Affinely Smooth Developable Varieties of Low Gauss Rank
Extended Version

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Abstract

We study projective varieties whose image of Gauss map has dimension less or equal to four and which are smooth outside the hyperplane at infinity. We describe their geometric structure, and show in particular that they are uniruled by linear spaces which are larger than a priori expected.

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Let $X \subset \mathbb{P}^N$ be an irreducible projective variety of dimension n. Its Gauss map is the rational map

$$\gamma : X - - \rightarrow \mathbb{G}(n, N), \quad x \longmapsto \mathbb{T}_x X,$$

which assigns to every smooth point of X its projective tangent space as a point of the Grassmannian of n–planes in $\mathbb{P}^N$. The variety X is called developable if the dimension of the image of the Gauss map — the Gauss rank r of X — is less than n.

In this article we wish to study smooth affine varieties $X \subset \mathbb{C}^N$, which are developable. However, to describe their geometric structure it will be necessary to consider their behavior at infinity. Therefore, we view X as a projective variety in $\mathbb{P}^N$ which is smooth outside the hyperplane at infinity H_∞. We will call such a variety *affinely smooth*.

The fundamental result about developable varieties is that a general fiber of the Gauss map is a linear space of dimension $d = n - r$. X is singular along a hypersurface of a Gauss fiber F, the *focal hypersurface* of F. The closure of the union of all these focal hypersurfaces is the *focal variety* X_f of X.

The affine smoothness of X forces the focal hypersurfaces to be the intersection of the Gauss fiber and the hyperplane at infinity, in particular the focal

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variety lies in H_∞. For Gauss rank 1 Hartman and Nirenberg proved the following theorem which was reproven and extended in various geometric settings by several authors [HN, A, NP, O].

Theorem. *An affinely smooth developable variety of Gauss rank 1 is a cone over a smooth curve whose vertex lies in the hyperplane at infinity.*

Akivis and Goldberg proved that an affinely smooth developable variety whose second fundamental form has a regular pencil of quadrics with distinct eigenvalues is always a cone [AG_3]. In contrast to this, Bourgain and Wu worked out an example of Gauss rank 2 which is not a cone [W]. Later, Akivis and Goldberg showed that this example is projectively equivalent to an earlier example of Sacksteder [AG_1, S]. Vitter as well as Dajczer and Gromoll proved that an affinely smooth developable variety of Gauss rank 2 is a union of $(n-1)$–planes if it is not a cone [DG, V]. This was refined in [P_1] to the following statement.

Theorem. *Let $X \subset \mathbb{P}^N$ be an affinely smooth developable variety of dimension n and Gauss rank 2 which is not a cone. Then there exists a unique curve C in the hyperplane at infinity such that X is the union of a one–dimensional family of $(n-1)$–planes that contain the $(n-2)$–th osculating planes of the curve C.*

Vitter also introduced the following concept: Let F be a general Gauss fiber and V its intersection with the hyperplane at infinity H_∞. Then the closure of the union of the linear Gauss fibers which intersect H_∞ in V is the *Gauss fiber cone with vertex V*. The $(n-1)$–planes in the above theorem are in fact the Gauss fiber cones. Wu and Zheng proved the existence of nontrivial Gauss fiber cones for $r = 3, 4$ [WZ].

Theorem. *Let X be an affinely smooth developable variety of dimension n and Gauss rank r less or equal to four. Then X has nontrivial Gauss fiber cones, i.e., they are of dimension greater than $d = n - r$.*

A priori these Gauss fiber cones are only cones with a $(d-1)$–dimensional vertex. Here we want to show that very often these Gauss fiber cones are linear spaces. Wu and Zheng gave also a criterion for this, but it applies in only a few cases [WZ, Theorem 2]. From our structure theorems we obtain in particular the following generalization of the theorem of Vitter and Dajczer–Gromoll.

Theorem. *Let $X \subset \mathbb{P}^N$ be an affinely smooth developable variety of dimension n and Gauss rank less or equal to four which is not a cone. Then X is a union of $(d+1)$–planes, where $d = n - r$.*

Unfortunately, the above mentioned method of Wu and Zheng for constructing a counter example cannot be modified to provide also a counter example to this theorem for $r \geq 5$, since it only produces quadrics which are necessarily uniruled by large linear subspaces. However, analyzing the 7–dimensional example $X \subset \mathbb{P}^8$ for Gauss rank $r = 5$, one sees that the appearing $(d+1)$–planes are not the union of Gauss fibers which indicates that they are artifacts of X being a quadric and a general affinely smooth developable variety X for $r \geq 5$ will not have them.

The main purpose of this article is to describe the structure of affinely smooth developable varieties of Gauss rank 3 and 4. We need to recall three definitions:

The *dual variety* X^* of a developable variety with Gauss fiber dimension d is *degenerate* if its dimension is less than the expected one, $N-1-d$.

At a general point of $x \in X$ there exists a linear subspace $\mathcal{A}$ of nilpotent matrices of the endomorphisms of the tangent space $\mathbb{T}_x X$ modulo the linear Gauss fiber F_x through x. We call $\mathcal{A} \subset \mathrm{End}(\mathbb{T}_x X/F_x)$ the *fiber movement system at infinity*. Its invariants, $l = \max\{\operatorname{rank} A \mid A \in \mathcal{A}\}$, the rank of a general matrix, and $b = \dim \sum_{A\in\mathcal{A}} \operatorname{Im} A$, the dimension of the span of all images of $A \in \mathcal{A}$, were already used to show that the focal variety X_f has dimension $d+l-1 \leq n-2$ and Gauss rank b [P_1, Theorem 3].

A variety $X \subset \mathbb{P}^N$ of dimension n and Gauss rank r is a *twisted* $(n-1)$–*plane of type* $(k_1, \ldots, k_r) \in \mathbb{N}^r$ with $\sum k_\varrho = n-r$ if it can be constructed in the following way: There exist r curves $C_\varrho \subset \mathbb{P}^N$ and a correspondence between them, i.e., a curve $C \subseteq C_1 \times \ldots \times C_r$ which projects surjectively onto each factor, such that X is the union of the one–dimensional family of $(n-1)$–planes that are the span of the k_ϱ–th osculating spaces to the curves C_ϱ at corresponding points. Hereby, we use that the zeroth osculating space is the point itself and the first the tangent line.

Any variety that is the union of a one–dimensional family of codimension one planes is a twisted plane of some type. Furthermore, the focal variety of a twisted plane of type $(k_1, \ldots, k_r)$ is a twisted plane of type $(k_1-1, \ldots, k_r-1)$ over the same curves, where the possibly appearing negative numbers and the corresponding directing curves have to be left out.

With this definition the affinely smooth developable varieties of Gauss rank 2 which are not cones are twisted $(n-1)$–planes of type $(0, n-2)$ where the last curve lies in H_∞.

Finally, we can state our structure theorem for Gauss rank 3. An analogous one for Gauss rank 4 can be found in Section 3.

Theorem. *Let $X \subset \mathbb{P}^N$ be an affinely smooth developable variety of Gauss rank 3 which is not a cone. With the fiber movement system $\mathcal{A}$ belonging to a general point of X, we define the following invariants of X:*

$$\begin{aligned} a &= \dim \mathcal{A} \\ l &= \max\{\operatorname{rank} A \mid A \in \mathcal{A}\} = \textit{rank of general matrix of } \mathcal{A}. \end{aligned}$$

According to the values of these invariants, we have the following geometric descriptions of X:

$\underline{l=1,\ a=1}$*: The focal variety X_f of X is the $(n-4)$–th osculating scroll of a unique curve $C \subset H_\infty$. X is the union of the one–dimensional family of Gauss fiber cones that are $(n-1)$–dimensional cones whose vertices are the $(n-3)$–th osculating spaces to the curve C.*

If X has a degenerated dual variety, then X is a twisted $(n-1)$–plane of type $(0,0,n-3)$ where the last directing curve lies in H_∞.

$\underline{l=1,\ a=2}$*: X is a twisted $(n-1)$–plane of type $(0,k_2,k_3)$ with $k_2, k_3 \geq 1$ where the last two directing curves lie in H_∞. Its Gauss fiber cones are the $(n-1)$–planes.*

$\underline{l = 2}$ *: The focal variety of X has dimension $n-2$ and Gauss rank 2. Further, it has an asymptotic $(n-3)$–plane in each tangent space. The variety X itself is the union of the two–dimensional family $\mathcal{G}$ of the $(n-2)$-dimensional linear Gauss fiber cones, each of which contains an asymptotic plane of X_f.*

X can also be seen as the union of a one–dimensional family of Gauss rank 1 varieties. To be precise, let Y be an integral manifold of the asymptotic distribution on X_f and $\mathcal{G}'$ be the one-dimensional subfamily of $\mathcal{G}$ which contains the asymptotic $(n-3)$–planes of X_f along Y. Define the variety $Z \subseteq X$ to be the union of the $(n-2)$–planes of $\mathcal{G}'$. Then Z has dimension $n-1$ and Gauss rank 1, and its Gauss fibers are the family $\mathcal{G}'$.

If $a = 1$, then $\dim X = 4$, *otherwise* $\dim X \geq 5$.

A direct computation shows that the descriptions in the above Theorem can also be read as ways how to construct a variety of the corresponding type if the occurring objects are chosen general enough. However, while the focal variety of the constructed variety will lie in H_∞, additional singularities — even outside H_∞ — may occur. Further, if in the $l = 2$ case the asymptotic submanifolds of X_f, which is supposed to become the focal variety of X, are linear, additional technical conditions must be imposed on the family $\mathcal{G}$.

1 The Setup

The structure theorems will be proven with the help of Cartan's moving frame method, for an introduction see the books [AG$_2$, L]. We will use the notations of [P$_1$], which we will recall briefly.

