + 6(\alpha_{11} - \alpha_{10}) \\
a_{00} + \gamma_2 &= a_{10} + 6\alpha_{10} \\
a_{01} + \gamma_2 &= a_{10} + \gamma_1 + 6(\alpha_{10} - \alpha_{01}) && \text{if} \quad a_{01} + \gamma_2 \geq a_{10} + \gamma_1 \\
a_{01} + \gamma_2 &= a_{11} + 6(\alpha_{11} - \alpha_{01}) && \text{else.}
\end{aligned}$$

A higher order expression for the term T^5 corresponding to the second syzygy can be derived from (+) (with $\varepsilon = 0$ and $\eta = 1$) by multiplying the three equations with φ^2, $\varphi t^{6\alpha_{01}}$, $t^{6\alpha_{02}}$ respectively and adding them to obtain

$$0 = t^{6\alpha_{10}}m_{10} + \sum_{j=1}^{3} u_{0j}m_{0j} + \sum_{j=1}^{3} u_{1j}m_{1j}$$

with $v(u_{1j}m_{1j}) > \gamma_2$. The usual argument leads to a higher order expression for T^5.

If $a_{01} + \gamma_2 \geq a_{10} + \gamma_1$, then we can also derive a higher order expression for T^6 from (+) by multiplying the equations by $t^{6(q-\alpha_{01})}$, φ^2, $\varphi t^{6(\alpha_{02}-\alpha_{01})}$ adding them and proceeding as before. A higher order expression for the term T^4 can now be found by reducing it and solving the remaining coefficients for the variables λ_k^{11}, which occur only in T^4.

At last when $a_{01} + \gamma_2 < a_{10} + \gamma_1$, we use the variables λ_k^{11} to get a higher order expression for $T^{6\prime}$. We claim that a higher order expression for T^4 can be found trivially because its order is greater than $c(\Delta)$. From $a_{01} + \gamma_2 < a_{10} + \gamma_1$ we conclude $a_{01} + \gamma_2 \leq a_{10} + \gamma_1 - 6$ and $a_{12} \leq a_{01} + \gamma_2 + \gamma_1 \leq a_{10} + 2\gamma_1 - 6$, thus we have $c(\Delta) \leq a_{10} + 2\gamma_1 - 2 \cdot 6 + 1 < a_{10} + \gamma_1$.

5 Betti numbers

For any plane singularity X the Jacobi factor J_X is δ_X–dimensional. More precisely, the subset $\mathrm{Pic}^0(X)$ of free modules of J_X is biregular to $\mathbb{C}^{\delta_X}$ and J_X is its closure. Rego proved that the number of components of $J_x \setminus \mathrm{Pic}^0(X)$ equals the multiplicity of the singularity X minus one [R]. Such results and more follow from purely combinatorial reasoning for singularities which possess an affine cell decomposition. We start our discussion with some notations:

Definition 19 *For the semi–group* $\Gamma = \langle p, q\rangle \subset \mathbb{N}$, $\gcd(p,q) = 1$*, we denote the* 0*–normalized semi–modules by* $\mathrm{Mod}(\Gamma)$*. The dimension of a* Γ*–semi–module* Δ *with* p*–basis* $(a_0 = 0, a_1, \ldots, a_{p-1})$ *is defined as*

$$\dim\Delta := \sum_{j=0}^{p-1} \left(g_\Delta(a_j) - g_\Delta(a_j+q)\right) = \sum_{j=0}^{p-1} \#\left([a_j, a_j+q[\setminus\Delta\right).$$

Analogously, for the semi–group $\Gamma = \langle 4, 2q, 2q+s\rangle \subset \mathbb{N}$, $\gcd(2, qs) = 1$*, we denote the* admissible 0*–normalized semi–modules by* $\mathrm{Mod}(\Gamma)$*. The dimension of an admissible* Γ*–semi–module* Δ *with* 2×2*–basis* $(a_{00} = 0, a_{01}; a_{10}, a_{11})$ *is defined as*

$$\dim\Delta := \sum_{i,j=0}^{1} g_\Delta(a_{ij}) - g_\Delta(\gamma_1) - g_\Delta(a_{01}+n),$$

where $n := \min\left(\{s\} \cup (\Delta \cap [\gamma_1, \infty[\cap(1+2\mathbb{N}))\right)$.

The codimension of Δ *is* $\mathrm{codim}\,\Delta := \delta_\Gamma - \dim\Delta$*, where* $\delta_\Gamma := \dim\Gamma$*. Thereby, the semi–modules are splitted into the disjoint subsets*

$$\mathrm{Mod}_d(\Gamma) := \{\Delta \in \mathrm{Mod}(\Gamma) \mid \dim\Delta = d\}$$

or dually

$$\mathrm{Mod}^d(\Gamma) := \{\Delta \in \mathrm{Mod}(\Gamma) \mid \mathrm{codim}\,\Delta = d\}.$$

Either geometrically from the next theorem or combinatorically from the proofs of the following Theorems, we will see that the values of the functions dim and codim lie in the range $[0, \delta_\Gamma]$ and $\mathrm{Mod}^0(\Gamma) = \{\Gamma\}$ as well as $\mathrm{Mod}_0(\Gamma) = \{\mathbb{N}\}$.

As an immediate consequence of the affine cell decomposition of the Jacobi factors and the remarks in Section 1 we have

Theorem 20 *Let* X *be a unibranched plane singularity with characteristic Puiseux exponents* (p,q) *or* $(4, 2q, s)$*. Let* Γ *be its associated semi–group and and* J_X *its Jacobi factor. Then the odd (co–)homology groups of* J_X *are zero, and the even (co–)homology group are free abelian groups with Betti numbers*

$$h_{2d}(J_X) = \#Mod_d(\Gamma) \qquad \text{and} \qquad h^{2d}(J_X) = \#Mod^d(\Gamma).$$

It is easy to write a computer program that computes all Γ–semi–modules together with their dimension. We discuss the results for the singularities with Puiseux exponents (p,q) first. For the singularities with Puiseux exponents $(2,q)$ and $(3,q)$ one obtains the following list, which has an obvious construction rule.

(p,q)	δ_X	$e(X)$	h^0	h^2	h^4	h^6	h^8									
(2,3)	1	2	1	1												
(2,5)	2	3	1	1	1											
(2,7)	3	4	1	1	1	1										
(2,9)	4	5	1	1	1	1	1									
(2,11)	5	6	1	1	1	1	1	1								
(3,2)	1	2	1	1												
(3,4)	3	5	1	2	1	1										
(3,5)	4	7	1	2	2	1	1									
(3,7)	6	12	1	2	3	2	2	1	1							
(3,8)	7	15	1	2	3	3	2	2	1	1						
(3,10)	9	22	1	2	3	4	3	3	2	2	1	1				
(3,11)	10	26	1	2	3	4	4	3	3	2	2	1	1			
(3,13)	12	35	1	2	3	4	5	4	4	3	3	2	2	1	1	
(3,14)	13	40	1	2	3	4	5	5	4	4	3	3	2	2	1	1

The Betti numbers for the singularities A_{2k}, E_6, E_8, i.e., for the singularities with the characteristic Puiseux exponents $(2,q)$, $(3,4)$, and $(3,5)$, have been computed by Cook [C] and Warmt [W1]. For $p \geq 4$ an explicit formula for the Betti numbers seems difficult to find. A long list of examples is included so that the reader may try himself.

(p,q)	δ_X	$e(X)$	$h^0 h^2 h^4 h^6 h^8 \ldots$
(4,3)	3	5	1 2 1 1
(4,5)	6	14	1 3 3 3 2 1 1
(4,7)	9	30	1 3 5 5 5 4 3 2 1 1
(4,9)	12	55	1 3 6 7 8 7 7 5 4 3 2 1 1
(4,11)	15	91	1 3 6 9 10 11 10 10 8 7 5 4 3 2 1 1
(4,13)	18	140	1 3 6 10 12 14 14 14 13 12 10 8 7 5 4 3 2 1 1
(4,15)	21	204	1 3 6 10 14 16 18 18 18 17 16 14 12 10 8 7 5 4 3 2 1 1
(4,17)	24	285	1 3 6 10 15 18 21 22 23 22 22 20 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,19)	27	385	1 3 6 10 15 20 23 26 27 28 27 27 25 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,21)	30	506	1 3 6 10 15 21 25 29 31 33 33 33 32 31 29 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,23)	33	650	1 3 6 10 15 21 27 31 35 37 39 39 39 38 37 35 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,25)	36	819	1 3 6 10 15 21 28 33 38 41 44 45 46 45 45 43 42 39 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,27)	39	1015	1 3 6 10 15 21 28 35 40 45 48 51 52 53 52 52 50 49 46 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,29)	42	1240	1 3 6 10 15 21 28 36 42 48 52 56 58 60 60 60 59 58 56 54 51 48 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,31)	45	1496	1 3 6 10 15 21 28 36 44 50 56 60 64 66 68 68 68 67 66 64 62 59 56 52 48 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(5,2)	2	3	1 1 1
(5,3)	4	7	1 2 2 1 1
(5,4)	6	14	1 3 3 3 2 1 1
(5,6)	10	42	1 4 6 7 7 5 5 3 2 1 1
(5,7)	12	66	1 4 7 9 10 9 8 6 5 3 2 1 1
(5,8)	14	99	1 4 8 11 13 13 12 10 9 6 5 3 2 1 1
(5,9)	16	143	1 4 9 13 16 17 17 15 13 11 9 6 5 3 2 1 1
(5,11)	20	273	1 4 10 16 22 25 28 27 26 23 21 17 15 11 9 6 5 3 2 1 1
(5,12)	22	364	1 4 10 17 24 29 33 34 34 31 29 25 22 18 15 11 9 6 5 3 2 1 1
(5,13)	24	476	1 4 10 18 26 33 38 41 42 40 38 34 31 26 23 18 15 11 9 6 5 3 2 1 1
(5,14)	26	612	1 4 10 19 28 37 43 48 50 50 48 44 41 36 32 27 23 18 15 11 9 6 5 3 2 1 1
(5,16)	30	969	1 4 10 20 31 43 53 61 67 70 71 68 66 60 56 50 45 38 34 27 23 18 15 11 9 6 5 3 2 1 1
(5,17)	32	1197	1 4 10 20 32 45 57 67 75 80 83 82 80 75 71 64 59 52 46 39 34 27 23 18 15 11 9 6 5 3 2 1 1
(5,18)	34	1463	1 4 10 20 33 47 61 73 83 90 95 96 95 91 87 80 75 67 61 53 47 39 34 27 23 18 15 11 9 6 5 3 2 1 1
(5,19)	36	1771	1 4 10 20 34 49 65 79 91 100 107 110 111 108 104 98 92 84 78 69 62 54 47 39 34 27 23 18 15 11 9 6 5 3 2 1 1
(5,21)	40	2530	1 4 10 20 35 52 71 89 106 119 131 138 144 144 143 138 133 124 118 108 100 90 82 71 64 54 47 39 34 27 23 18 15 11 9 6 5 3 2 1 1
(5,22)	42	2990	1 4 10 20 35 53 73 93 112 128 142 152 160 163 164 160 156 148 141 131 123 112 103 92 83 72 64 54 47 39 34 27 23 18 15 11 9 6 5 3 2 1 1
(5,23)	44	3510	1 4 10 20 35 54 75 97 118 137 153 166 176 182 185 183 180 173 166 156 148 136 127 115 105 93 84 72 64 54 47 39 34 27 23 18 15 11 9 6 5 3 2 1 1
(5,24)	46	4095	1 4 10 20 35 55 77 101 124 146 164 180 192 201 206 207 205 199 193 183 174 163 153 140 130 117 106 94 84 72 64 54 47 39 34 27 23 18 15 11 9 6 5 3 2 1 1