Let $X \subset \mathbb{P}^N$ be an irreducible variety which is smooth outside the hyperplane at infinity $H_\infty \subset \mathbb{P}^N$. Denote by n the dimension of X and by d the dimension of a general Gauss fiber. We adapt the frame such that

$\{e_0\}$	is a general point of X,
$\{e_0, \dots, e_d\}$	is the linear Gauss fiber F of X through $\{e_0\}$,
$\{e_0, \dots, e_n\}$	is the tangent space $\mathbb{T}_{e_0}X$ of X in $\{e_0\}$,
$\{e_1, \dots, e_N\}$	is the hyperplane at infinity H_∞,
$\{e_1, \dots, e_d\}$	is the Gauss fiber cone vertex.

Here we use the curly brackets to indicate the linear span of the enclosed elements. Using the index ranges $1 \leq \delta, \varepsilon \leq d$, $d+1 \leq i, j \leq n$, and $n+1 \leq \mu, \nu \leq N$, the infinitesimal movement of the frame is given by

$$\begin{aligned} de_0 &= \omega^0 e_0 + \omega^\delta e_\delta + \omega^i e_i \\ de_\delta &= \qquad\quad \omega_\delta^\varepsilon e_\varepsilon + \omega_\delta^i e_i \\ de_i &= \qquad\quad \omega_i^\delta e_\delta + \omega_i^j e_j + \omega_i^\mu e_\mu \\ de_\mu &= \qquad\quad \omega_\mu^\delta e_\delta + \omega_\mu^i e_i + \omega_\mu^\nu e_\nu. \end{aligned}$$

Note that the Gauss fiber cone vertex $\{e_1, \dots, e_d\}$ is fixed if $de_\delta = 0$ modulo $\{e_1, \dots, e_d\}$, i.e., if $\omega_\delta^i = 0$ for all δ, i. This distribution is integrable, and an integral manifold is a Gauss fiber cone.

By differentiating $\omega^\mu = \omega^\mu_\delta = 0$ and using Cartan's lemma, one finds functions $a^i_{\delta j}$, q^μ_{ij} such that

$$\omega^i_\delta = a^i_{\delta j}\omega^j \qquad \text{and} \qquad \omega^\mu_i = q^\mu_{ij}\omega^j.$$

Let $A_\delta = (a^i_{\delta j})^i_j$, $\mathcal{A} = \{A_\delta\}$, $Q^\mu = (q^\mu_{ij})_{ij}$, and $\mathcal{Q} = \{Q^\mu\}$. These invariantly defined linear subspaces, $\mathcal{A}$ and $\mathcal{Q}$, of the endomorphisms of $\mathbb{T}_{e_0}X/F$ resp. of bilinear forms on $\mathbb{T}_{e_0}X/F$ are called the *fiber movement system (at infinity)* resp. the *(nondegenerated part of) the second fundamental form* of X in $\{e_0\}$. Due to our assumption that X is smooth outside H_∞, the matrices $A \in \mathcal{A}$ are nilpotent. Furthermore, the matrices Q and QA for $A \in \mathcal{A}$, $Q \in \mathcal{Q}$ are symmetric. This holds for any developable variety and follows from the symmetry of the second fundamental form along the Gauss fiber. Such linear systems $\mathcal{A}$, $\mathcal{Q}$ were studied by Wu and Zheng [WZ, Proposition 2 and 3]. Their results were refined to the following classification in [P_1, Proposition 2].

Proposition. *Let $\mathcal{A}$ be a nontrivial linear system of endomorphisms of $\mathbb{C}^r$ and $\mathcal{Q}$ a linear system of symmetric bilinear forms of $\mathbb{C}^r$ with*

1. *every $A \in \mathcal{A}$ is nilpotent,*
2. *the bilinear form $Q(\,\cdot\,, A(\,\cdot\,))$ is symmetric for every $A \in \mathcal{A}$ and $Q \in \mathcal{Q}$,*
3. $\operatorname{Sing}\mathcal{Q} = \{v \in \mathbb{C}^r \mid Q(v, \mathbb{C}^r) = 0\ \forall Q \in \mathcal{Q}\} = 0$.

Let l be the rank of a general matrix of $\mathcal{A}$. Then there exists a basis of $\mathbb{C}^r$ such that $\mathcal{A}$ is contained in the following linear systems of matrices

$r \setminus l$	1	2	3
3	$\begin{pmatrix} 0 & 0 & s \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & s & * \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix}$	
4	$\begin{pmatrix} 0 & 0 & 0 & s \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & s & * \\ 0 & 0 & * & s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & s & t & * \\ 0 & 0 & 0 & s \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 \end{pmatrix}$ †)	$\begin{pmatrix} 0 & s & t & * \\ 0 & 0 & s+* & t \\ 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 \end{pmatrix}$

†) *If the system $\mathcal{Q}$ contains a matrix of full rank, then $t = u$, otherwise $t = 0$.*

The linear system $\mathcal{A}$ always contains the matrix with $s = 1$ and all other entries set to zero.

In particular, the linear system $\mathcal{A}$ has a nontrivial common kernel.

The systems $\mathcal{Q}$ which belong to the above systems $\mathcal{A}$ have also been computed in [WZ] or [P_1] and will be recalled when needed.

Before we treat the different cases separately, we will show that if $l \geq 2$, $\dim \mathcal{A} = 1$, and X is not a cone, then $\dim X = r + 1$. We adapt the frame such that $\operatorname{rank} A_1 = l$ and $A_\varepsilon = 0$ for $2 \leq \varepsilon \leq d$. Then we differentiate $\omega^i_\varepsilon = 0$ to obtain

$$0 = d\omega^i_\varepsilon = -\omega^i_1 \wedge \omega^1_\varepsilon.$$

Since there are $l \geq 2$ linear independent 1–forms ω^i_1, this implies $\omega^1_\varepsilon = 0$ and $de_\varepsilon = 0 \bmod \{e_\varepsilon\}$. Therefore, $\{e_\varepsilon\}$ is a fixed linear space and X — as the union of the linear spaces $\{e_1, e_\varepsilon\}$ — is a cone over it. Thus if X is not a cone, we must have $d = 1$ and $\dim X = r + 1$.

2 The Proof for Gauss Rank 3

Now we treat the different cases — according to the invariants of the linear system $\mathcal{A}$ at a general point — separately.

Case $l = 1,\ a = 1$. We adapt the frame such that

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ Q^{n+1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & q^{n+1}_1 & q^{n+1}_2 \\ 1 & q^{n+1}_2 & q^{n+1}_3 \end{pmatrix},\ Q^\mu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q^\mu_1 & q^\mu_2 \\ 0 & q^\mu_2 & q^\mu_3 \end{pmatrix},$$

and $A_\varepsilon = 0$, where $2 \leq \varepsilon \leq d$, $n+2 \leq \mu \leq N$. In particular, we have $\omega^{n-2}_\varepsilon = \omega^{n-1}_1 = \omega^n_1 = 0$ and $\omega^{n-2}_1 = \omega^n$. We differentiate these equalities to obtain some useful relations:

$$\begin{aligned}
0 = d\omega^{n-2}_\varepsilon &= -\omega^{n-2}_1 \wedge \omega^1_\varepsilon = -\omega^n \wedge \omega^1_\varepsilon && \Rightarrow \omega^1_\varepsilon = f_1\omega^n \\
0 = d\omega^{n-1}_1 &= -\omega^{n-1}_{n-2} \wedge \omega^{n-2}_1 = \omega^n \wedge \omega^{n-1}_{n-2} && \Rightarrow \omega^{n-1}_{n-2} = f_2\omega^n \\
0 = d\omega^n_1 &= -\omega^n_{n-2} \wedge \omega^{n-2}_1 = \omega^n \wedge \omega^n_{n-2} && \Rightarrow \omega^n_{n-2} = f_3\omega^n \\
0 = d(\omega^{n-2}_1 - \omega^n) &= -\omega^{n-2}_1 \wedge \omega^1_1 - \omega^{n-2}_{n-2} \wedge \omega^{n-2}_1 + \omega^n \wedge \omega^0 + \omega^n_i \wedge \omega^i \\
&= \omega^{n-1} \wedge (-\omega^n_{n-1}) + \omega^n \wedge (\ldots) && \Rightarrow \omega^n_{n-1} = f_4\omega^{n-1} + f_5\omega^n
\end{aligned}$$

for some suitable functions $f_1, \ldots, f_5$.

Now we can examine the focal variety X_f of X. Its dimension is $n-3$ since from

$$de_1 = \omega^\varepsilon_1 e_\varepsilon + \omega^{n-2}_1 e_{n-2} \mod \{e_1\}$$

and the fact that X_f contains the linear space $\{e_1, \ldots, e_d\}$, we see that the tangent space of X_f at the general point e_1 is $\{e_1, \ldots, e_{n-2}\}$. The second fundamental form of X_f is

$$\begin{aligned}
\mathrm{I\!I}_{X_f, e_1} = d^2 e_1 &= \omega^{n-2}_1(\omega^{n-1}_{n-2} e_{n-1} + \omega^n_{n-2} e_n + \omega^{n+1}_{n-2} e_{n+1}) \\
&= (\omega^{n-2}_1)^2(f_2 e_{n-1} + f_3 e_n + e_{n+1}) \mod \{e_1, \ldots, e_{n-2}\}.
\end{aligned}$$

Thus X_f has Gauss rank 1. Since X is not a cone, X_f is not a cone. Therefore X_f has to be a $(d-1)$–th osculating scroll of a unique curve $C \subset H_\infty$.

We turn to the one–dimensional family of Gauss fiber cones of X given by the distribution $\omega^i_\delta = 0$ for all i, δ, i.e. $\omega^n = 0$. Each of which is a priori a cone with a $(d-1)$–dimensional vertex, but we will show that it is a cone with a d–dimensional vertex. Since

$$de_0 = \omega^1 e_1 + \omega^\varepsilon e_\varepsilon + \omega^{n-2} e_{n-2} + \omega^{n-1} e_{n-1} \mod \{e_0, \omega^n\},$$

the tangent space of the Gauss fiber cone G at e_0 is $\{e_0, \ldots, e_{n-1}\}$. The second fundamental form of G — using the index range $n-2 \leq k \leq n-1$ — is

$$\begin{aligned}\mathrm{II}_{G,e_0} &= \omega^k \omega_k^n e_n + \omega^k \omega_k^{n+1} e_{n+1} + \omega^{n-1} \omega_{n-1}^{\mu} e_{\mu} \\ &= (\omega^{n-1})^2 (f_4 e_n + q_1^{n+1} e_{n+1} + q_1^{\mu} e_{\mu}) \mod \{e_1, \ldots, e_{n-2}, \omega^n\}.\end{aligned}$$

Thus G has only Gauss rank 1, and its Gauss fibers are $\{e_0, \ldots, e_{n-2}\}$. The linear space $\{e_1, \ldots, e_{n-2}\}$, which is the tangent space to X_f at any of the smooth points of $\{e_1, \ldots, e_d\}$, is fixed on G because

$$de_1 = de_{\varepsilon} = de_{n-2} = 0 \mod \{e_1, \ldots, e_{n-2}, \omega^n\}.$$

Therefore, the Gauss fiber cone is the union of a one–dimensional family of $(d+1)$–planes containing the d–th osculating space to the curve C; hence, it is a cone with the d–th osculating space of C as vertex.

We treat the special case where X has a degenerate dual variety. This is equivalent to the fact that the linear system $\mathcal{Q}$ of the second fundamental form contains only matrices of rank less than 3 [L, 7.3], i.e., $q_1^{n+1} = q_1^{\mu} = 0$, and due to $\operatorname{Sing}\mathcal{Q} = 0$ we may assume $q_2^{n+1} = 0$ and $q_2^{\mu} = 1$. We claim that in this case the Gauss fiber cones are $(n-1)$–planes. This will be implied if the second fundamental form of each Gauss fiber cone vanishes. By our above computations it only remains to show that $f_4 = 0$. We get this by differentiating $\omega_{n-1}^{n+1} = 0$:

$$\begin{aligned}0 = d\omega_{n-1}^{n+1} &= -\omega_{n-2}^{n+1} \wedge \omega_{n-1}^{n-2} - \omega_n^{n+1} \wedge \omega_{n-1}^{n} - \omega_{\mu}^{n+1} \wedge \omega_{n-1}^{\mu} \\ &= -f_4 \omega^{n-2} \wedge \omega^{n-1} + \omega^n \wedge (\ldots).\end{aligned}$$

Summarizing the above computations, we see that X is the union of the one–dimensional family of $(n-1)$–planes, the linear Gauss fiber cones, containing the d–th osculating space of the curve $C \subset H_{\infty}$.

Case $l = 1,\ a = 2$. Here we have $\operatorname{Im}\mathcal{A} = \ker\mathcal{A}$, and the statement follows from [P_1, Corollary 11] in view of [WZ, Theorem 2] or [P_1, Theorem 6].

Case $l = 2$. We adapt the frame such that

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ A_{\varepsilon} = \begin{pmatrix} 0 & 0 & t_{\varepsilon} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ Q^{n+1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\text{and } Q^{\mu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q_1^{\mu} \\ 0 & q_1^{\mu} & q_2^{\mu} \end{pmatrix},$$

where $2 \leq \varepsilon \leq d$, $n+2 \leq \mu \leq N$ and $t_{\varepsilon} = 0$ if $a = 1$.

The Gauss fiber cones of X are the integral manifolds of the distribution $\omega_{\delta}^{i} = 0$ for all δ, i, i.e., of the distribution $\omega^{n-1} = \omega^n = 0$. We claim that a Gauss fiber cone G is a linear space. It is enough to show that the second fundamental form of G vanishes. On G we have

$$\begin{aligned}de_0 &= \omega^1 e_1 + \omega^{\varepsilon} e_{\varepsilon} + \omega^{n-2} e_{n-2} \mod \{e_0, \omega^{n-1}, \omega^n\} \\ \mathrm{II}_{G,e_0} &= d^2 e_0 = (\omega^1 \omega_1^{n-1} + \omega^{n-2} \omega_{n-2}^{n-1}) e_{n-1} + \omega^{n-2} \omega_{n-2}^{n} e_n + \omega^{n-2} \omega_{n-2}^{n+1} e_{n+1} \\ &\qquad \mod \{e_0, \ldots, e_{n-2}, \omega^{n-1}, \omega^n\}.\end{aligned}$$

We know that $\omega_1^{n-1} = \omega_{n-2}^{n+1} = \omega^n$ vanish on G. To compute ω_{n-2}^{n-1} and ω_{n-2}^n, we differentiate $\omega_1^n = 0$, $\omega_1^{n-1} = \omega_{n-2}^{n+1}$, and $\omega_1^{n-1} = \omega^n$. With the index range $n-2 \leq k \leq n-1$ we have

$$\begin{aligned}
0 &= d\omega_1^n = -\omega_k^n \wedge \omega_1^k = \omega^{n-1} \wedge \omega_{n-2}^n + \omega^n \wedge \omega_{n-1}^n \\
0 &= d(\omega_1^{n-1} - \omega_{n-2}^{n+1}) = -\omega_1^{n-1} \wedge \omega_1^1 - \omega_k^{n-1} \wedge \omega_1^k + \omega_i^{n+1} \wedge \omega_{n-2}^i + \omega_{n+1}^{n+1} \wedge \omega_{n-2}^{n+1} \\
&= \omega^{n-2} \wedge \omega_{n-2}^n + \omega^{n-1} \wedge (2\omega_{n-2}^{n-1}) + \omega^n \wedge (\ldots) \\
0 &= d(\omega_1^{n-1} - \omega^n) = -\omega_1^{n-1} \wedge \omega_1^1 - \omega_k^{n-1} \wedge \omega_1^k + \omega^n \wedge \omega^0 + \omega_i^n \wedge \omega^i \\
&= \omega^{n-2} \wedge (-\omega_{n-2}^n) + \omega^{n-1} \wedge (\omega_{n-2}^{n-1} - \omega_{n-1}^n) + \omega^n \wedge (\ldots).
\end{aligned}$$

From the first equation we get by Cartan's Lemma

$$\omega_{n-2}^n = f_1\omega^{n-1} + f_2\omega^n \quad \text{and} \quad \omega_{n-1}^n = f_2\omega^{n-1} + f_3\omega^n.$$

From the second we obtain

$$2\omega_{n-2}^{n-1} = f_1\omega^{n-2} + f_4\omega^{n-1} + f_5\omega^n.$$

Plugging this into the third equation, we find $f_1 = 0$; hence,

$$\omega_{n-2}^n = f_2\omega^n, \quad \omega_{n-1}^n = f_2\omega^{n-1} + f_3\omega^n, \quad \omega_{n-2}^{n-1} = \frac{f_4}{2}\omega^{n-1} + \frac{f_5}{2}\omega^n.$$

All these terms vanish on the Gauss fiber cone G and therefore also the second fundamental form of G, i.e., G is a linear space.

Now we turn to the irreducible focal variety X_f of X. The point e_1 is a general point of X_f, and the tangent space $\mathbb{T}_{e_1}X_f$ is the image of

$$de_1 = \omega_1^\varepsilon e_\varepsilon + \omega_1^{n-2}e_{n-2} + \omega_1^{n-1}e_{n-1} \mod \{e_1\}.$$

Since X_f is the union of the linear spaces $\{e_1, \ldots, e_d\}$, the tangent space $\mathbb{T}_{e_1}X_f$ must contain this linear space $\{e_1, \ldots, e_d\}$. Because ω_1^{n-2} and ω_1^{n-1} are linear independent, the tangent space $\mathbb{T}_{e_1}X_f$ is $\{e_1, \ldots, e_{n-1}\}$, and hence the dimension of X_f is $n-2$.

We can compute the second fundamental form of X_f easily as

$$\begin{aligned}
\mathrm{I\!I}_{X_f,e_1} &= \omega_1^{n-2}(\omega_{n-2}^n e_n + \omega_{n-2}^{n+1}e_{n+1}) + \omega_1^{n-1}(\omega_{n-1}^n e_n + \omega_{n-1}^{n+1}e_{n+1} + \omega_{n-1}^\mu e_\mu) \\
&= (2f_2\omega_1^{n-2}\omega_1^{n-1} + f_3(\omega_1^{n-1})^2)e_n + 2\omega_1^{n-2}\omega_1^{n-1}e_{n+1} + q_1^\mu(\omega_1^{n-1})^2 e_\mu \\
&\qquad\qquad \mod \{e_1, \ldots, e_{n-1}\}.
\end{aligned}$$

Thus X_f is of Gauss rank 2 and has $\{e_1, \ldots, e_d\}$ as Gauss fiber. Further, it has the asymptotic space $\{e_1, \ldots, e_{n-2}\} = \{\omega_1^{n-1}\}^\perp$. This asymptotic space is the intersection of the linear Gauss fiber cone $G = \{e_0, \ldots, e_{n-2}\}$ with the hyperplane at infinity. We can consider this asymptotic distribution $\omega_1^{n-1} = 0 \Leftrightarrow \omega^n = 0$ on X_f as well as on X. By the Theorem of Frobenius it is completely integrable on both varieties since

$$\begin{aligned}
d\omega_1^{n-1} &= -\omega_1^{n-1} \wedge \omega_1^1 - \omega_{n-2}^{n-1} \wedge \omega_1^{n-2} - \omega_{n-1}^{n-1} \wedge \omega_1^{n-1} \\
&= -(\tfrac{f_4}{2}\omega_1^{n-2} + \tfrac{f_5}{2}\omega_1^{n-1}) \wedge \omega_1^{n-2} = 0 \mod \{\omega_1^{n-1}\}.
\end{aligned}$$

Now let Y and $Z \supset Y$ be integral manifolds of this distribution on X_f resp. X. Then Z is the union of the Gauss fiber cones G that contain the tangent spaces of Y or equivalently the asymptotic planes of X_f along Y. It remains to show that Z has Gauss rank 1 and has the Gauss fiber cones G as Gauss fibers. We compute the second fundamental form of Z. On Z we have

$$\begin{aligned} de_0 &= \omega^1 e_1 + \omega^\varepsilon e_\varepsilon + \omega^{n-2} e_{n-2} + \omega^{n-1} e_{n-1} \mod \{e_0, \omega^n\} \\ \mathrm{II}_{Z,e_0} &= d^2 e_0 = (\omega^{n-2}\omega^n_{n-2} + \omega^{n-1}\omega^n_{n-1}) e_n + (\omega^{n-2}\omega^{n+1}_{n-2} + \omega^{n-1}\omega^{n+1}_{n-1}) e_{n+1} \\ &\quad + \omega^{n-1}\omega^\mu_{n-1} e_\mu = (\omega^{n-1})^2 (f_2 e_n + e_{n+1}) \mod \{e_0, \ldots, e_{n-1}, \omega^n\}. \end{aligned}$$

Clearly, the singular locus of II_{Z,e_0} is the linear space $\{e_0, \ldots, e_{n-2}\}$, the Gauss fiber cone of X.