(p,q)	δ_X	$e(X)$	$h^0 h^2 h^4 h^6 h^8 \ldots$
(6,5)	10	42	1 4 6 7 7 5 5 3 2 1 1
(6,7)	15	132	1 5 10 14 17 16 16 14 11 9 7 5 3 2 1 1
(6,11)	25	728	1 5 14 26 39 50 59 63 64 62 59 53 47 41 34 28 23 18 13 10 7 5 3 2 1 1
(6,13)	30	1428	1 5 15 30 49 67 85 97 106 108 109 104 99 90 82 71 63 53 45 36 30 23 18 13 10 7 5 3 2 1 1
(6,17)	40	4389	1 5 15 34 61 95 131 167 200 227 248 259 266 264 260 249 237 220 204 184 167 147 131 113 98 82 70 57 47 37 30 23 18 13 10 7 5 3 2 1 1
(6,19)	45	7084	1 5 15 35 65 105 151 199 247 290 328 355 376 385 389 385 376 361 344 322 299 275 250 226 202 179 157 137 117 100 84 70 57 47 37 30 23 18 13 10 7 5 3 2 1 1
(7,2)	3	4	1 1 1 1
(7,3)	6	12	1 2 3 2 2 1 1
(7,4)	9	30	1 3 5 5 5 4 3 2 1 1
(7,5)	12	66	1 4 7 9 10 9 8 6 5 3 2 1 1
(7,6)	15	132	1 5 10 14 17 16 16 14 11 9 7 5 3 2 1 1
(7,8)	21	429	1 6 15 25 35 40 43 44 40 37 32 28 22 18 13 11 7 5 3 2 1 1
(7,9)	24	715	1 6 16 29 43 54 62 66 66 63 58 51 45 37 31 24 19 14 11 7 5 3 2 1 1
(7,10)	27	1144	1 6 17 33 52 70 84 93 97 97 92 86 77 69 58 50 40 33 25 20 14 11 7 5 3 2 1 1
(7,11)	30	1768	1 6 18 37 61 86 108 124 135 139 138 132 124 112 101 88 76 63 53 42 34 26 20 14 11 7 5 3 2 1 1
(7,12)	33	2652	1 6 19 41 70 102 133 159 178 190 194 193 184 174 159 145 127 112 95 81 66 55 43 35 26 20 14 11 7 5 3 2 1 1
(7,13)	36	3876	1 6 20 45 80 120 162 199 229 249 262 265 261 251 237 218 200 179 159 137 119 100 84 68 56 44 35 26 20 14 11 7 5 3 2 1 1
(7,15)	42	7752	1 6 21 50 95 151 216 280 341 389 429 453 468 468 462 443 423 394 367 332 302 268 238 205 179 151 129 106 88 70 58 44 35 26 20 14 11 7 5 3 2 1 1
(8,3)	7	15	1 2 3 3 2 2 1 1
(8,5)	14	99	1 4 8 11 13 13 12 10 9 6 5 3 2 1 1
(8,7)	21	429	1 6 15 25 35 40 43 44 40 37 32 28 22 18 13 11 7 5 3 2 1 1
(8,9)	28	1430	1 7 21 41 65 86 102 115 118 118 113 106 96 85 73 63 53 42 34 26 20 15 11 7 5 3 2 1 1
(8,11)	35	3978	1 7 23 51 90 135 180 220 251 272 282 282 275 262 244 223 200 177 154 132 111 92 75 61 47 37 28 21 15 11 7 5 3 2 1 1
(8,13)	42	9690	1 7 25 61 117 190 273 357 435 501 551 584 600 600 588 566 535 498 458 415 372 329 288 249 213 179 150 123 100 80 64 49 38 28 21 15 11 7 5 3 2 1 1
(9,2)	4	5	1 1 1 1 1
(9,4)	12	55	1 3 6 7 8 7 7 5 4 3 2 1 1
(9,5)	16	143	1 4 9 13 16 17 17 15 13 11 9 6 5 3 2 1 1
(9,7)	24	715	1 6 16 29 43 54 62 66 66 63 58 51 45 37 31 24 19 14 11 7 5 3 2 1 1
(9,8)	28	1430	1 7 21 41 65 86 102 115 118 118 113 106 96 85 73 63 53 42 34 26 20 15 11 7 5 3 2 1 1
(9,10)	36	4862	1 8 28 63 112 167 219 268 303 326 338 338 331 314 293 268 245 215 190 162 139 116 97 77 63 48 38 28 22 15 11 7 5 3 2 1 1
(9,11)	40	8398	1 8 29 69 129 203 282 360 428 482 520 541 547 538 519 489 456 416 376 334 295 254 219 184 155 127 104 82 66 50 39 29 22 15 11 7 5 3 2 1 1

At least, we are able to describe their asymptotic behavior for $q \to \infty$. The following two theorems determine the first $\lfloor \frac{q}{p} \rfloor + 1$ and the last $q - \lceil \frac{q}{p} \rceil + 1$ of the $\delta_X + 1 = (p-1)(q-1)/2 + 1$ Betti numbers. In particular, all the Betti numbers for the singularities with characteristic Puiseux exponents $(2, q)$ or $(3, q)$ are described.

Theorem 21 *Let X be a unibranched plane singularity with Puiseux exponents (p, q) and J_X its Jacobi factor. Then the even Betti numbers $h^0(J_X), h^2(J_X), \ldots, h^{2\lfloor \frac{q}{p} \rfloor}(J_X)$ of the cohomology of J_X are the same as the first $\lfloor \frac{q}{p} \rfloor + 1$ coefficients of the power series*

$$P := \frac{1}{(1-t)^{p-1}}.$$

Proof. P is the Poincare series of the polynomial ring in the $p-1$ variables $t_1, \ldots, t_{p-1}$. Let Mon^d be the set of monomials of degree d in this ring. For $\Gamma = \langle p, q \rangle$ and $d \leq \lfloor \frac{q}{p} \rfloor$ we define the map

$$\begin{array}{rcl} \Phi_d : \mathrm{Mon}^d & \longrightarrow & \mathrm{Mod}^d(\Gamma) \\ \prod_{j=1}^{p-1} t_j^{r_j} & \longmapsto & \left\langle a_j := jq - \left(\sum_{i=1}^{j} r_i\right) p \,\middle|\, j = 0 \ldots p-1 \right\rangle. \end{array}$$

The theorem is proved when we have shown that Φ_d is well–defined and bijective. Note that (a_j) is a p–basis, thus the map is injective. To see that the map is well–defined, we need to show that a semi–module Δ with a p–basis like above has really codimension d. From $\sum_{i=1}^{p-1} r_i \leq \lfloor \frac{q}{p} \rfloor$ we see that $0 = a_0 < a_1 < a_2 < \ldots < a_{p-1}$ and $a_j + q < a_{j+2}$; hence, defining for any interval $I \subseteq \mathbb{N}$

$$S^j(I) := \{n \in I \mid n = iq \bmod p \text{ for some } i \in \{0, 1, \ldots, j\}\}$$

we find for $\Delta = \bigcup (a_j + p\mathbb{N})$

$$[a_j, a_j + q[\, \cap \Delta = S^j([a_j, a_j + q[) \cup \{a_j + q - kp \mid 0 < k \leq r_{j+1}\}$$

where the union is disjoint. We compare Δ with the semi–module Γ, which has the p–basis (jq). Here we have

$$[jq, jq + q[\, \cap \Gamma = S^j([qj, qj + q[).$$

Because $jq \equiv a_j \bmod p$, we obviously have $\# S^j([a_j, a_j + q[) = \# S^j([qj, qj + q[)$. Therefore,

$$\#\left([a_j, a_j + q[\, \cap \Delta\right) = \#\left([jq, jq + q[\, \cap \Gamma\right) + r_{j+1},$$

and the dimension formula implies that $\operatorname{codim} \Delta = \sum_{j=1}^{p-1} r_j = d$.

It remains to prove that the maps Φ_d are surjective, i.e., we need to show that the dimension of any semi–module not in the image of any $\Phi_0, \ldots, \Phi_{\lfloor \frac{q}{p} \rfloor}$ has codimension greater than $\lfloor \frac{q}{p} \rfloor$. Let Δ be any 0–normalized semi–module with p–basis $(a_j = jq - \alpha_j p)$. Set $r_0 := 0$ and $r_j := \alpha_j - \alpha_{j-1} \geq 0$ for $0 < j \leq p-1$.

The semi–module Δ lies in the image of Φ_d for some $d \leq \lfloor \frac{q}{p} \rfloor$ iff $d = \sum_{j=1}^{p-1} r_j$. Therefore, for the surjectivity of the Φ_d, $d \leq \lfloor \frac{q}{p} \rfloor$, it is enough to show that

$$\dim \Delta \geq \min \left\{ \sum_{j=1}^{p-1} r_j, \left\lceil \frac{q}{p} \right\rceil \right\};$$

however, we will show the stronger statement

$$\dim \Delta \geq \sum_{j=1}^{p-1} \min \left\{ r_j, \left\lceil \frac{q}{p} \right\rceil \right\}.$$

We prove this by successively reducing the vector $r = (r_j)$ to zero, where the statement is trivial. Let k be the least integer with $r_k \neq 0$. We define the semi–module Δ' to be the one that corresponds to the vector $r' = (0, \ldots, 0, r_k - 1, r_{k+1}, \ldots, r_{p-1})$, i.e., Δ' has the p–basis (a'_j) with $a'_j = a_j$ for $j < k$ and $a'_j = a_j + p$ for $j \geq k$. Our estimate is proven when we have shown that $\dim \Delta' \geq \dim \Delta$ with strict inequality when $r_k \leq \lceil \frac{q}{p} \rceil$. Set $I_j := [a_j, a_j + q[$ and $I'_j := [a'_j, a'_j + q[$. Then $I_j = I'_j = [jq, jq + q[$ for $j < k$ and their disjoint union is $[0, kq[$. Because $\Delta' \subset \Delta$, we have $\#([0, kq[\setminus \Delta') \geq \#([0, kq[\setminus \Delta)$ as a first indication of $\dim \Delta' \geq \dim \Delta$.

For $j \geq k$ we have $I'_j = p + I_j$, and there is the natural injective map

$$\Psi_j : I_j \cap \Delta \longrightarrow I'_j \cap \Delta', \quad n \longmapsto n + p.$$

Because $\Delta' \setminus (p + \Delta) = \{0, q, 2q, \ldots, (k-1)q\}$ and $\# I'_j = q$, we have that either Ψ_j is bijective or $(I'_j \cap \Delta') \setminus \operatorname{Im} \Psi_j = \{lq\}$ for some $l < k$. In the later case we get $a'_j < lq$, in particular $a'_j \in [0, kq[$ and $a_j = a'_j - p \in [0, kq[$ as well. It follows that $a_j \in [0, kq[\cap (\Delta \setminus \Delta')$. Summarizing we have shown that either

$$\begin{aligned} &\#(I'_j \cap \Delta') = \#(I_j \cap \Delta) \qquad \text{or} \\ &\#(I'_j \cap \Delta') = \#(I_j \cap \Delta) + 1 \quad \text{and} \quad a_j \in [0, kq[\cap (\Delta \setminus \Delta'). \end{aligned}$$

Since

$$\begin{aligned} \dim \Delta \ &= \#([0, kq[\setminus \Delta) + \sum_{j=k}^{p-1} \#(I_j \setminus \Delta) \\ \dim \Delta' &= \#([0, kq[\setminus \Delta') + \sum_{j=k}^{p-1} \#(I'_j \setminus \Delta'), \end{aligned}$$

we conclude that $\dim \Delta' \geq \dim \Delta$.

Now assume that $r_k \leq \lceil \frac{q}{p} \rceil$. Then $a'_k = kq - (r_k - 1)p > (k-1)q$ and the interval I'_k cannot contain any of the $0, q, \ldots, (k-1)q$, thus $\#(I'_k \cap \Delta') = \#(I_k \cap \Delta)$. Since we have $a_k = kq - r_k p \in [0, kq[\cap (\Delta \setminus \Delta')$ as well, it follows that $\dim \Delta' > \dim \Delta$. □

Theorem 22 *Let X be a unibranched plane singularity with Puiseux exponents (p, q) and J_X its Jacobi factor. Set $n := q - \lceil \frac{q}{p} \rceil$. Then the even Betti numbers*

$h_0(J_X), h_2(J_X), \ldots, h_{2n}(J_X)$ *of the homology of* J_X *are the same as the first* $n+1$ *coefficients of the power series*

$$P := \frac{1}{\prod\limits_{i=1}^{p-1}(1-t^i)}.$$

Proof. P is the Poincare series of the weighted polynomial ring in the $p-1$ variables $t_1, \ldots, t_{p-1}$ where the weighted degree of t_i is i. Let Mon be the set of all monomials and Mon_d the monomials of weighted degree d in this ring. The strategy of this proof is to define an obviously surjective map from Mon into the set of $\langle p\rangle$–semi–modules, $\mathrm{Mod}(\langle p\rangle)$,

$$\Psi : \mathrm{Mon} \longrightarrow \mathrm{Mod}(\langle p\rangle),$$

and then show that it induces a bijection between Mon_d and $\mathrm{Mod}_d(\Gamma)$ for $d \leq n$.

For a $\langle p\rangle$–semi–module Δ we have also a notion of a p–basis. It is the unique set $\{b_0 = 0, b_1, \ldots, b_{p-1}\}$ such that $\Delta = \bigcup_{j=0}^{p-1}(b_j + p\mathbb{N})$. Whenever possible we will assume that the b_j are ordered by $0 = b_0 < b_1 < \ldots < b_{p-1}$. Now the map Ψ is defined in the following way: Let $m = \prod_{j=1}^{p-1} t_j^{r_j}$ be a monomial of weighted degree $d = \sum_{j=1}^{p-1} r_j j$. Then $\Psi(m)$ is the unique $\langle p\rangle$–semi–module Δ which possesses an ordered p–basis $\{b_j\}$ with $\#([b_{j-1}, b_j] \setminus \Delta) = r_j$ for $j = 1 \ldots r-1$, i.e., there are r_j gaps in Δ between the basis elements b_{j-1} and b_j. A p–basis for such a Δ can be constructed inductively: Having found $b_0 = 0, b_1, \ldots, b_{j-1}$ let b_j be the position of the (r_j+1)–th gap in $\bigcup_{i=0}^{j-1}(a_i + p\mathbb{N})$ after b_{j-1}. Obviously, Ψ is bijective. The following table illustrates this map for $p = 3$. The module Δ is represented as a sequence of members of Δ, "$\bullet$", and gaps of Δ, "$\circ$"; the elements of the 3–basis are underlined.

wdeg	Mon	Δ									
0	1	$\underline{\bullet}$	$\underline{\bullet}$	$\underline{\bullet}$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
1	t_1	$\underline{\bullet}$	$\circ$	$\underline{\bullet}$	$\bullet$	$\underline{\bullet}$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
2	t_1^2	$\underline{\bullet}$	$\circ$	$\circ$	$\bullet$	$\underline{\bullet}$	$\underline{\bullet}$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
	t_2	$\underline{\bullet}$	$\underline{\bullet}$	$\circ$	$\bullet$	$\bullet$	$\underline{\bullet}$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
3	t_1^3	$\underline{\bullet}$	$\circ$	$\circ$	$\bullet$	$\circ$	$\underline{\bullet}$	$\bullet$	$\underline{\bullet}$	$\bullet$	$\bullet$
	t_1t_2	$\underline{\bullet}$	$\circ$	$\underline{\bullet}$	$\bullet$	$\circ$	$\bullet$	$\bullet$	$\underline{\bullet}$	$\bullet$	$\bullet$
4	t_1^4	$\underline{\bullet}$	$\circ$	$\circ$	$\bullet$	$\circ$	$\circ$	$\bullet$	$\underline{\bullet}$	$\underline{\bullet}$	$\bullet$
	$t_1^2t_2$	$\underline{\bullet}$	$\circ$	$\circ$	$\bullet$	$\underline{\bullet}$	$\circ$	$\bullet$	$\bullet$	$\underline{\bullet}$	$\bullet$
	t_2^2	$\underline{\bullet}$	$\underline{\bullet}$	$\circ$	$\bullet$	$\bullet$	$\circ$	$\bullet$	$\bullet$	$\underline{\bullet}$	$\bullet$

Several arguments of this proof are based on a comparison of an arbitrary $\langle p\rangle$–semi–module Δ with the $\langle p\rangle$–semi–modules $\Delta_r := \Psi(t_1^r)$. Note that

$$\Delta_r = \left\{0, p, 2p, \ldots, \left\lfloor \frac{r}{p-1} \right\rfloor p, r + \left\lfloor \frac{r}{p-1} \right\rfloor + 1, r + \left\lfloor \frac{r}{p-1} \right\rfloor + 2, \ldots \right\},$$

and the conductor of Δ_r is $r + \lceil \frac{r}{p-1} \rceil$. The most important case is the one for $r = n$. Here one finds $\Delta_n = p\mathbb{N} \cup (q + \mathbb{N})$ and the conductor is q or $q-1$. The essential comparison property of the Δ_r is

(†) *Let m be a monomial of weighted degree d and $\Delta = \Psi(m)$ be the corresponding $\langle p\rangle$–semi–module, then $c(\Delta) \le c(\Delta_d)$.*

We prove the claim (†) by induction. Assuming it holds for $\Delta = \Psi(t_1^{r_1} \cdots t_\varrho^{r_\varrho})$ we will show that for $\Delta' = \Psi(t_1^{r_1} \cdots t_\varrho^{r_\varrho+1})$ we have $c(\Delta') \le c(\Delta_{d+\varrho})$ as well. (Some or all of the r_i may be zero.) First we consider the modules Δ_d and $\Delta_{d+\varrho}$. Let lp be the smallest p–multiple with $lp > c(\Delta_d) =: c$. Then we have the following partition of Δ_d:

$$\begin{array}{ll} \Delta_d = \Delta_{d+\varrho} \cup \{c+1, c+2, \ldots, c+\varrho\} & \text{if } c \in p\mathbb{N} \subset \Delta_d \\ \Delta_d = \Delta_{d+\varrho} \cup \{c, c+1, \ldots, lp-1\} \cup \{lp+1, lp+2, \ldots, c+\varrho\} & \text{if } c+\varrho \ge lp \\ \Delta_d = \Delta_{d+\varrho} \cup \{c, c+1, \ldots, c+\varrho-1\} & \text{else.} \end{array}$$

Therefore, we have

$$c(\Delta_{d+\varrho}) = \begin{cases} c(\Delta_d) + \varrho & \text{for } [c(\Delta_d), c(\Delta_d)+\varrho] \cap p\mathbb{N} = \emptyset \\ c(\Delta_d) + \varrho + 1 & \text{else.} \end{cases}$$

The claim (†) is proved when we have shown that

$$c(\Delta') \le \begin{cases} c(\Delta) + \varrho & \text{for } [c(\Delta), c(\Delta)+\varrho] \cap p\mathbb{N} = \emptyset \\ c(\Delta) + \varrho + 1 & \text{else,} \end{cases}$$

because on the one hand if $c(\Delta) < c(\Delta_d)$ then $c(\Delta') \le c(\Delta) + \varrho \le c(\Delta_{d+\varrho})$ and on the other hand if $c(\Delta) = c(\Delta_d)$ then $c(\Delta') \le c(\Delta_{d+\varrho})$ is obvious from the above.

Let $\{b_j\}$ be an ordered p–basis of Δ. We know for the ordered p–basis of Δ' that $b'_j = b_j$ for $j < \varrho$ and $b'_j > b_j$ for $j \ge \varrho$. By the definition of Δ', Δ and Δ' differ only by one element, an additional gap in Δ between the $(\varrho-1)$–th and ϱ–th element of the p–basis $\{b'_j\}$ of Δ' — the last gap in Δ' at all. By the definition of b_ϱ this must be b_ϱ, i.e., $\Delta = \Delta' \cup \{b_\varrho\}$. In particular, $c(\Delta') = b_\varrho + 1$. To get an estimate for $c(\Delta)$ from below, consider Δ in the interval between $b_\varrho - \varrho - 1$ and b_ϱ

$$[b_\varrho - \varrho - 1, b_\varrho[\setminus \Delta = [b_\varrho - \varrho - 1, b_\varrho[\setminus \bigcup_{j=0}^{p-1} (b_j + p\mathbb{N}) = [b_\varrho - \varrho - 1, b_\varrho[\setminus \bigcup_{j=0}^{\varrho-1} (b_j + p\mathbb{N}).$$

Because the interval $[b_\varrho - \varrho - 1, b_\varrho[$ are $\varrho + 1$ consecutive numbers, the above set is nonempty, thus there is a gap in Δ greater or equal to $b_\varrho - \varrho - 1$. Hence, $c(\Delta) \ge b_\varrho - \varrho$, and $c(\Delta') \le c(\Delta) + \varrho + 1$. If this inequality is not strict, we find

$$[c(\Delta), c(\Delta) + \varrho[= [b_\varrho - \varrho, b_\varrho[\subset \bigcup_{j=0}^{\varrho-1} (b_j + p\mathbb{N}).$$

This can only happen if $[c(\Delta), c(\Delta)+\varrho[\cap (b_j + p\mathbb{N}) \ne \emptyset$ for all $j = 0, \ldots, \varrho-1$; in particular, with $j = 0$ we find the claimed estimate $c(\Delta') \le c(\Delta) + \varrho$ for $[c(\Delta), c(\Delta) + \varrho[\cap p\mathbb{N} = \emptyset$, and the statement (†) is proved.

The statement (†) has two immediate consequences. If $d \le n$ then $c(\Delta) \le c(\Delta_d) \le c(\Delta_n) \le q$ and thus the $\langle p\rangle$–semi–module is trivially a Γ–semi–module as well. Further, the dimension of any Γ–semi–module Δ with $c(\Delta) \leq q$ is

$$\dim \Delta = \sum_{j=0}^{p-1} g_\Delta(b_j) = \sum_{j=0}^{p-1} \sum_{i=j+1}^{p-1} r_i = \sum_{j=1}^{p-1} r_j j.$$

Hence, if m is the unique monomial with $\Psi(m) = \Delta$ then $\dim \Delta = \operatorname{wdeg} m$. This shows that the image of Mon_d under Ψ lies in $\mathrm{Mod}_d(\Gamma)$. Therefore, we obtain injective maps

$$\Psi_d : \mathrm{Mon}_d \longrightarrow \mathrm{Mod}_d(\Gamma).$$

The proof of the Theorem is finished when we have shown that they are surjective as well.