3 The Case of Gauss Rank 4

Here we prove the structure theorem for Gauss rank 4. Unfortunately, the descriptions will not always be detailed enough to yield methods for the constructions of varieties of the corresponding type.

Theorem. *Let $X \subset \mathbb{P}^N$ be an affinely smooth developable variety of Gauss rank 4 and Gauss fiber dimension $d = n - 4$ which is not a cone. With the fiber movement system $\mathcal{A}$ belonging to a general point of X, we define the following invariants of X:*

$$\begin{aligned} a &= \dim \mathcal{A} \\ b &= \dim \textstyle\sum_{A \in \mathcal{A}} \operatorname{Im} A \\ l &= \max\{\operatorname{rank} A \mid A \in \mathcal{A}\} = \textit{rank of general matrix of } \mathcal{A}. \end{aligned}$$

According to the values of these invariants, we have the following geometric descriptions of X:

1. *($l = 1,\ a = 1$) The focal variety X_f of X is the $(d-1)$–th osculating scroll of a unique curve $C \subset H_\infty$. X is the union of the one–dimensional family of Gauss fiber cones that are $(n-1)$–dimensional cones whose vertices are the d–th osculating spaces to the curve C.*

2. *($l = 1,\ a = 2$) X_f is a twisted $(d-1)$–plane of type (k_1, k_2) of two curves at infinity. X is the union of the one–dimensional family of Gauss fiber cones, which are $(n-1)$–dimensional cones over the $(d+1)$–planes of the twisted $(d+1)$–plane of type (k_1+1, k_2+1) of the same curves as the one above.*

3. *($l = 1,\ a = 3$) X is a twisted $(n-1)$–plane of type $(0, k_2, k_3, k_4)$ with $k_2, k_3, k_4 \geq 1$ where the last three curves lie in H_∞. Its Gauss fiber cones are the $(n-1)$–planes.*

4. *($l = 2,\ \mathcal{A}^2 = 0$) The focal variety X_f is an $(n-3)$–dimensional variety of Gauss rank 2. X is the union of the two–dimensional family of the linear Gauss fiber cones, which contain the tangent spaces to X_f.*

5. *($l=2,\ a=1,\ \mathcal{A}^2\neq 0,\ X^*$ nondegenerate) X has dimension 5 and is the union of the two–dimensional family of the three–dimensional Gauss fiber cones, which are uniruled by 2–planes. The 2–planes of a Gauss fiber cone with vertex $V\subset H_\infty$ intersect the tangent spaces $\mathbb{T}_vX$, $v\in V$, of the focal variety X_f in codimension one.*

6. *($l=2,\ a=b=2,\ \mathcal{A}^2\neq 0,\ X^*$ nondegenerate) X is the union of its two–dimensional family of Gauss fiber cones, each of which is an $(n-2)$–dimensional cone over a d–plane in H_∞. Such a d–plane is asymptotic in the tangent spaces of X_f along the Gauss fiber cone vertex.*

7. *($l=2,\ b=3,\ \mathcal{A}^2\neq 0,\ X^*$ nondegenerate) X_f is a twisted d–plane of type (k_1,k_2,k_3). X is the union of the three–dimensional family of linear Gauss fiber cones, which contain a d–plane of this twisted d–plane.*

8. *($l=2,\ \mathcal{A}^2\neq 0,\ X^*$ degenerate) X is the union of its two–dimensional family of linear Gauss fiber cones. X_f has dimension $n-2$ and Gauss rank b. The linear Gauss fiber cone for a general vertex $V\subset X_f$ intersects the tangent spaces $\mathbb{T}_vX_f$, $v\in V$, in a fixed linear space of dimension $n-3$.*

   X can also be seen as the union of a one–dimensional family of Gauss rank 1 varieties, whose Gauss fibers are the linear Gauss fiber cones of X.

9. *($l=3$) X is the union of a two–dimensional family of Gauss rank 1 varieties whose Gauss fibers are the linear Gauss fiber cones of X. The intersection of the linear Gauss fiber cone with H_∞ is an asymptotic plane in the tangent spaces of X_f.*

For $l\geq 2$ the condition $a=1$ is equivalent to $\dim X=5$.

To prove this theorem, we have again to treat the different cases separately.

Case $l=1,\ a=1$. The proof is analogous to the one for Gauss rank 3.

Case $l=1,\ a=2$. We adapt the frame such that

$$A_1=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\ A_d=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix},\ Q^\mu=\begin{pmatrix}0&0&0&q_1^\mu\\0&0&0&q_2^\mu\\0&0&q_3^\mu&q_4^\mu\\q_1^\mu&q_2^\mu&q_4^\mu&q_5^\mu\end{pmatrix},$$

and $A_\varepsilon=0$ with the index ranges $2\leq\varepsilon\leq d-1$, $n+1\leq\mu\leq N$, in particular $\omega_\delta^{n-1}=\omega_\delta^n=0$ and $\omega_1^{n-3}=\omega_1^{n-2}=\omega^n$.

We start by examining a general Gauss fiber cone G of X, given by the integrable distribution $\omega^n=0$. Using the index range $n-3\leq k\leq n-1$, its tangent space at a general point e_0 is the image of

$$de_0=\omega^1e_1+\omega^\varepsilon e_\varepsilon+\omega^de_d+\omega^ke_k\mod\{e_0,\omega^n\},$$

i.e., it is $\{e_0,\dots,e_{n-1}\}$. Its second fundamental form can be computed as

$$\mathrm{I\!I}_{G,e_0}=d^2e_0=\omega^k\omega_k^ne_n+\omega^k\omega_k^\mu e_\mu\mod\{e_0,\dots,e_{n-1},\omega^n\}.$$

To determine the unknown forms ω_k^n, we differentiate $\omega_1^n=\omega_d^n=0$,

$$\begin{aligned}0=d\omega_1^n&=-\omega_{n-3}^n\wedge\omega_1^{n-3}=\omega^n\wedge\omega_{n-3}^n\ \Rightarrow\ \omega_{n-3}^n=f_1\omega^n\\0=d\omega_d^n&=-\omega_{n-2}^n\wedge\omega_d^{n-2}=\omega^n\wedge\omega_{n-2}^n\ \Rightarrow\ \omega_{n-2}^n=f_2\omega^n,\end{aligned}$$

and further $\omega_1^{n-3} = \omega^n$,

$$
\begin{aligned}
0 &= d(\omega_1^{n-3} - \omega^n) = -\omega_1^{n-3} \wedge \omega_1^1 - \omega_{n-3}^{n-3} \wedge \omega_1^{n-3} + \omega^n \wedge \omega^0 + \omega_i^n \wedge \omega^i \\
&= -\omega^{n-1} \wedge \omega_{n-1}^n - \omega^n \wedge (\dots) \quad \Rightarrow \quad \omega_{n-1}^n = f_3 \omega^{n-1} + f_4 \omega^n.
\end{aligned}
$$

Therefore,

$$
\mathrm{I\!I}_{G,e_0} = (\omega^{n-1})^2 (f_3 e_n + q_3^\mu e_\mu) \mod \{e_0, \dots, e_{n-1}, \omega^n\}.
$$

We see that G is either of Gauss rank 1 with Gauss fibers $\{e_0, \dots, e_{n-2}\}$ or even linear. We claim that in any case G is a cone over $\{e_1, \dots, e_{n-2}\}$. This is equivalent to the fact that $\{e_1, \dots, e_{n-2}\}$ is a fixed linear space on G, which follows from

$$
\begin{aligned}
de_1 &= de_\varepsilon = de_d = 0 && \mod \{e_1, \dots, e_{n-2}, \omega^n\} \\
de_{n-3} &= \omega_{n-3}^{n-1} e_{n-1} + \omega_{n-3}^n e_n + \omega_{n-3}^\mu e_\mu = 0 && \mod \{e_1, \dots, e_{n-2}, \omega^n\} \\
de_{n-2} &= \omega_{n-2}^{n-1} e_{n-1} + \omega_{n-2}^n e_n + \omega_{n-2}^\mu e_\mu = 0 && \mod \{e_1, \dots, e_{n-2}, \omega^n\}.
\end{aligned}
$$

Here we used $\omega_{n-3}^{n-1} = \omega_{n-2}^{n-1} = 0 \mod \{\omega^n\}$, which can be derived analogously to $\omega_{n-3}^n = \omega_{n-2}^n = 0 \mod \{\omega^n\}$.

From [P_1, Theorem 9] we know that X_f is the union of the one–dimensional family of $(d-1)$–planes $\{e_1, \dots, e_d\}$ and has Gauss rank 2. By the classification of Gauss rank 2 varieties [P_2] it is therefore a twisted $(d-1)$–plane of type (k_1, k_2) of two curves $C_1, C_2 \subset H_\infty$. The movement of the $(d-1)$–planes $\{e_1, \dots, e_d\}$ is characterized by the span of $\{e_1, \dots, e_d\}$ and the image of $de_1, \dots, de_d$. (This is the associated curve $\Phi^{(1)}$ of $\Phi = \{e_1, \dots, e_d\}$ in the notation of [P_1, Section 4].) We have

$$
\begin{aligned}
de_1 &= \omega^n e_{n-3} && \mod \{e_1, \dots, e_d\} \\
de_\varepsilon &= 0 && \mod \{e_1, \dots, e_d\} \\
de_d &= \omega^n e_{n-2} && \mod \{e_1, \dots, e_d\},
\end{aligned}
$$

hence the common image is the $(d+1)$–plane $\{e_1, \dots, e_{n-2}\}$. This $(d+1)$–plane is on the one hand by the above computation a $(d+1)$–plane of the twisted $(d+1)$-plane of type (k_1+1, k_2+1) of the curves $C_1, C_2 \subset H_\infty$ and on the other hand a vertex of the Gauss fiber cone G.

Case $l = 1,\ a = 3$. Here we have $\operatorname{Im} \mathcal{A} = \ker \mathcal{A}$, and the statement follows from [P_1, Corollary 11] in view of [WZ, Theorem 2] or [P_1, Theorem 6].

Case $l = 2,\ \mathcal{A}^2 = 0$. Here we are again in the situation that $\operatorname{Im} \mathcal{A} = \ker \mathcal{A}$; hence, the Gauss fiber cones are linear. By [P_1, Theorem 9] X_f has Gauss rank 2, and one easily checks that every Gauss fiber cone contains a tangent space of X_f.

Case $l = 2,\ \mathcal{A}^2 \neq 0,\ X^*$ *nondegenerate*. It was shown in [P_1, Appendix] that the linear system $\mathcal{Q}$ belonging to this linear system $\mathcal{A}$ has the form

$$
\begin{pmatrix}
0 & 0 & 0 & q_1 \\
0 & q_1 & 0 & q_3 \\
0 & 0 & q_2 & q_4 \\
q_1 & q_3 & q_4 & q_5
\end{pmatrix}.
$$

By the assumption that X^* is nondegenerate, $\mathcal{Q}$ contains a matrix of full rank [L, 7.3], i.e., one with $q_1 q_2 \neq 0$. By scaling this matrix we may assume $q_1 = 1$. Then the transformation

$$T = \begin{pmatrix} 1 & -\frac{1}{2}q_3 & -q_4 & \frac{3}{8}q_3^2 - \frac{1}{2}q_5 \\ 0 & 1 & 0 & -\frac{1}{2}q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

takes Q to a matrix with $q_3 = q_4 = q_5 = 0$ and maps $\mathcal{A}$ to itself. Hence, we can adapt the frame such that

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ A_\varepsilon = \begin{pmatrix} 0 & 0 & s_\varepsilon & t_\varepsilon \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_\varepsilon \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ Q^{n+1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & q_2^{n+1} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\text{and} \quad Q^\mu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_3^\mu \\ 0 & 0 & q_2^\mu & q_4^\mu \\ 0 & q_3^\mu & q_4^\mu & q_5^\mu \end{pmatrix}$$

with $q_2^{n+1} \neq 0$ and the index ranges $2 \leq \varepsilon \leq d$ and $n+2 \leq \mu \leq N$. Again, to prove our geometric statements, we need to express several 1–forms in terms of the semi–basic forms. We use the index range $n-3 \leq k \leq n-2$ and start by differentiating $\omega_1^{n-1} = \omega_1^n = 0$,

$$\begin{aligned} 0 &= d\omega_1^n = -\omega_k^n \wedge \omega_1^k = \omega^{n-2} \wedge \omega_{n-3}^n + \omega^n \wedge \omega_{n-2}^n \\ 0 &= d\omega_1^{n-1} = -\omega_\varepsilon^{n-1} \wedge \omega_1^\varepsilon - \omega_k^{n-1} \wedge \omega_1^k = \omega^{n-2} \wedge \omega_{n-3}^{n-1} + \omega^n \wedge (\omega_{n-2}^{n-1} - s_\varepsilon \omega_1^\varepsilon), \end{aligned}$$

and obtain

$$\begin{aligned} \omega_{n-3}^n &= f_1 \omega^{n-2} + f_2 \omega^n \\ \omega_{n-2}^n &= f_2 \omega^{n-2} + f_3 \omega^n \\ \omega_{n-3}^{n-1} &= f_4 \omega^{n-2} + f_5 \omega^n \\ \omega_{n-2}^{n-1} &= s_\varepsilon \omega_1^\varepsilon + f_5 \omega^{n-2} + f_6 \omega^n. \end{aligned}$$

Next from $\omega_1^{n-2} = \omega_{n-3}^{n+1}$ we get

$$\begin{aligned} 0 &= d(\omega_1^{n-2} - \omega_{n-3}^{n+1}) = -\omega_1^{n-2} \wedge \omega_1^1 - \omega_k^{n-2} \wedge \omega_1^k + \omega_i^{n+1} \wedge \omega_{n-3}^i + \omega_{n+1}^{n+1} \wedge \omega_{n-3}^{n+1} \\ &= \omega^{n-2} \wedge (2\omega_{n-3}^{n-2} - f_1 \omega^{n-3} - f_4 q_2^{n+1} \omega^{n-1}) + \omega^n \wedge (\dots) \end{aligned}$$

and further

$$\omega_{n-3}^{n-2} = \frac{f_1}{2} \omega^{n-3} + f_7 \omega^{n-2} + \frac{f_4 q_2^{n+1}}{2} \omega^{n-1} + f_8 \omega^n.$$

Finally, by differentiating $\omega_1^{n-2} = \omega^n$,

$$\begin{aligned} 0 &= d(\omega_1^{n-2} - \omega^n) = -\omega_1^{n-2} \wedge \omega_1^1 - \omega_k^{n-2} \wedge \omega_1^k + \omega^n \wedge \omega^0 + \omega_i^n \wedge \omega^i \\ &= -\tfrac{3}{2} f_1 \omega^{n-3} \wedge \omega^{n-2} + \omega^{n-1} \wedge (-\omega_{n-1}^n - \tfrac{1}{2} f_4 q_2^{n+1} \omega^{n-2}) + \omega^n \wedge (\dots), \end{aligned}$$

we find $f_1 = 0$ and

$$\omega^n_{n-1} = -\frac{f_4 q_2^{n+1}}{2}\omega^{n-2} + f_9\omega^{n-1} + f_{10}\omega^n.$$

Now we consider the distribution $\omega^{n-2} = \omega^{n-1} = \omega^n = 0$. For $b = 3$ this is the integrable distribution whose integral manifolds are the Gauss fiber cones. For $b = 2$ this distribution is only a subdistribution of that distribution, which is given this time by $\omega^{n-2} = \omega^n = 0$. However, this distribution $\omega^{n-2} = \omega^{n-1} = \omega^n = 0$ is also integrable for $b = 2$ by the Theorem of Frobenius, since we already know that $d\omega^{n-2} = d\omega^n = 0 \bmod \{\omega^{n-2}, \omega^n\}$ and

$$d\omega^{n-1} = -\omega^{n-1} \wedge \omega^0 - \omega^{n-1}_\varepsilon \wedge \omega^\varepsilon - \omega^{n-1}_i \wedge \omega^i = 0 \mod \{\omega^{n-2}, \omega^{n-1}, \omega^n\}$$

by $\omega^{n-1}_{n-3} = 0 \bmod \{\omega^{n-2}, \omega^n\}$.

Let L be an integrable manifold of this distribution. We want to prove that L is linear. Its tangent space at a general point e_0 is $\{e_0, \dots, e_{n-3}\}$ as the image of

$$de_0 = \omega^1 e_1 + \omega^\varepsilon e_\varepsilon + \omega^{n-3} e_{n-3} \mod \{e_0, \omega^{n-2}, \omega^{n-1}, \omega^n\}.$$

Its second fundamental form,

$$\begin{aligned} \mathrm{I\!I}_{L,e_0} = d^2 e_0 &= (\omega^1\omega^{n-2}_1 + \omega^{n-3}\omega^{n-2}_{n-3})e_{n-2} + (\omega^\varepsilon\omega^{n-1}_\varepsilon + \omega^{n-3}\omega^{n-1}_{n-3})e_{n-1} \\ &+\omega^{n-3}\omega^n_{n-3}e_n + \omega^{n-3}\omega^{n+1}_{n-3}e_{n+1} = 0 \bmod \{e_0, \dots, e_{n-3}, \omega^{n-2}, \omega^{n-1}, \omega^n\} \end{aligned}$$

vanishes by $\omega^{n-2}_{n-3} = \omega^{n-1}_{n-3} = \omega^n_{n-3} = 0 \bmod \{\omega^{n-2}, \omega^{n-1}, \omega^n\}$; hence, L is linear.

Unfortunately, we cannot prove much more about the structure of X in case of $a = 1$. We can only note that the linear spaces $L = \{e_0, \dots, e_{n-3}\}$ of the Gauss fiber cone G intersect the tangent space to X_f at the point e_1, which is $\{e_1, \dots, e_{n-2}\}$, in codimension one. It seems possible that this intersection moves, while L moves in G. However, if it does not, which will be a special case, then X has an analogous geometric description as the one which will be given for the case $a = b = 2$.