For the surjectivity of Ψ_d with $d \leq n$, we must show that for any $\Delta \in \mathrm{Mod}_d(\Gamma)$ the unique monomial m with $\Psi(m) = \Delta$ has weighted degree d. By the above argument this is clear if $c(\Delta) \leq q$. Thus to prove the surjectivity of the $\Psi_0, \ldots, \Psi_n$, it is enough to show that for any $\Delta \in \mathrm{Mod}(\Gamma)$ with $\dim \Delta \leq n$ we have $c(\Delta) \leq q$. We will prove that $c(\Delta) > q$ implies $\dim \Delta > n$ by an inductive process like above. We close the last gap in the semi–module Δ to obtain the semi–module Δ', thereby reducing the conductor. We will show that $\dim \Delta' \leq \dim \Delta$ and $\dim \Delta > n$ if $c(\Delta') \leq q < c(\Delta)$.

If $\{b_j\}$ is an ordered p–basis of Δ, then its conductor is $c := b_{p-1} - p + 1$. The semi–module Δ' has an unordered p–basis $\{b'_j\}$ with $b'_j = b_j$ for $j < p-1$ and $b'_{p-1} = b_{p-1} - p$. The conductor of Δ' is $c' := \max\{b_{p-2} - p + 1, b_{p-1} - 2p + 1\}$; in particular $c - p \leq c' < c$. Since $b_{p-1} \geq c$ and $b'_{p-1} \geq c'$, we have $[b_{p-1}, b_{p-1} + q[\subset \Delta, \Delta'$ and $[b'_{p-1}, b'_{p-1} + q[\subset \Delta'$, and the dimensions of Δ and Δ' can be computed very similarly as

$$\dim \Delta = \sum_{j=0}^{p-1} \#(I_j \setminus \Delta) \text{ resp. } \dim \Delta = \sum_{j=0}^{p-1} \#(I_j \setminus \Delta') \text{ with } I_j = [b_j, b_j + q[\,.$$

Due to $\Delta' = \Delta \cup \{b'_{p-1}\}$, we get

$$\dim \Delta = \dim \Delta' + \#J \qquad \text{with } \; J := \left\{ j \in \{0, \ldots, p-1\} \mid b'_{p-1} \in I_j \right\},$$

showing $\dim \Delta \geq \dim \Delta'$.

Now, let us assume additionally that $c' \leq q < c$. We need to show that $\dim \Delta > n$. We claim that

$$(\dagger\dagger) \qquad \#J \geq \#([c', q] \setminus p\mathbb{N}).$$

Knowing this we can easily finish the proof. Choose $l \in \mathbb{N}$ such that $c(\Delta_l) = c'$, then $\dim \Delta_l \leq \dim \Delta'$ by ($\dagger$). Further, $\Delta_n \setminus \Delta_l = [c', q[\, \setminus p\mathbb{N}$; hence, $\dim \Delta_n = \dim \Delta_l + \#([c', q[\, \setminus p\mathbb{N})$. Putting this together, we get

$$\dim \Delta = \dim \Delta' + \#J \geq \dim \Delta_l + \#([c', q] \setminus p\mathbb{N}) = \dim \Delta_n + 1 = n + 1.$$

Finally, it remains to prove ($\dagger\dagger$). For each of the $k \in [c', b'_{p-1}[$ find the index j_k with $b_{j_k} \equiv k \bmod p$. Then $j_k \in J$ is equivalent to $b'_{p-1} \in [b_{j_k}, b_{j_k} + q[$ or to $b_{j_k} > b'_{p-1} - q$. Since $b'_{p-1} \notin \Delta$, we find $b'_{p-1} - q \notin \Delta$ and $b'_{p-1} - q \notin \Delta'$ as well.

Therefore, $j_k \in J$ is in fact equivalent to $b_{j_k} \geq b'_{p-1} - q$. This implies that only $b'_{p-1} - q$ of the $b'_{p-1} - c'$ integers in $[c', b'_{p-1}[$ can fail to have a corresponding j index that lies in J; in particular, $\#J \geq q - c'$, nearly proving (††). If we actually have $\#J = q - c'$ then $b'_{p-1} - q$ of the b_{j_k} must be less than $b'_{p-1} - q$; thus we must have $b_j = j$ for $j = 0 \dots b'_{p-1} - q - 1$. Let $l \in [c', b'_{p-1}[$ be the integer with $l \equiv b_0 = 0 \mod p$. If l were greater than q, we would have $b_{b'_{p-1}-l} = b'_{p-1} - l \equiv b'_{p-1} \mod p$, contradicting the definition of a p–basis. Therefore, $l \leq q$ and $[c', q] \cap p\mathbb{N} \neq \emptyset$, proving (††). □

We turn to the singularities with characteristic Puiseux exponents $(4, 2q, s)$. With the help of a computer program one obtains the following list of Betti numbers:

$(4,2q,s)$	δ_X	$e(X)$	$h^0 h^2 h^4 h^6$ h^8 $\dots$
(4,6,7)	8	23	1 3 4 4 4 3 2 1 1
(4,6,9)	9	27	1 3 4 4 4 4 3 2 1 1
(4,6,11)	10	31	1 3 4 4 4 4 4 3 2 1 1
(4,6,13)	11	35	1 3 4 4 4 4 4 4 3 2 1 1
(4,10,11)	14	76	1 3 6 8 9 9 9 8 7 5 4 3 2 1 1
(4,10,13)	15	85	1 3 6 8 9 9 9 9 8 7 5 4 3 2 1 1
(4,10,15)	16	94	1 3 6 8 9 9 9 9 9 8 7 5 4 3 2 1 1
(4,10,17)	17	103	1 3 6 8 9 9 9 9 9 9 8 7 5 4 3 2 1 1
(4,14,15)	20	178	1 3 6 10 13 15 16 16 16 15 14 12 10 8 7 5 4 3 2 1 1
(4,14,17)	21	194	1 3 6 10 13 15 16 16 16 16 15 14 12 10 8 7 5 4 3 2 1 1
(4,14,19)	22	210	1 3 6 10 13 15 16 16 16 16 16 15 14 12 10 8 7 5 4 3 2 1 1
(4,14,21)	23	226	1 3 6 10 13 15 16 16 16 16 16 16 15 14 12 10 8 7 5 4 3 2 1 1
(4,18,19)	26	345	1 3 6 10 15 19 22 24 25 25 25 24 23 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,18,21)	27	370	1 3 6 10 15 19 22 24 25 25 25 25 24 23 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,18,23)	28	395	1 3 6 10 15 19 22 24 25 25 25 25 25 24 23 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,18,25)	29	420	1 3 6 10 15 19 22 24 25 25 25 25 25 25 24 23 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,22,23)	32	593	1 3 6 10 15 21 26 30 33 35 36 36 36 35 34 32 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,22,25)	33	629	1 3 6 10 15 21 26 30 33 35 36 36 36 36 35 34 32 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,22,27)	34	665	1 3 6 10 15 21 26 30 33 35 36 36 36 36 36 35 34 32 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,22,29)	35	701	1 3 6 10 15 21 26 30 33 35 36 36 36 36 36 36 35 34 32 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,26,27)	38	938	1 3 6 10 15 21 28 34 39 43 46 48 49 49 49 48 47 45 43 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,26,29)	39	987	1 3 6 10 15 21 28 34 39 43 46 48 49 49 49 49 48 47 45 43 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,26,31)	40	1036	1 3 6 10 15 21 28 34 39 43 46 48 49 49 49 49 49 48 47 45 43 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,26,33)	41	1085	1 3 6 10 15 21 28 34 39 43 46 48 49 49 49 49 49 49 48 47 45 43 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,30,31)	44	1396	1 3 6 10 15 21 28 36 43 49 54 58 61 63 64 64 64 63 62 60 58 55 52 48 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,30,33)	45	1460	1 3 6 10 15 21 28 36 43 49 54 58 61 63 64 64 64 64 63 62 60 58 55 52 48 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,30,35)	46	1524	1 3 6 10 15 21 28 36 43 49 54 58 61 63 64 64 64 64 64 63 62 60 58 55 52 48 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,30,37)	47	1588	1 3 6 10 15 21 28 36 43 49 54 58 61 63 64 64 64 64 64 64 63 62 60 58 55 52 48 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,34,35)	50	1983	1 3 6 10 15 21 28 36 45 53 60 66 71 75 78 80 81 81 81 80 79 77 75 72 69 65 61 56 52 48 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,34,37)	51	2064	1 3 6 10 15 21 28 36 45 53 60 66 71 75 78 80 81 81 81 81 80 79 77 75 72 69 65 61 56 52 48 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,34,39)	52	2145	1 3 6 10 15 21 28 36 45 53 60 66 71 75 78 80 81 81 81 81 81 80 79 77 75 72 69 65 61 56 52 48 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1
(4,34,41)	53	2226	1 3 6 10 15 21 28 36 45 53 60 66 71 75 78 80 81 81 81 81 81 81 80 79 77 75 72 69 65 61 56 52 48 44 40 37 33 30 27 24 21 19 16 14 12 10 8 7 5 4 3 2 1 1

The table leads to the following

Conjecture 23 *For a unibranched plane singularity with characteristic Puiseux exponents* $(4, 2q, s)$ *the* $\delta_X + 1 = 2q + (s-1)/2$ *even Betti numbers of the cohomology of its Jacobi factor are as follows:*

1. *the first* $(q+1)/2$ *even Betti numbers* $h^0(J_X), h^2(J_X), \ldots, h^{q-1}(J_X)$ *are the same as the first* $(q+1)/2$ *coefficients of the power series* $(1-t)^{-3}$.
2. *the last* $(3q+1)/2$ *even Betti numbers* $h^{2\delta_x}(J_X), h^{2\delta_x-2}(J_X), \ldots, h^{s+q-2}(J_X)$ *are the same as the first* $(3q+1)/2$ *coefficients of the power series* $\prod_{j=1}^{3}(1-t^j)^{-1}$.
3. $h^{2(q-1)}(J_X) = h^{2q}(J_X) = \ldots = h^{s+1}(J_X) = (q+1)^2/4$.
4. *For* $l = 1, \ldots, (q-3)/2$: $h^{2(q-1-l)}(J_X) = h^{2(q-l)}(J_X) - l$.
5. *For* $l = 1, \ldots, (q-5)/2$: $h^{s+1+2l}(J_X) = h^{s-1+2l}(J_X) - \lceil \frac{l}{2} \rceil$.

Part 1 and Part 2 of the conjecture are proven in Theorems 24 and 25. They describe $2q+1$ of the $2q+(s-1)/2$ Betti numbers. Unfortunately, we are not able to prove the remaining parts of the Conjecture. However, the conjecture implies that the sequence of Betti numbers of $J_{X'}$ for a singularity X' with Puiseux exponents $(4, 2q, s+2)$ can be obtained from the sequence for a singularity X with Puiseux exponents $(4, 2q, s)$ by inserting $(q+1)^2/4 = e(J_{X'}) - e(J_X)$ after $h^{2(q-1)}(J_X)$. We prove this partially by showing in Theorems 26 and 27 that the first $(s-q)/2+2$ and the last $(q+s)/2+1$ numbers of the above two sequences are the same, thus determining all or at least $s+3$ of the Betti numbers.