For this case $a = b = 2$, where $s_\varepsilon = 0$ and some $t_\varepsilon \neq 0$, we will now show that the Gauss fiber cone G is a cone over the d–plane $\{e_1, \dots, e_{n-3}\}$, i.e., all linear spaces L inside G contain this d–plane. We show that this d–plane is fixed on the Gauss fiber cone, which was given by the distribution $\omega^{n-2} = \omega^n = 0$. Because of

$$\begin{aligned} de_1 &= \omega^{n-2}_1 e_{n-2} = 0 && \bmod \{e_1, \dots, e_{n-3}, \omega^{n-2}, \omega^n\} \\ de_\varepsilon &= 0 && \bmod \{e_1, \dots, e_{n-3}, \omega^{n-2}, \omega^n\} \\ de_{n-3} &= \omega^{n-2}_{n-3}e_{n-2} + \omega^{n-1}_{n-3}e_{n-1} + \omega^n_{n-3}e_n + \omega^{n+1}_{n-3}e_{n+1} && \\ &= \tfrac{1}{2} f_4 q_2^{n+1}\omega^{n-1}e_{n-2} && \bmod \{e_1, \dots, e_{n-3}, \omega^{n-2}, \omega^n\}, \end{aligned}$$

it remains to show that $f_4 = 0$. This follows from differentiating $\omega^{n-1}_\varepsilon = 0$:

$$0 = d\omega^{n-1}_\varepsilon = -\omega^{n-1}_{n-3} \wedge \omega^{n-3}_\varepsilon = -f_4 t_\varepsilon q_2^{n+1}\omega^{n-2} \wedge \omega^n.$$

We turn to the focal variety X_f. Its tangent space at the general point e_1 is $\{e_1, \ldots, e_{n-2}\}$ as the image of

$$de_1 = \omega_1^\varepsilon e_\varepsilon + \omega_1^{n-3} e_{n-3} + \omega_1^{n-2} e_{n-2} \mod \{e_1\}.$$

Since its second fundamental form is

$$\begin{aligned} \mathrm{I\!I}_{X_f, e_1} &= d^2 e_1 = (\omega_1^{n-3}\omega_{n-3}^{n-1} + \omega_1^{n-2}\omega_{n-2}^{n-1}) e_{n-1} + (\omega_1^{n-3}\omega_{n-3}^{n} + \omega_1^{n-2}\omega_{n-2}^{n}) e_n \\ &\quad + (\omega_1^{n-3}\omega_{n-3}^{n+1} + \omega_1^{n-2}\omega_{n-2}^{n+1}) e_{n+1} + \omega_1^{n-2}\omega_{n-2}^{\mu} e_\mu \\ &= \omega_1^{n-3}\omega_1^{n-2}(2f_5 e_{n-1} + 2f_2 e_n + 2e_{n+1}) \\ &\quad + (\omega_1^{n-2})^2 (f_6 e_{n-1} + f_3 e_n + q_3^\mu e_\mu) \mod \{e_1, \ldots, e_{n-2}\}, \end{aligned}$$

the vertex of the Gauss fiber cone, $\{e_1, \ldots, e_{n-3}\}$, is an asymptotic space of X_f at the general point e_1 — and hence all points — of the a priori Gauss fiber cone vertex $\{e_1, \ldots, e_d\}$.

At last, we treat the case of $b = 3$, i.e., some $s_\varepsilon \neq 0$. Here we want to show that in addition to $f_1 = 0$ one has $f_4 = f_7 = 0$ and $f_9 = f_2$. We get this by differentiating $\omega_\varepsilon^n = \omega_\varepsilon^{n-2} = 0$:

$$\begin{aligned} 0 = d\omega_\varepsilon^n &= -\omega_{n-3}^n \wedge \omega_\varepsilon^{n-3} - \omega_{n-1}^n \wedge \omega_\varepsilon^{n-1} \\ &= \tfrac{1}{2} f_4 q_2^{n+1} s_\varepsilon \omega^{n-2} \wedge \omega^n + (f_2 - f_9) s_\varepsilon \omega^{n-1} \wedge \omega^n \\ 0 = d\omega_\varepsilon^{n-2} &= -\omega_1^{n-2} \wedge \omega_\varepsilon^1 - \omega_{n-3}^{n-2} \wedge \omega_\varepsilon^{n-3} - \omega_{n-1}^{n-2} \wedge \omega_\varepsilon^{n-1} \\ &= -s_\varepsilon f_7 \omega^{n-2} \wedge \omega^{n-1} + \omega^n \wedge (\ldots). \end{aligned}$$

The key observation for this case is that the distribution $\omega^n = 0$ is integrable by the Theorem of Frobenius since

$$d\omega^n = -\omega^n \wedge \omega^0 - \omega_i^n \wedge \omega^i = 0 \bmod \{\omega^n\}$$

using what we just proved. With the index range $n-3 \leq k \leq n-1$ we have on an integral manifold Y

$$\begin{aligned} de_0 &= \omega^1 e_1 + \omega^\varepsilon e_\varepsilon + \omega^k e_k \mod \{e_0, \omega^n\}, \\ \mathrm{I\!I}_{Y, e_0} &= d^2 e_0 = \omega^k \omega_k^n e_n + \omega^k \omega_k^{n+1} e_{n+1} + \omega^k \omega_k^\mu e_\mu \\ &= (\omega^{n-2})^2 (f_2 e_n + e_{n+1}) + (\omega^{n-1})^2 (f_2 e_n + q_2^{n+1} e_{n+1} + q_2^\mu e_\mu) \\ &\qquad \mod \{e_0, \ldots, e_{n-1}, \omega^n\}. \end{aligned}$$

Therefore, Y has Gauss rank 2, its Gauss fibers are the linear Gauss fiber cones L of X, and possesses a pair of conjugate $(n-2)$–planes. We claim that Y is in fact a cone over $\{e_1, \ldots, e_{n-3}\} \subset L$. We must show that this space is fixed on Y. This follows from

$$\begin{aligned} de_1 &= \omega_1^{n-2} e_{n-2} = 0 && \mod \{e_1, \ldots, e_{n-3}, \omega^n\} \\ de_\varepsilon &= 0 && \mod \{e_1, \ldots, e_{n-3}, \omega^n\} \\ de_{n-3} &= \omega_{n-3}^{k+1} e_{k+1} + \omega_{n-3}^{n+1} e_{n+1} = 0 && \mod \{e_1, \ldots, e_{n-3}, \omega^n\}. \end{aligned}$$

(This distribution, $\omega^n = 0$, is also integrable in the case of $a = b = 2$. There one can also show that Y is a variety of Gauss rank 2 whose Gauss fibers are the linear Gauss fiber cones of X and has a pair of conjugate $(n-2)$–planes in general tangent space. However, it seems that Y does not have to be a cone with an d–dimensional vertex.)

One can also consider the distribution $\omega^n = 0$ on the focal variety X_f. Its integral manifold Z has the tangent space $\{e_1, \ldots, e_{n-3}\}$ and must be linear by the above consideration. Hence, X_f is uniruled by codimension one planes. Since it has Gauss rank 3 by [P$_1$, Theorem 9], it is a twisted d–plane of type (k_1, k_2, k_3) of three curves in H_∞ [P$_1$, Section 4].

Case $l = 2$, $\mathcal{A}^2 \neq 0$, X^* degenerate. Since X^* is degenerate, the linear system $\mathcal{Q}$ cannot contain a matrix of full rank [L, 7.3]. From [P$_1$, Appendix] we know that elements of $\mathcal{Q}$ are of the form

$$\begin{pmatrix} 0 & 0 & 0 & q_1 \\ 0 & q_1 & 0 & q_3 \\ 0 & 0 & 0 & q_4 \\ q_1 & q_3 & q_4 & q_5 \end{pmatrix}.$$

Due to $\operatorname{Sing} \mathcal{Q} = 0$, the linear system $\mathcal{Q}$ must contain the matrices

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & q_3 \\ 0 & 0 & 0 & 0 \\ 1 & q_3 & 0 & q_5 \end{pmatrix} \quad \text{and} \quad Q' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_3' \\ 0 & 0 & 0 & 1 \\ 0 & q_3' & 1 & q_5' \end{pmatrix}.$$

A transformation of the basis of $\mathbb{C}^4$ with

$$T = \begin{pmatrix} 1 & -\frac{1}{2}q_3 & 0 & \frac{3}{8}q_3^2 - \frac{1}{2}q_5 \\ 0 & 1 & 0 & -\frac{1}{2}q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

takes the entries q_3, q_5 to zero while leaving the elements of $\mathcal{A}$ fixed. Therefore we may adapt the frame such that

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ A_\varepsilon = \begin{pmatrix} 0 & 0 & 0 & s_\varepsilon \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_\varepsilon \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ Q^{n+1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\text{and} \quad Q^\mu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_1^\mu \\ 0 & 0 & 0 & q_2^\mu \\ 0 & q_1^\mu & q_2^\mu & q_3^\mu \end{pmatrix}$$

with the index ranges $2 \leq \varepsilon \leq d$ and $n+2 \leq \mu \leq N$, in addition we will use the index range $n-3 \leq k \leq n-2$. To prove our statements about the structure of X we will need the following relations

$$\begin{aligned} &\omega^n_{n-3},\, \omega^n_{n-1} &&= 0 \quad \operatorname{mod} \{\omega^n\} \\ &\omega^{n-2}_{n-3},\, \omega^{n-2}_{n-1},\, \omega^{n-1}_{n-3},\, \omega^n_{n-2} &&= 0 \quad \operatorname{mod} \{\omega^{n-2}, \omega^n\}, \end{aligned} \tag{$*$}$$

which we derive now. Differentiating $\omega_1^{n-1} = \omega_1^n = 0$,

$$
\begin{aligned}
0 &= d\omega_1^{n-1} = -\omega_\varepsilon^{n-1} \wedge \omega_1^\varepsilon - \omega_k^{n-1} \wedge \omega_1^k = \omega^{n-2} \wedge \omega_{n-3}^{n-1} + \omega^n \wedge (\ldots)\\
0 &= d\omega_1^n \ \ = -\omega_k^n \wedge \omega_1^k \qquad\qquad\qquad\ \ = \omega^{n-2} \wedge \omega_{n-3}^n + \omega^n \wedge \omega_{n-2}^n,
\end{aligned}
$$

we find $\omega_{n-3}^n = f_1\omega^{n-2} + f_2\omega^n$ and $\omega_{n-3}^{n-1}, \omega_{n-2}^n = 0 \mod \{\omega^{n-2}, \omega^n\}$. Now we compare the differentials of $\omega_1^{n-2} = \omega^n$ and $\omega_1^{n-2} = \omega_{n-3}^{n+1}$:

$$
\begin{aligned}
0 &= d(\omega_1^{n-2} - \omega^n) = -\omega_1^{n-2} \wedge \omega_1^1 - \omega_k^{n-2} \wedge \omega_1^k + \omega^n \wedge \omega^0 + \omega_i^n \wedge \omega^i\\
&= \omega^{n-2} \wedge (\omega_{n-3}^{n-2} + f_1\omega^{n-3}) + \omega^{n-1} \wedge (-\omega_{n-1}^n) + \omega^n \wedge (\ldots)\\
0 &= d(\omega_1^{n-2} - \omega_{n-3}^{n+1}) = -\omega_1^{n-2} \wedge \omega_1^1 - \omega_k^{n-2} \wedge \omega_1^k + \omega_i^{n+1} \wedge \omega_{n-3}^i + \omega_{n+1}^{n+1} \wedge \omega_{n-3}^{n+1}\\
&= \omega^{n-2} \wedge (2\omega_{n-3}^{n-2} - f_1\omega^{n-3}) + \omega^n \wedge (\ldots).
\end{aligned}
$$

From $\omega_{n-3}^{n-2} + f_1\omega^{n-3} = 0 \mod \{\omega^{n-2}, \omega^{n-1}, \omega^n\}$ and $2\omega_{n-3}^{n-2} - f_1\omega^{n-3} = 0 \mod \{\omega^{n-2}, \omega^n\}$ we conclude $f_1 = 0$ and $\omega_{n-3}^{n-2} = 0 \mod \{\omega^{n-2}, \omega^n\}$. In addition we now realize that $\omega_{n-1}^n = 0 \mod \{\omega^{n-1}, \omega^n\}$. Finally, differentiating $\omega_{n-1}^{n+1} = 0$,

$$
\begin{aligned}
0 &= d\omega_{n-1}^{n+1} = -\omega_{n-3}^{n+1} \wedge \omega_{n-1}^{n-3} - \omega_{n-2}^{n+1} \wedge \omega_{n-1}^{n-2} - \omega_n^{n+1} \wedge \omega_{n-1}^n - \omega_\mu^{n+1} \wedge \omega_{n-1}^\mu\\
&= -\omega^{n-3} \wedge \omega_{n-1}^n - \omega^{n-2} \wedge \omega_{n-1}^{n-2} - \omega^n \wedge (\ldots)
\end{aligned}
$$

yields $\omega_{n-1}^n = 0 \mod \{\omega^{n-3}, \omega^{n-2}, \omega^n\}$, hence $\omega_{n-1}^n = 0 \mod \{\omega^n\}$. We also see now that $\omega_{n-1}^{n-2} = 0 \mod \{\omega^{n-2}, \omega^n\}$.

A Gauss fiber cone G of X is given by the integrable distribution $\omega^{n-2} = \omega^n = 0$. Its tangent space at e_0 is the image of

$$
de_0 = \omega^1 e_1 + \omega^\varepsilon e_\varepsilon + \omega^{n-3} e_{n-3} + \omega^{n-1} e_{n-1} \mod \{e_0, \omega^{n-2}, \omega^n\},
$$

i.e., it is $\{e_0, \ldots, e_{n-3}, e_{n-1}\}$. Its second fundamental form

$$
\begin{aligned}
\mathrm{I\!I}_{G,e_0} = d^2 e_0 &= (\omega^{n-3}\omega_{n-3}^{n-2} + \omega^{n-1}\omega_{n-1}^{n-2}) e_{n-2} + (\omega^{n-3}\omega_{n-3}^n + \omega^{n-1}\omega_{n-1}^n) e_n\\
&+ \omega^{n-3}\omega_{n-3}^{n+1} e_{n+1} + \omega^{n-1}\omega_{n-1}^\mu e_\mu = 0 \mod \{e_0, \ldots, e_{n-3}, e_{n-1}, \omega^{n-2}, \omega^n\}
\end{aligned}
$$

vanishes by $(*)$. Therefore G is the linear space $\{e_0, \ldots, e_{n-3}, e_{n-1}\}$. From [P$_1$, Theorem 9] we know that X_f has Gauss rank b, and its tangent space at the general point e_1 is $\{e_1, \ldots, e_{n-2}\}$. The linear Gauss fiber cone intersects this tangent space $\mathbb{T}_{e_1} X_f$ in the linear space $\{e_1, \ldots, e_{n-3}\}$, which is a fixed d–plane along the Gauss fiber cone vertex $V = \{e_1, \ldots, e_d\} \subset X_f$ since

$$
de_1 = de_\varepsilon = de_{n-3} = 0 \mod \{e_1, \ldots, e_{n-3}, \omega^{n-2}, \omega^n\}.
$$

Now we note that the distribution $\omega^n = 0$ is integrable by the theorem of Frobenius, since

$$
d\omega^n = -\omega^n \wedge \omega^0 - \omega_i^n \wedge \omega^i = 0 \mod \{\omega^n\}
$$

by $(*)$. An integral manifold Y has in a general point e_0 the tangent space $\{e_0, \ldots, e_{n-1}\}$ because — using the index range $n-3 \le k \le n-1$ — one has

$$
de_0 = \omega^1 e_1 + \omega^\varepsilon e_\varepsilon + \omega^k e_k \mod \{e_0, \omega^n\}.
$$

Its second fundamental form is by $(*)$

$$\begin{aligned}\mathrm{I\!I}_{Y,e_0} &= d^2e_0 = \omega^k(\omega_k^n e_n + \omega_k^{n+1}e_{n+1} + \omega_k^\mu e_\mu)\\ &= (\omega^{n-2})^2(fe_n + e_{n+1}) \mod \{e_0,\dots,e_{n-1},\omega^n\}.\end{aligned}$$

Thus it has Gauss rank 1, and its Gauss fibers, $\{e_0,\dots,e_{n-3},e_{n-1}\}$, are the linear Gauss fiber cones of X.

Case $l=3$. We start with adapting the frame such that

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix},\ A_\varepsilon = \begin{pmatrix} 0 & 0 & t_\varepsilon & u_\varepsilon\\ 0 & 0 & s_\varepsilon & t_\varepsilon\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\ Q^{n+1} = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix},$$

$$\text{and}\quad Q^\mu = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & q_1^\mu\\ 0 & 0 & q_1^\mu & q_2^\mu\\ 0 & q_1^\mu & q_2^\mu & q_3^\mu\end{pmatrix}$$

with the index ranges $2\le\varepsilon\le d$ and $n+2\le\mu\le N$. In addition, we will use the index range $n-3\le k\le n-1$. Our first task is to express ω_j^i for $i>j$ in terms of the forms $\{\omega^{n-3},\dots,\omega^n\}$ for later uses. Differentiating $\omega_1^n=0$ yields

$$0 = d\omega_1^n = -\omega_k^n\wedge\omega_1^k = \omega^{n-2}\wedge\omega_{n-3}^n + \omega^{n-1}\wedge\omega_{n-2}^n + \omega^n\wedge\omega_{n-1}^n,$$

thus

$$\begin{aligned}\omega_{n-3}^n &= f_1\omega^{n-2} + f_2\omega^{n-1} + f_3\omega^n\\ \omega_{n-2}^n &= f_2\omega^{n-2} + f_4\omega^{n-1} + f_5\omega^n\\ \omega_{n-1}^n &= f_3\omega^{n-2} + f_5\omega^{n-1} + f_6\omega^n.\end{aligned}$$

Using the relation $\omega_1^{n-1}=\omega^n$, we obtain modulo *omega*^{n}

$$\begin{aligned}0 = d(\omega_1^{n-1}-\omega^n) &= -\omega_1^{n-1}\wedge\omega_1^1 - \omega_k^{n-1}\wedge\omega_1^k + \omega^n\wedge\omega^0 + \omega_i^n\wedge\omega^i\\ &= \omega^{n-3}\wedge(-\omega_{n-3}^n) + \omega^{n-2}\wedge(\omega_{n-3}^{n-1}-\omega_{n-2}^n) + \omega^{n-1}\wedge(\omega_{n-2}^{n-1}-\omega_{n-1}^n)\end{aligned}$$

and therefore

$$\begin{aligned}\omega_{n-3}^{n-1} &= -f_1\omega^{n-3} + g_1\omega^{n-2} + g_2\omega^{n-1} + g_3\omega^n\\ \omega_{n-2}^{n-1} &= -f_2\omega^{n-3} + (g_2-f_4+f_3)\omega^{n-2} + g_4\omega^{n-1} + g_5\omega^n.\end{aligned}$$

Next we consider $\omega_{n-3}^{n+1}=\omega^n$ and its differential modulo ω^n

$$\begin{aligned}0 = d(\omega^n-\omega_{n-3}^{n+1}) &= -\omega^n\wedge\omega^0 - \omega_i^n\wedge\omega^i + \omega_i^{n+1}\wedge\omega_{n-3}^i + \omega_{n+1}^{n+1}\wedge\omega_{n-3}^{n+1}\\ &= \omega^{n-3}\wedge(2\omega_{n-3}^n) + \omega^{n-2}\wedge(\omega_{n-3}^{n-1}+\omega_{n-2}^n) + \omega^{n-1}\wedge(\omega_{n-3}^{n-2}+\omega_{n-1}^n)\end{aligned}$$