Theorem 24 *Let* X *be a unibranched plane singularity with Puiseux exponents* $(4, 2q, s)$ *and* J_X *its Jacobi factor. Then the even Betti numbers* $h^0(J_X), h^2(J_X), \ldots, h^{2\lfloor \frac{q}{2} \rfloor}(J_X)$ *of the cohomology of* J_X *are the same as the first* $\lfloor \frac{q}{2} \rfloor + 1$ *coefficients of the power series*

$$P := \frac{1}{(1-t)^3}.$$

Proof. The proof is similar to the proof of Theorem 21. For $d \leq \lfloor \frac{q}{2} \rfloor$ we construct a bijection between the monomials of degree d of the polynomial ring in the variables t_1, t_2, t_3 and the admissible semi–modules of $\Gamma = \langle 4, \gamma_1 = 2q, \gamma_2 = \gamma_1 + s \rangle$ of codimension d

$$\begin{array}{cccc} \Phi_d : & \mathrm{Mon}^d & \longrightarrow & \mathrm{Mod}^d(\Gamma) \\ & t_1^{r_1} t_2^{r_2} t_3^{r_3} & \longmapsto & \left\langle \begin{array}{l} a_{00} = 0, a_{01} = \gamma_1 - 4r_1, a_{10} = \gamma_2 - 4(r_1 + r_2), \\ a_{11} = \gamma_2 + \gamma_1 - 4(r_1 + r_2 + r_3) \end{array} \right\rangle . \end{array}$$

The maps Φ_d are well–defined if we can show that a semi–module Δ with a 2×2–basis like the one on the right hand side is admissible and of correct dimension. Admissibility is obvious as $a_{01}+s = a_{10}+4r_2 \in \Delta$. We compute its codimension by comparing it with the Γ–semi–module Γ itself. Due to $r_1 + r_2 + r_3 < q/2$, we have the following ordering

$$a_{00} = 0 < a_{01} \leq \gamma_1 < s < a_{10} \leq a_{01} + s \leq \gamma_2 < a_{11}.$$

As $a_{10}, a_{11} > s$, we find $n = \min((\{s, a_{10}, a_{11}\} + 4\mathbb{N}) \cap [\gamma_1, \infty[) = s$, and the dimension of Δ can be computed as

$$\dim \Delta = \#([0, \gamma_1[\setminus \Delta) + \#([a_{01}, a_{01} + s[\setminus \Delta) + \#([a_{10}, \infty[\setminus \Delta).$$

The semi–module Γ is $\Phi_0(1)$; hence, an analogous formula holds for it as well. Because of the above ordering we get the following partitions

$$\begin{aligned} [0, \gamma_1[\setminus \Gamma &= [0, \gamma_1[\setminus \Delta & &\cup \{a_{01} + 4k \mid 0 \le k < r_1\} \\ -4r_1 + ([\gamma_1, \gamma_1 + s[\setminus \Gamma) &= [a_{01}, a_{01} + s[\setminus \Delta & &\cup \{a_{10} + 4k \mid 0 \le k < r_2\} \\ -4(r_1 + r_2) + ([\gamma_2, \infty[\setminus \Gamma) &= [a_{10}, \infty[\setminus \Delta & &\cup \{a_{11} + 4k \mid 0 \le k < r_3\}. \end{aligned}$$

Therefore, $\operatorname{codim} \Delta = \dim \Gamma - \dim \Delta = r_1 + r_2 + r_3 = d$ as desired.

The maps Φ_d are clearly injective, thus it remains to show that they are surjective, too. We must prove that the modules which are not in the image of some $\Phi_0, \ldots, \Phi_{\lfloor \frac{q}{2} \rfloor}$ have codimension greater than $\lfloor \frac{q}{2} \rfloor$. Since there are two types of admissible modules, this falls naturally into two parts.

Let us assume that we have an admissible module Δ with $a_{01} + s \in \Delta$. Since $a_{01} + s \equiv a_{10} \bmod 4$, we get $a_{01} + s \ge a_{10}$. By the relation between the elements of a 2×2–basis — see the paragraph below Definition 6 — we find r_1, r_2, r_3 with

$$a_{00} = 0, \quad a_{01} = \gamma_1 - 4r_1, \quad a_{10} = \gamma_2 - 4(r_1 + r_2), \quad a_{11} = \gamma_2 + \gamma_1 - 4(r_1 + r_2 + r_3).$$

We claim the following rough estimate

$$(\star) \qquad \operatorname{codim} \Delta \ge r_1 + \min\{r_2, q + 1 - r_3\} + \min\left\{r_3, \left\lceil \tfrac{q}{2} \right\rceil\right\}.$$

This implies in particular that if $r_1 + r_2 + r_3 > \lfloor \frac{q}{2} \rfloor$ then $\operatorname{codim} \Delta > \lfloor \frac{q}{2} \rfloor$, i.e., any admissible semi–module Δ with $a_{01} + s \in \Delta$ that is not in the image of some $\Phi_0, \ldots, \Phi_{\lfloor \frac{q}{2} \rfloor}$ has a codimension greater than $\lfloor \frac{q}{2} \rfloor$.

We prove the claim $(\star)$ by modifying Γ into Δ in three steps. The first step consists of the remark that the module $\Delta' = \Phi_{r_1}(t_1^{r_1})$ was described above in detail. In particular, we found $\operatorname{codim} \Delta' = r_1$ and for its 2×2–basis (a'_{ij}) the ordering

$$a'_{00} = 0 < a'_{01} = a_{01} < \gamma_1 < s < a'_{10} = a_{01} + s = a_{01} + n' \le \gamma_2 < a'_{11} = a'_{10} + \gamma_1.$$

In the second step we consider the semi–module $\tilde{\Delta}$ with 2×2–basis $\tilde{a}_{00} = 0$, $\tilde{a}_{01} = a_{01}$, $\tilde{a}_{10} = a'_{10} - 4r_2 = a_{10}$, and $\tilde{a}_{11} = a'_{11} - 4r_2 = a_{10} + \gamma_1$, i.e., $\tilde{\Delta}$ is obtained from Δ' by closing the $2r_2$ gaps $a'_{10} - 4k, a'_{11} - 4k$ for $k = 1, \ldots, r_2$. We write the dimension formulas as

$$\begin{aligned} \dim \Delta' &= (g_{\Delta'}(0) + g_{\Delta'}(a_{01}) - g_{\Delta'}(\gamma_1)) + g_{\Delta'}(a'_{10}) - g_{\Delta'}(a_{01} + n') \\ \dim \tilde{\Delta} &= (g_{\tilde{\Delta}}(0) + g_{\tilde{\Delta}}(a_{01}) - g_{\tilde{\Delta}}(\gamma_1)) \quad + g_{\tilde{\Delta}}(\tilde{a}_{10}) - g_{\tilde{\Delta}}(a_{01} + \tilde{n}). \end{aligned}$$

Since $0 < a_{01} \le \gamma_1$ the closing of any $2r_2$ gaps in Δ' decreases the term in the brackets for Δ' to the terms for $\tilde{\Delta}$ by at least $2r_2$. Next note that $a_{01} + n' \ge a'_{10}$ by the definition of n', thus the only gaps greater than or equal to $a_{01} + n'$ in Δ' are those which are $a'_{11} \bmod 4$ and analogously for $\tilde{\Delta}$. Since $\tilde{n} \le n' = s$, we can estimate the length of the intervals $[a_{01}+\tilde{n}, \tilde{a}_{11}[$ and $[a_{01}+n', a'_{11}[$ by $\tilde{a}_{11}-(a_{01}+\tilde{n}) \geq a'_{11}-4r_2-(a_{01}+s) = \gamma_1 - 4r_2$ and $a'_{11}-(a_{01}+s) = \gamma_1$; hence, $g_{\tilde{\Delta}}(a_{01}+\tilde{n}) \geq g_{\Delta'}(a_{01}+n') - \min\{r_2, \lfloor \frac{\gamma_1}{4} \rfloor\}$. Summing up, we get as an intermediate result

$$\dim \Delta' \geq \dim \tilde{\Delta} + r_2 + \max\left\{0, r_2 - \left\lfloor \frac{q}{2} \right\rfloor\right\} + g_{\Delta'}(a_{10}) - g_{\tilde{\Delta}}(\tilde{a}_{10}).$$

The only gaps in Δ' after a'_{10} are the $\lfloor \frac{q}{2} \rfloor$ gaps which are equal to a'_{11} mod 4. $\tilde{\Delta}$ has also $\lfloor \frac{q}{2} \rfloor$ gaps equal to $\tilde{a}_{11}$ mod 4 after $\tilde{a}_{10}$, but may have in addition some that are equal to a_{01} mod 4 if $\tilde{a}_{10} < a_{01}-4$. In this case set $\tilde{r}_2 = \lceil \frac{s}{4} \rceil$ then $a'_{10} - 4\tilde{r}_2 \in]a_{01}-4, a_{01}[$ and thus $a_{01}-4(r_2-\tilde{r}_2), a_{01}-4(r_2-\tilde{r}_2-1), \ldots, a_{01}-4$ are the addition gaps. Therefore, $g_{\tilde{\Delta}}(\tilde{a}_{10}) - g_{\Delta'}(a_{10}) = \max\{0, r_2 - \lceil \frac{s}{4} \rceil\}$ and in the whole $\operatorname{codim} \tilde{\Delta} \geq \operatorname{codim} \Delta' + r_2 = r_1 + r_2$.

In the final step we compare the codimensions of $\tilde{\Delta}$ and Δ itself. The only difference in the 2×2–bases of $\tilde{\Delta}$ and Δ is that $a_{11} = \tilde{a}_{11} - 4r_3$, i.e., we are closing the r_3 gaps $\tilde{a}_{11}-4, \ldots, \tilde{a}_{11}-4r_3 = a_{11}$ in $\tilde{\Delta}$. By the same argument as before, the term $g_{\tilde{\Delta}}(0) + g_{\tilde{\Delta}}(a_{01}) - g_{\tilde{\Delta}}(\gamma_1)$ is at least r_3 greater than $g_{\Delta}(0) + g_{\Delta}(a_{01}) - g_{\Delta}(\gamma_1)$. Due to $a_{10} \leq a_{01} + \tilde{n}$, all closed gaps equal to or after $a_{01}+\tilde{n}$ are closed gaps after a_{10} as well, thus using $n \leq \tilde{n}$

$$g_{\tilde{\Delta}}(a_{10}) - g_{\tilde{\Delta}}(a_{01}+\tilde{n}) \geq g_{\Delta}(a_{10}) - g_{\Delta}(a_{01}+\tilde{n}) \geq g_{\Delta}(a_{10}) - g_{\Delta}(a_{01}+n).$$

However, for the first time there may be gaps in Δ after a_{11}, and we obtain as an intermediate result only

$$\operatorname{codim} \Delta \geq r_1 + r_2 + r_3 - g_{\Delta}(a_{11}).$$

We can count the gaps after a_{11} precisely. There $\max\{0, r_3 - \lceil \frac{q}{2} \rceil\}$ equal to a_{10} modulo 4 and $\max\{0, r_3 + r_2 - \lceil \frac{\gamma_2}{4} \rceil\}$ equal to a_{01} modulo 4. Using $\lceil \frac{\gamma_2}{4} \rceil = \lceil \frac{2q+s}{4} \rceil \geq q+1$, we obtain ($\star$) by

$$\begin{aligned}\operatorname{codim} \Delta &\geq r_1 + (r_2 - \max\{0, r_2 + r_3 - (q+1)\}) + \left(r_3 - \max\left\{0, r_3 - \left\lceil \tfrac{q}{2} \right\rceil\right\}\right) \\ &= r_1 + \min\{r_2, q+1-r_3\} + \min\left\{r_3, \left\lceil \tfrac{q}{2} \right\rceil\right\}.\end{aligned}$$

The second type of admissible modules are those with $s \in \Delta$. Let us assume in addition that $a_{10} > a_{01}+s$ for a semi–module Δ, otherwise we have $a_{01}+s \in \Delta$ and Δ is admissible of first type as well. We show that for all these semi–modules $\operatorname{codim} \Delta \geq \lceil \frac{q}{2} \rceil$ by comparing Δ with the following simple semi–module

$$\overline{\Delta} = \langle \overline{a}_{00} = 0, \overline{a}_{01} = \gamma_1, \overline{a}_{10} = \gamma_2, \overline{a}_{11} = s = \gamma_2 + \gamma_1 - 4q \rangle.$$

We find $\overline{n} = s$ and $\dim \overline{\Delta} = g_{\overline{\Delta}}(0) + g_{\overline{\Delta}}(s) = g_{\Gamma}(0) - q + \lfloor \frac{q}{2} \rfloor = \dim \Gamma - \lceil \frac{q}{2} \rceil$; hence $\operatorname{codim} \overline{\Delta} = \lceil \frac{q}{2} \rceil$. We will modify $\overline{\Delta}$ in three steps into Δ and show that the codimension does not decrease during these modifications.

First let $\Delta' = \langle 0, a_{01}, \overline{a}_{10}, \overline{a}_{11} \rangle$. If α_{01} is chosen such that $a_{01} = \gamma_1 - 4\alpha_{01}$, then we are closing the α_{01} gaps $a_{01}, a_{01}+4, \ldots, \gamma_1 - 4$ in $\overline{\Delta}$. Hence, $g_{\Delta'}(k) = g_{\overline{\Delta}}(k)$ for $k \geq \gamma_1$. Further $\overline{n} = n' = s$ and $g_{\Delta'}(0) = g_{\overline{\Delta}}(0) - \alpha_{01}$, $g_{\Delta'}(a_{01}) = g_{\overline{\Delta}}(\overline{a}_{01}) + 2\alpha_{01}$, $g_{\Delta'}(a_{01}+s) = \alpha_{01}$, as well as $g_{\overline{\Delta}}(\overline{a}_{01}+s) = 0$. Plugging this into the dimension formulas yields $\dim \overline{\Delta} = \dim \Delta'$.

Next we modify Δ' into $\tilde{\Delta} = \langle 0, a_{01}, a_{10}, \overline{a}_{11} \rangle$. Setting $\alpha_{10} = (\overline{a}_{10} - a_{10})/4$, this means that we are closing the α_{10} gaps $a_{10}, a_{10}+4, \ldots, \overline{a}_{10} - 4$ of Δ'. We have the ordering $a_{10} > a_{01} + s > s = \overline{a}_{11}$ thus $\tilde{n} = s$ and $g_{\tilde{\Delta}}(k) = g_{\Delta'}(k) - \alpha_{10}$ for $k \leq a_{10}$. In addition $g_{\tilde{\Delta}}(a_{10}) = g_{\Delta'}(\overline{a}_{10}) = 0$, because a_{10} resp. $\overline{a}_{10}$ are the greatest elements of the 2×2–bases of $\tilde{\Delta}$ resp. Δ'. From the dimension formula we obtain $\operatorname{codim} \tilde{\Delta} = \operatorname{codim} \Delta' + \alpha_{10} \geq \lceil \frac{q}{2} \rceil$.

Finally, the semi–module Δ is obtained from $\tilde{\Delta}$ by closing the $(s - a_{11})/4 =: \beta$ gaps $a_{11}, a_{11} - 4, \ldots, s - 4 = \overline{a}_{11} - 4$. By definition $n \leq \tilde{n} = s$ and because no gaps after s were closed, we obtain $g_\Delta(a_{01} + n) \geq g_\Delta(a_{01} + \tilde{n}) = g_{\tilde{\Delta}}(a_{01} + \tilde{n})$. As usual, $g_{\tilde{\Delta}}(0) + g_{\tilde{\Delta}}(a_{01}) - g_{\tilde{\Delta}}(\gamma_1)$ is at least β greater than the corresponding term for Δ. From the semi–module property $a_{11} \geq a_{10} - \gamma_1$ we conclude $a_{11} > a_{01} + s - \gamma_1 > a_{01}$. Thus the only gaps after a_{11} resp. $\overline{a}_{11}$ are those which are equal to a_{10} modulo 4, and we find $g_\Delta(a_{11}) = g_{\tilde{\Delta}}(\overline{a}_{11}) + \beta$. Summing up, we obtain $\operatorname{codim} \Delta \geq \operatorname{codim} \tilde{\Delta} \geq \lceil \frac{q}{2} \rceil$ again. □

Theorem 25 *Let X be a unibranched plane singularity with Puiseux exponents $(4, 2q, s)$ and J_X its Jacobi factor. Set $k := (3q-1)/2$. Then the even Betti numbers $h_0(J_X), h_2(J_X), \ldots, h_{2k}(J_X)$ of the homology of J_X are the same as the first $k+1$ coefficients of the power series*

$$P := \frac{1}{(1-t)(1-t^2)(1-t^3)}.$$

Proof. The beginning of this proof is the same as the one of the proof of Theorem 22 with p replaced by 4 and q replaced by $\gamma_1 = 2q$. Of all the modules $\Delta_r = \psi(t_1^r)$ the following two will be of special importance at the end:

$$\begin{aligned} \Delta_k \quad &= 4\mathbb{N} \cup (\gamma_1 + \mathbb{N}) \qquad \text{with } c(\Delta_k) = \gamma_1 \quad \text{and} \\ \Delta_{k-1} &= 4\mathbb{N} \cup (\gamma_1 - 2 + \mathbb{N}) \quad \text{with } c(\Delta_{k-1}) = \gamma_1 - 2. \end{aligned}$$

The last step of the proof, where one proves that the maps $\Psi_d : \mathrm{Mon}_d \to \mathrm{Mod}_d(\Gamma)$ are surjective has to be modified due to the different dimension formula. As before we show that $c(\Delta) > \gamma_1$ implies $\dim \Delta > k$ by an inductive process. Let Δ' be the Γ–semi–module obtained from Δ by closing the last gap. If $\{b_0 = 0, b_1, b_2, b_3\}$ is an ordered 4–basis of Δ then its conductor is $c := b_3 - 3$. Because $c > \gamma_1$, b_3 is the element a_{10} or a_{11} is a 2×2–basis of Δ. The module Δ' has the unordered 4–basis $\{b'_j\}$ with $b'_0 = 0$, $b'_1 = b_1$, $b'_2 = b_2$, and $b'_3 = b_3 - 4$, and its conductor is $c' = \max\{b_2 - 3, b_3 - 3\}$. The dimension formula says

$$\dim \Delta = (g_\Delta(0) + g_\Delta(a_{01}) - g_\Delta(\gamma_1)) + g_\Delta(a_{10}) + g_\Delta(a_{11}) - g_\Delta(a_{01} + n)$$

and analogously for Δ'. Since we are closing one gap in Δ greater than γ_1, the term in the brackets decreases by one for Δ'. Because $b_3 = a_{10}$ or $b_3 = a_{11}$, $g_\Delta(a_{11})$ resp. $g_\Delta(a_{10})$ decreases by one or stays the same. By definition $b_3 > c$ and $b'_3 \geq c'$, hence $g_\Delta(b_3) = g_{\Delta'}(b'_3) = 0$. The number n may stay the same or be reduced by at most 4. If $n' = n$, then obviously $g_\Delta(a_{01} + n)$ decreases by one or stays the same. If $n' < n$, then b'_3 must be the smallest odd number in Δ'. Hence, the smallest odd number in Δ is $b'_3 + 2 = b_3 - 2 = b_2 = n$. Due to $a_{01} \geq 2$ and $\gamma_1 + 2\mathbb{N} \subset \Delta, \Delta'$, we find $0 \leq g_\Delta(a_{01} + n) \leq g_\Delta(b_3) = 0$ and $0 \leq g_{\Delta'}(a_{01} + n') \leq g_{\Delta'}(b'_3) = 0$, showing $g_\Delta(a_{01} + n) = g_{\Delta'}(a_{01} + n') = 0$. Summing up the changes, we obtain $\dim \Delta \geq \dim \Delta'$.

Now, let us assume additionally that $c' \le \gamma_1 < c$, i.e., we are closing the last gap, $\gamma_1 + 1$ or $\gamma_1 + 3$, greater than γ_1. Choose the index J such that $a_{1J} = b_3$. We first consider the case where the last gap is $\gamma_1 + 3$. Here we have $a_{1,1-J} \le \gamma_1 + 1$, thus $g_\Delta(a_{1,1-J})$ decreases by one during this process. The above discussion yields $\dim \Delta > \dim \Delta'$. Due to $c(\Delta') = \gamma_1$, we get $\dim \Delta > \dim \Delta' \ge \dim \Delta_k = k$ by (†).

Finally, we consider the other case, where the last gap of Δ is $\gamma_1 + 1$. Because we always have $n \ge \gamma_1 + 1$ and $a_{01} + n \ge \gamma_1 + 3$, we get $g_\Delta(a_{01} + n) = g_{\Delta'}(a_{01} + n') = 0$. Therefore, by the above discussion $\dim \Delta > \dim \Delta'$ and if $c(\Delta') = \gamma_1$, we can finish the proof like above. However, $c(\Delta')$ may as well be $\gamma_1 - 2$. Here $a_{1,1-J} < \gamma_1 < b_3'$ thus $g_{\Delta'}(a_{1,1-J}) = g_\Delta(a_{1,1-J}) - 1$ and the above discussion yields $\dim \Delta > \dim \Delta' + 1$. Using $c(\Delta_{k-1}) = \gamma_1 - 2$ and (†), we obtain $\dim \Delta > \dim \Delta' + 1 \ge \dim \Delta_{k-1} + 1 = k$. □

Theorem 26 *Let X and X' be unibranched plane singularities with Puiseux exponents $(4, 2q, s)$ resp. $(4, 2q, s')$ with $s' \ge s$ and J_X resp. $J_{X'}$ their Jacobi factors. Set $k := (s - q)/2 + 1$. Then the first $k + 1$ even Betti numbers of the cohomology of J_X and $J_{X'}$ are the same, i.e., $h^{2d}(J_X) = h^{2d}(J_{X'})$ for $d = 0, \dots, k$.*

Proof. Let Γ and Γ' be the semi–groups corresponding to the singularities. By induction we may assume $s' = s + 2$. We are going to show that the following map is well–defined and bijective for $d \le k$

$$\begin{array}{rccc} \Phi_d : & \mathrm{Mod}^d(\Gamma) & \longrightarrow & \mathrm{Mod}^d(\Gamma') \\ & \Delta = \langle 0, a_{01}, a_{11}, a_{11} \rangle & \longmapsto & \Delta' = \langle 0, a_{01}, a'_{10} = a_{10} + 2, a'_{11} = a_{11} + 2 \rangle \end{array}$$

If the 2×2–basis of Δ is written as $a_{00} = 0$, $a_{01} = \gamma_1 - 4\alpha_{01}$, $a_{10} = \gamma_2 - 4\alpha_{10}$, $a_{11} = \gamma_2 + \gamma_1 - 4\alpha_{11}$ then Δ' is the Γ'–semi–module whose 2×2–basis has the same α_{ij}. Δ' is admissible, because $s = a_{11} + 4l \in \Delta$ or $a_{01} + s = a_{10} + 4l \in \Delta$ implies $s' = a'_{11} + 4l \in \Delta'$ or $a_{01} + s' = a'_{10} + 4l \in \Delta'$. The injectivity of the map is trivial, its well–definedness and surjectivity will follow from the statement

(⊛) Let Δ be an admissible Γ–semi–module and J the index with $a_{1J} = \min\{a_{10}, a_{11}\}$ then

$$\begin{array}{lll} a) & \operatorname{codim} \Delta \le k & \Longrightarrow a_{10}, a_{11} > \gamma_1 \quad\;\;\; \text{or } (a_{1J}, a_{1,1-J}) = (\gamma_1 - 1, 2\gamma_1 - 1) \\ b) & \operatorname{codim} \Delta \le k - 1 & \Longrightarrow a_{10}, a_{11} > \gamma_1 + 2 \text{ or } (a_{1J}, a_{1,1-J}) = (\gamma_1 + 1, 2\gamma_1 + 1) \end{array}$$

Assume we have proven (⊛). For the well–definedness of Φ_d we need to show that $\operatorname{codim} \Delta' = \operatorname{codim} \Delta$ or equivalently $\dim \Delta' = \dim \Delta + 1$. If $a_{10}, a_{11} > \gamma_1$, then we obtain Δ' from Δ by inserting a gap and nongap after γ_1, more precisely $\Delta' = (\Delta \cap [0, \gamma_1]) \cup (2 + (\Delta \cap [\gamma_1, \infty[))$. Here $n = \min\{s, a_{10}, a_{11}\}$ and $n' = \min\{s', a'_{10}, a'_{11}\}$. In the dimension formula for Δ' the term $g_{\Delta'}(0) + g_{\Delta'}(a_{01}) - g_{\Delta'}(\gamma_1)$ is by one greater than the corresponding term for Δ because of the extra gap after γ_1. In contrast $g_{\Delta'}(a_{10}) + g_{\Delta'}(a_{11}) - g_{\Delta'}(a_{01} + n')$ is the same as the term for Δ, because everything is shifted by 2. Hence, $\dim \Delta' = \dim \Delta + 1$ as desired.

If $(a_{1J}, a_{1,1-J}) = (\gamma_1 - 1, 2\gamma_1 - 1)$ then $(a'_{1J}, a'_{1,1-J}) = (\gamma_1 + 1, 2\gamma_1 + 1)$, $n = \gamma_1 + 3$, and $n' = \gamma_1 + 1$. Obviously, $\#([a_{01}, \gamma_1[\setminus \Delta) = \#([a_{01}, \gamma_1[\setminus \Delta') + \varepsilon$

where $\varepsilon = 0$ if $a_{01} = \gamma_1$ and $\varepsilon = 1$ otherwise. Further, $g_\Delta(0) = g_{\Delta'}(0) + 1$, $g_\Delta(a_{10}) = g_{\Delta'}(a'_{10})$, and $g_\Delta(a_{11}) = g_{\Delta'}(a'_{11})$. Thus the interesting terms are $g_\Delta(a_{01}+n) = g_\Delta(2\gamma_1 - 4\alpha_{01} + 3)$ and $g_{\Delta'}(a_{01}+n') = g_{\Delta'}(2\gamma_1 - 4\alpha_{01} + 1)$. As shifting by 2 gives a bijection between $[2\gamma_1 - 4\alpha_{01}+3, \infty[$ and $[2\gamma_1 - 4\alpha_{01}+5, \infty[$, which respects membership in Δ resp. Δ', we have $g_{\Delta'}(a_{01}+n') = g_\Delta(a_{01} + n) + \#([2\gamma_1 - 4\alpha_{01}+1, 2\gamma_1 - 4\alpha_{01} + 5[\setminus \Delta')$. From $a_{01}, a'_{1J} \leq \gamma_1 + 1$ we see that the only possible gap in in the above interval must be equal to $a'_{1,1-J} = 2\gamma_1 + 1$ modulo 4, i.e., it can only be $2\gamma_1 - 4\alpha_{01} + 1$. For this to be a gap, we must have $\alpha_{01} > 0$, hence $g_{\Delta'}(a_{01}+n') = g_\Delta(a_{01}+n) + \varepsilon$. This shows that we always have $\dim \Delta' = \dim \Delta + 1$.

The surjectivity follows now, too. Let $\Delta' \in \mathrm{Mod}^d(\Gamma')$ with $d \leq k$. As $k = k' - 1$ we may apply *b)* to Δ' to obtain $a'_{10}, a'_{11} > \gamma_1 + 2$ or $(a'_{10}, a'_{11}) = (\gamma_1 + 1, 2\gamma_1 + 1)$. Thus Δ' is the image of $\Delta = \langle 0, a_{01}, a'_{10} - 2, a'_{11} - 2\rangle$ under Φ_d — that Δ has the correct dimension was shown above.

We prove the statement ($\circledast$) by first considering two special types of semi–modules and then compare the other modules with them. Define

$$\Delta^{10}_\alpha := \langle 0, a_{01} = \gamma_1, a_{10} = \gamma_2 - 4\alpha, a_{11} = \gamma_2 + \gamma_1 - 4\alpha\rangle \text{ for } \left\lceil \tfrac{q}{2} \right\rceil \leq \alpha \leq \left\lfloor \tfrac{\gamma_2}{4} \right\rfloor$$

$$\Delta^{11}_\alpha := \langle 0, a_{01} = \gamma_1, a_{10} = \gamma_2 - 4\alpha, a_{11} = s - 4\alpha\rangle \qquad \text{for } 0 \leq \alpha \leq \left\lfloor \tfrac{s}{4} \right\rfloor$$

the definition is such that in Δ^{1J}_α the minimum of a_{10} and a_{11} is a_{1J} and $a_{1,1-J} = a_{1J} + \gamma_1$. Their dimension is computed easily: Using $n \geq a_{1J} \Rightarrow n + \gamma_1 \geq a_{1,1-J} \geq c(\Delta^{1J}_\alpha)$, we find $g_{\Delta^{1J}_\alpha}(\gamma_1 + n) = 0$

$$\dim \Delta^{10}_\alpha = g_{\Delta^{10}_\alpha}(0) + g_{\Delta^{10}_\alpha}(a_{10}) = \dim \Gamma - 2\alpha + \left\lfloor \tfrac{q}{2} \right\rfloor$$
$$\Longrightarrow \operatorname{codim} \Delta^{10}_\alpha = 2\alpha - \left\lfloor \tfrac{q}{2} \right\rfloor$$

$$\dim \Delta^{11}_\alpha = g_{\Delta^{11}_\alpha}(0) + g_{\Delta^{11}_\alpha}(a_{11}) = \dim \Gamma - (2\alpha + q) + \left\lfloor \tfrac{q}{2} \right\rfloor$$
$$\Longrightarrow \operatorname{codim} \Delta^{11}_\alpha = 2\alpha + \left\lceil \tfrac{q}{2} \right\rceil$$

We claim that for these two types of semi–modules $a_{1J} \leq \gamma_1 - 1$ implies $\operatorname{codim} \Delta^{1J}_\alpha \geq k$. If $s \equiv 1 \bmod 4$, then $a_{1J} \leq \gamma_1 - 1$ is equivalent to $\alpha \geq (s+3)/4$ for Δ^{10}_α and $\alpha \geq (s - 2q + 1)/4$ for Δ^{11}_α and their codimension is bounded by

$$\operatorname{codim} \Delta^{1J}_\alpha \geq \min\left\{ \tfrac{s+3}{2} - \left\lfloor \tfrac{q}{2} \right\rfloor, \tfrac{s-2q+1}{2} + \left\lceil \tfrac{q}{2} \right\rceil \right\} = \tfrac{s+1}{2} - \left\lfloor \tfrac{q}{2} \right\rfloor = k.$$

An analogous consideration for $s \equiv 3 \bmod 4$ yields the same result. Obviously, $a_{1J} \leq \gamma_1 + 1$ implies $\operatorname{codim} \Delta^{1J}_\alpha \geq k - 1$ in the same way. Finally, note that the codimension is strictly increasing in α.

Now ($\circledast$) follows from this and the following comparison statement, which we prove in a moment:

($\circledast\circledast$) Let Δ be an admissible Γ–semi–module with 2×2–basis (a_{ij}). Let J be the index with $a_{1J} = \min\{a_{10}, a_{11}\}$. Assume that $a_{1J} \leq \gamma_1 + 1$. Let Δ^{1J} be the unique special semi–module like above with the same a_{1J}. Then $\operatorname{codim} \Delta \geq \operatorname{codim} \Delta^{1J}$ with strict inequality if $a_{1J} + \gamma_1 > a_{1,1-J}$.

For example, we show ($\circledast$ *a*). Let Δ be a semi–module as in ($\circledast\circledast$) with $a_{1J} \leq \gamma_1 - 1$ then $\Delta^{1J} = \langle 0, \gamma_1, a_{1J}, a_{1J} + \gamma_1\rangle$. Now ($\circledast\circledast$) implies $\operatorname{codim} \Delta \geq$

codim $\Delta^{1J} \geq k$ and equality holds only for $a_{1,1-J} = a_{1J} + \gamma_1$ and $a_{1J} = \gamma_1 - 1$. This is the statement ($\circledast$ a).

To prove the claim ($\circledast\circledast$) we modify Δ^{1J} in two steps into Δ and watch for the dimension changes. The 2×2–basis of Δ^{1J} is by definition $(0, \gamma_1; a_{1J}, a_{1J} + \gamma_1)$ — up to the oder of the last two elements. Let $\alpha_{01} := (\gamma_1 - a_{01})/4$ and define $\tilde{\Delta} = \Delta^{1J} \cup \{a_{01}, \ldots, \gamma_1 - 4\}$, i.e., $\tilde{\Delta}$ has a 2×2–basis $(0, a_{01}; a_{1J}, a_{1J} + \gamma_1)$. We compare its dimension

$$\dim \tilde{\Delta} = g_{\tilde{\Delta}}(0) + \#([a_{01}, \gamma_1[\setminus \tilde{\Delta}) + g_{\tilde{\Delta}}(a_{1J}) - g_{\tilde{\Delta}}(a_{01} + \tilde{n})$$

with the dimension $g_{\Delta^{1J}}(0) + g_{\Delta^{1J}}(a_{1J})$ of Δ^{1J}. Since we are closing α_{01} gaps, we find $g_{\tilde{\Delta}}(0) = g_{\Delta^{1J}}(0) - \alpha_{01}$ and $g_{\tilde{\Delta}}(a_{1J}) \leq g_{\Delta^{1J}}(a_{1J})$. In the interval $[a_{01}, \gamma_1[$ there are α_{01} gaps in $\tilde{\Delta}$ equal to $a_{1J} + \gamma_1$ modulo 4. The only other possible gaps in this interval have to be equal to a_{1J} modulo 4. Let l be one of them. Then $l + \gamma_1 \equiv a_{1J} + \gamma_1 \mod 4$ and $a_{01} < l < \min\{\gamma_1, a_{10}\}$ implies $a_{01} + \gamma_1 < l + \gamma_1 < \min\{a_{1J} + \gamma_1, 2\gamma_1\}$; hence, $l + \gamma_1$ is also a gap in $\tilde{\Delta}$. Now we have either $\tilde{n} = \gamma_1 + 1$ or $\tilde{n}$ is the smallest number greater than γ_1 and equal to a_{1J} mod 4. In the first case we have trivially $a_{01} + \tilde{n} \leq \gamma_1 + l$; in the second case $a_{01} + \tilde{n}$ is the smallest number greater than $a_{01} + \gamma_1$ and equal to $a_{1J} + \gamma_1$ mod 4, and we get again $a_{01} + \tilde{n} \leq l + \gamma_1$. Therefore, we found for any of the gaps in $([a_{01}, \gamma_1[\setminus \Delta) \cap (a_{1J} + 4\mathbb{Z})$ a gap that contributes to $g_{\tilde{\Delta}}(a_{01} + \tilde{n})$. Summing up the changes, we obtain $\dim \tilde{\Delta} \leq \dim \Delta^{1J}$.

We obtain Δ from $\tilde{\Delta}$ by closing the $\eta := (a_{1J} + \gamma_1 - a_{1,1-J})/4$ gaps $\{a_{1,1-J}, a_{1,1-J} + 4, \ldots, a_{1J} + \gamma_1 - 4\}$. Due to our assumption $a_{1J} < a_{1,1-J}$, the computation of the dimension of Δ is easy. Obviously, $g_\Delta(0) = g_{\tilde{\Delta}}(0) - \eta$, $g_\Delta(a_{1J}) = g_{\tilde{\Delta}}(a_{1J}) - \eta$, and $\#([a_{01}, \gamma_1[\setminus \Delta) \leq \#([a_{01}, \gamma_1[\setminus \tilde{\Delta})$. Because $n \leq \tilde{n}$, we find $g_\Delta(a_{01} + n) \geq g_\Delta(a_{01} + \tilde{n}) \geq g_{\tilde{\Delta}}(a_{01} + \tilde{n}) - \eta$. Finally, $g_\Delta(a_{1,1-J})$ maybe nonzero this time, but there can only be gaps equal to a_{01} modulo 4 after $a_{1,1-J}$, thus $g_\Delta(a_{1,1-J}) \leq \eta$. In fact, $g_\Delta(a_{1,1-J}) \leq \max\{\eta - 1, 0\}$, using $a_{01} < a_{1J} + \gamma_1$. Summation yields $\dim \Delta + \min\{\eta, 1\} \leq \dim \tilde{\Delta} \leq \dim \Delta^{1J}$. □

Theorem 27 *Let X and X' be unibranched plane singularities with Puiseux exponents $(4, 2q, s)$ resp. $(4, 2q, s')$ with $s' \geq s$ and J_X resp. $J_{X'}$ their Jacobi factors. Set $k := (q + s)/2$. Then the first $k + 1$ even Betti numbers of the homology of J_X and $J_{X'}$ are the same, i.e., $h_{2d}(J_X) = h_{2d}(J_{X'})$ for $d = 0, \ldots, k$.*

Proof. We will prove that $\mathrm{Mod}_d(\Gamma) = \mathrm{Mod}_d(\Gamma')$ for $d = 0, \ldots, k$, where Γ and Γ' are the semi–groups of the singularities X resp. X'. By induction we may restrict to the case $s' = s + 2$. We claim the following:

$$(\ddagger) \qquad \begin{array}{ll} a) \text{ For } \Delta \in \mathrm{Mod}_d(\Gamma) \text{ with } d \leq k & : \; c(\Delta) \leq s + 1. \\ b) \text{ For } \Delta \in \mathrm{Mod}_d(\Gamma) \text{ with } d \leq k - 1 & : \; c(\Delta) \leq s - 1. \end{array}$$

The obvious consequence of a) is that any such Γ–semi–module Δ is a Γ'–semi–module as well. In fact, its dimension as a Γ–semi–module and Γ'–semi–module must be the same. Namely, the terms of the dimension formula depend only on the the 4–basis of Δ as a $\langle 4 \rangle$–semi–module — with the exception of the computation of n. However, if n differs for Δ as a Γ–semi–module and Γ'–semi–module then n must be s for Δ as a Γ–semi–module and even bigger for Δ as a

Γ'–semi–module; hence, $a_{01} + s \geq 2 + s > c(\Delta)$ shows that $g_\Delta(a_{01} + n) = 0$ in both cases. Therefore, we have an inclusion $\mathrm{Mod}_d(\Gamma) \subseteq \mathrm{Mod}_d(\Gamma')$.

To prove equality, apply $b)$ to a $\Delta \in \mathrm{Mod}_d(\Gamma')$. We find $c(\Delta) \leq s' - 1 = s + 1$, thus Δ is also a Γ–semi–module, and we have just shown that it has the same dimension d as a Γ–semi–module.

The claim (‡) is proven by comparing Δ with simpler semi–modules. For $c \in \mathbb{N} \setminus (1+\Gamma)$ define Δ_c as the Γ–semi–module $\Delta_c = \Gamma \cup (c + \mathbb{N})$, then $c(\Delta_c) = c$. The dimension of $\Delta_{s-1} = \Gamma + s\mathbb{N} = \langle 0, \gamma_1, s+2, s \rangle = \left\langle 0, \gamma_1, \gamma_2 - 4\left\lfloor \frac{q}{2} \right\rfloor, \gamma_2 + \gamma_1 - 4q \right\rangle$ is

$$\dim \Delta_{s-1} = g_{\Delta_{s-1}}(0) = g_\Gamma(0) - q - \left\lfloor \frac{q}{2} \right\rfloor = \frac{\gamma_2 + \gamma_1 - 3}{2} - q - \left\lfloor \frac{q}{2} \right\rfloor = k - 1.$$

Clearly, the dimensions of $\Delta_{s+1} = \Delta_{s-1} \setminus \{s\}$ and $\Delta_{s+3} = \Delta_{s-1} \setminus \{s, s+2\}$ are k resp. $k + 1$.

Now (‡) follows from the obvious fact that $\dim \Delta_c$ is monotone increasing and

(‡‡) for $\Delta \in \mathrm{Mod}(\Gamma)$ with $c := c(\Delta) \geq s$ we have $\dim \Delta \geq \dim \Delta_c$.

For example, assume $c \geq s + 2$. Since the even number $s + 1$ lies in $\Gamma \subset \Delta$ we find $c \geq s + 3$, thus $\dim \Delta \geq \dim \Delta_c \geq \dim \Delta_{s+3} = k + 1$.

It remains to prove (‡‡) by modifying Δ_c in two steps into Δ. Let $\{b_0 = 0, b_1, b_2, b_3\}$ be an ordered 4–basis of Δ_c as a $\langle 4 \rangle$–semi–module. By the definition of Δ_c we find $b_1 = \gamma_1$, $c = b_3 - 3$, and $b_2 = \gamma_2$ or $b_2 = b_3 - 2$. Let $\{e_0 = 0, e_1, e_2, e_3\}$ be a 4–basis of Δ which we order such that $e_i \equiv b_i \bmod 4$. Since the greatest element of the 4–basis of Δ as well as of Δ' is $c + 3$, one gets $e_3 = b_3 = c + 3 > s$.

Setting $\beta := (e_2 - b_2)/4$, we define $\tilde{\Delta}$ to be the semi–module obtained by closing the β gaps $e_2, e_2 + 4, \ldots, e_2 + 4(\beta - 1) = b_2 - 4$ in Δ_c. Let $\hat{\beta}$ be the number of these gaps that are less than γ_1. The dimensions of Δ_c and $\tilde{\Delta}$ are

$$\begin{aligned} \dim \Delta_c &= g_{\Delta_c}(0) + g_{\Delta_c}(b_2) - g_{\Delta_c}(\gamma_1 + s), \\ \dim \tilde{\Delta} &= g_{\tilde{\Delta}}(0) + g_{\tilde{\Delta}}(e_2) - g_{\tilde{\Delta}}(\gamma_1 + \tilde{n}), \end{aligned}$$

making use of $c \geq s \Rightarrow n_c = s$ for Δ_c. Clearly, $g_{\tilde{\Delta}}(0) = g_{\Delta_c}(0) - \beta$. Between e_2 and b_2 in $\tilde{\Delta}$ there are β gaps equal to $b_3 \bmod 4$ and $\max\{0, \hat{\beta} - 1\}$ gaps equal to $b_1 = \gamma_1 \bmod 4$, i.e., $g_{\tilde{\Delta}}(e_2) = g_{\Delta_c}(b_2) + \beta + \max\{0, \hat{\beta} - 1\}$. For the last term $g_{\tilde{\Delta}}(\gamma_1 + \tilde{n})$ we observe the following: If $\tilde{n} = s$ then $g_{\tilde{\Delta}}(\gamma_1 + \tilde{n}) \leq g_{\Delta_c}(\gamma_1 + s)$. If $\tilde{n} < s$ then $e_2 \leq \tilde{n}$ and from the semi–module property $e_3 \leq e_2 + \gamma_1 \leq \tilde{n} + \gamma_1$; hence, $s + \gamma_1 > \tilde{n} + \gamma_1 \geq c$ and $g_{\Delta_c}(\gamma_1 + s) = g_{\tilde{\Delta}}(\gamma_1 + \tilde{n}) = 0$. Summing up the above terms we get $\dim \tilde{\Delta} \geq \dim \Delta_c + \max\{0, \hat{\beta} - 1\}$.

We obtain the semi–module Δ from $\tilde{\Delta}$ by closing the $\eta := (e_1 - \gamma_1)/4$ gaps $e_1, e_1 + 4, \ldots, \gamma_1 - 4$. Again we need to compute the dimension

$$\dim \Delta = g_\Delta(0) + \#([e_1, \gamma_1[\, \setminus \Delta) + g_\Delta(e_2) - g_\Delta(e_1 + n).$$

Obviously, $g_\Delta(0) = g_{\tilde{\Delta}}(0) - \eta$. In the interval $[e_1, \gamma_1[$ there are η gaps equal to b_3 modulo 4 and $\max\{0, \eta - \hat{\beta}\}$ equal to e_2 modulo 4. The number of gaps after e_2 decreases from $\tilde{\Delta}$ to Δ by less than $\max\{0, \hat{\beta} - 1\}$. Therefore,

$$\begin{aligned}\dim \Delta &\geq \dim \tilde{\Delta} - \max\{0, \hat{\beta} - 1\} + \max\{0, \eta - \hat{\beta}\} - g_\Delta(e_1 + n) + g_{\tilde{\Delta}}(\gamma_1 + \tilde{n}) \\ &\geq \dim \Delta_c + \max\{0, \eta - \hat{\beta}\} - g_\Delta(e_1 + n) + g_{\tilde{\Delta}}(\gamma_1 + \tilde{n}).\end{aligned}$$

Note that $n = \tilde{n}$, since Δ and $\tilde{\Delta}$ have the same odd numbers. Further, $g_\Delta(\gamma_1 + n) = g_{\tilde{\Delta}}(\gamma_1 + n)$ because Δ and $\tilde{\Delta}$ differ only in numbers less than γ_1. Therefore, we need to count the gaps of Δ in the interval $[e_1 + n, \gamma_1 + n[$. The gaps in this interval are b_3 modulo 4, because Δ being admissible implies $b_2 \leq e_1 + s$ and $b_2 \leq e_1 + n$ using $n = \min((\{s\} \cup (b_2 + 4\mathbb{N})) \cap [\gamma_1, \infty[)$. In particular, $g_\Delta(e_1 + n) \leq g_{\tilde{\Delta}}(\gamma_1 + n) + \eta$ and the estimate $\dim \Delta \geq \dim \Delta_c$ is obvious for $\hat{\beta} = 0 \Leftrightarrow b_2 > \gamma_1$. However, for any of the $\min\{\eta, \hat{\beta}\}$ numbers $e_2 + 4l \in [e_1, \gamma_1[\cap \Delta$ we find $e_2 + \gamma_1 + 4l \in [e_1 + \gamma_1, 2\gamma_1[\cap \Delta \subset [e_1 + \gamma_1, \gamma_1 + n[\cap \Delta$. Since $e_2 + \gamma_1 + 4l \equiv b_3 \bmod 4$ at least $\min\{\eta, \hat{\beta}\}$ of the positions $[e_1 + n, \gamma_1 + n[\cap (b_3 + 4\mathbb{Z})$ are not gaps. This implies $g_\Delta(e_1 + n) \leq g_{\tilde{\Delta}}(\gamma_1 + \tilde{n}) + \eta - \min\{\eta, \hat{\beta}\}$ showing $\dim \tilde{\Delta} \geq \dim \Delta_c$ even in this case. $\square$

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