Plugging in the terms from above, we find $f_1=0$ by looking at the coefficient of $\omega^{n-3}\wedge\omega^{n-2}$ and further

$$\omega_{n-3}^{n-2} = 2f_2\omega^{n-3} + (g_2+f_4-f_3)\omega^{n-2} + h_1\omega^{n-1} + h_2\omega^n.$$

Finally, we show that $g_1 = f_2 = 0$ in addition to $f_1 = 0$. The differentials of $\omega_1^{n-2} = \omega^{n-1}$ and $\omega_{n-3}^n = f_2\omega^{n-1} + f_3\omega^n$ modulo $\{\omega^{n-1}, \omega^n\}$ are

$$\begin{aligned}0 = d(\omega_1^{n-2} - \omega^{n-1}) &= -\omega_{n-3}^{n-2} \wedge \omega_1^{n-3} + \omega_{n-3}^{n-1} \wedge \omega^{n-3} + \omega_{n-2}^{n-1} \wedge \omega^{n-2} \\ &= (3f_2 + g_1)\omega^{n-2} \wedge \omega^{n-3} \mod \{\omega^{n-1}, \omega^n\}\end{aligned}$$

$$\begin{aligned}0 = d(\omega_{n-3}^n - f_2\omega^{n-1} - f_3\omega^n) &= -\omega_{n-2}^n \wedge \omega_{n-3}^{n-2} + f_2\omega_{n-3}^{n-1} \wedge \omega^{n-3} \\ &+ f_2\omega_{n-2}^{n-1} \wedge \omega^{n-2} = f_2(g_1 - f_2)\omega^{n-2} \wedge \omega^{n-3} \mod \{\omega^{n-1}, \omega^n\}.\end{aligned}$$

From $0 = 3f_2 + g_1 = f_2(g_1 - f_2)$, we conclude $g_1 = f_2 = 0$.

Now we can examine X. We start by showing that a general Gauss fiber cone G, which is given by the integrable distribution $\omega^{n-2} = \omega^{n-1} = \omega^n = 0$, is linear. The tangent space of G at e_0 is the image of

$$de_0 = \omega^1 e_1 + \omega^\varepsilon e_\varepsilon + \omega^{n-3} e_{n-3} \mod \{e_0, \omega^{n-2}, \omega^{n-1}, \omega^n\},$$

i.e., it is $\{e_0, \dots, e_{n-3}\}$. The second fundamental form of G,

$$\mathrm{I\!I}_{G,e_0} = \omega^{n-3}\omega_{n-3}^{k+1} e_{k+1} + \omega^{n-3}\omega_{n-3}^{n+1} e_{n+1} \mod \{e_0, \dots, e_{n-3}, \omega^{n-2}, \omega^{n-1}, \omega^n\},$$

vanishes, because $\omega_{n-3}^{k+1} = 0 \mod \{\omega^{n-2}, \omega^{n-1}, \omega^n\}$ by our computations above. Thus G is the linear space $\{e_0, \dots, e_{n-3}\}$.

Next we note that the distribution $\omega^{n-1} = \omega^n = 0 \Leftrightarrow \omega_1^{n-2} = \omega_1^{n-1} = 0$ is integrable by the Theorem of Frobenius, since

$$\begin{aligned}d\omega_1^{n-2} &= -\omega_1^{n-2} \wedge \omega_1^1 - \omega_\varepsilon^{n-2} \wedge \omega_1^\varepsilon - \omega_k^{n-2} \wedge \omega_1^k = 0 \mod \{\omega^{n-1}, \omega^n\} \\ d\omega_1^{n-1} &= -\omega_1^{n-1} \wedge \omega_1^1 - \omega_k^{n-1} \wedge \omega_1^k = 0 \mod \{\omega^{n-1}, \omega^n\},\end{aligned}$$

where we used again that $\omega_{n-3}^{n-2}, \omega_{n-3}^{n-1} = 0 \mod \{\omega^{n-2}, \omega^{n-1}, \omega^n\}$.

Let Y be an integral manifold of this distribution. Its tangent space at the point e_0 is $\{e_0, \dots, e_{n-2}\}$ as the image of

$$de_0 = \omega^1 e_1 + \omega^\varepsilon e_\varepsilon + \omega^{n-3} e_{n-3} + \omega^{n-2} e_{n-2} \mod \{e_0, \omega^{n-1}, \omega^n\}.$$

Its second fundamental form is

$$\begin{aligned}\mathrm{I\!I}_{Y,e_0} = d^2 e_0 &= (\omega^{n-3}\omega_{n-3}^{n-1} + \omega^{n-2}\omega_{n-2}^{n-1})e_{n-1} + (\omega^{n-3}\omega_{n-3}^n + \omega^{n-2}\omega_{n-2}^n)e_n \\ &+ (\omega^{n-3}\omega_{n-3}^{n+1} + \omega^{n-2}\omega_{n-2}^{n+1})e_{n+1} + \omega^{n-2}\omega_{n-2}^\mu e_\mu \\ &= (g_2 - f_4 + f_3)(\omega^{n-2})^2 e_{n-1} \mod \{e_0, \dots, e_{n-2}, \omega^{n-1}, \omega^n\}.\end{aligned}$$

Therefore, Y has Gauss rank 1 and has $\{e_0, \dots, e_{n-3}\}$ — the linear Gauss fiber cones of X — as Gauss fibers.

Finally, we want to make a remark about how the linear Gauss fiber cones relate to the focal variety X_f. The second fundamental form of X_f can be easily computed to be

$$\begin{aligned}\mathrm{I\!I}_{X_f,e_1} = &(2f_3\omega_1^{n-3}\omega_1^{n-1} + f_4(\omega_1^{n-2})^2 + 2f_5\omega_1^{n-2}\omega_1^{n-1} + f_6(\omega_1^{n-1})^2)e_n \\ &+ (2\omega_1^{n-3}\omega_1^{n-1} + (\omega_1^{n-2})^2)e_{n+1} + (2q_1^\mu\omega_1^{n-2}\omega_1^{n-1} + q_2^\mu(\omega_1^{n-1})^2)e_\mu \\ &\mod \{e_1, \dots, e_{n-1}\}.\end{aligned}$$

We see that the intersection of the linear Gauss fiber cones with the hyperplane H_∞, the linear space $\{e_1, \dots, e_{n-3}\}$, is an asymptotic space of $\mathrm{I\!I}_{X_f,e_1}$, which is in fact fixed along the Gauss fiber cone vertex $\{e_1, \dots, e_d\}$.

4 Constructions for Gauss Rank 3

In this final section we will show that the descriptions in the structure theorem for Gauss rank 3 can be read as ways how to construct developable varieties with a prescribed focal variety. Since the twisted planes are well understood, only the following remains to be proved for the $l = 1$ case.

Proposition 1 *Let C be a curve and $\mathcal{G}$ a two–dimensional family of $(n-2)$–planes such that every $(n-3)$-th osculating plane of C is contained in a one–dimensional subfamily of $\mathcal{G}$. Let X be the union of the planes of $\mathcal{G}$. If $\mathcal{G}$ is chosen general enough, then X is a developable variety whose focal variety is the $(n-4)$-th osculating scroll of C.*

Proof. We will use the language of [FP] for the local computations, which is very convenient for constructions. Let $\varphi(s)$ be a local lifting of a parameterization of C. Then the $(n-3)$-th osculating planes of C are

$$\left\{\varphi, \ldots, \varphi^{(n-3)}\right\}.$$

We choose the family $\mathcal{G}$ by choosing a parameterized surface $\psi(s,t)$ and setting

$$\mathcal{G}_{(s,t)} := \left\{\varphi, \ldots, \varphi^{(n-3)}, \psi\right\}.$$

Then X is locally the image of

$$\begin{array}{rccc} \Phi: & (\mathbb{C}^2, 0) \times \mathbb{C}^{n-2} \times \mathbb{C} & \longrightarrow & \widehat{X} \\ & (s,t,\lambda,\mu) & \longmapsto & \sum_{i=0}^{n-3} \lambda_i \varphi^{(i)}(s) + \mu\psi(s,t). \end{array}$$

Its tangent space is

$$T_{\varphi(s,t,\lambda,\mu)}\widehat{X} = \left\{\varphi, \ldots, \varphi^{(n-3)}, \psi, \lambda_{n-3}\varphi^{(n-2)} + \mu\frac{\partial\psi}{\partial s}, \mu\frac{\partial\psi}{\partial t}\right\};$$

hence, it depends precisely on s, t, and the ratio $(\lambda_{n-3} : \mu)$ if ψ was chosen general enough. Therefore, X has Gauss rank 3. An adapted parameterization of X is given by $\Phi(s,t,u,\lambda,\lambda_{n-3}u)$, whose images for fixed (s,t,u) are the Gauss fibers of X. For general (s,t,u) the dimension of $T_{\Phi(s,t,u,\lambda,\lambda_{n-3}u)}\widehat{X}$ will drop only for $\lambda_{n-3} = 0$; hence, the focal variety of X is the image of $\Phi(s,t,u,\lambda_0,\ldots,\lambda_{n-4},0,0)$, i.e., the $(n-4)$-th osculating scroll of C. □

As already mentioned in the introduction the $l = 2$ case gets very technically if the prescribed focal variety is ruled by codimension one planes. Therefore, we restrict ourselves to show the following:

Proposition 2 *Let Y be a variety of dimension $n-2$ and Gauss rank 2 which has an asymptotic $(n-3)$–plane in each tangent space, but such that the integral submanifolds of this distribution are not linear. Choose a 2–dimensional family $\mathcal{G}$ containing the asymptotic $(n-2)$–planes of Y which has also the additional property described in the Theorem. Let X be the union of these planes. If the family $\mathcal{G}$ was chosen sufficiently general, X will be developable of Gauss rank 3 and $l = 2$, and its focal variety will lie inside H_∞.*

Proof. Let

$$\begin{array}{rccc} \Phi: & (\mathbb{C}^2,0)\times\mathbb{C}^{n-3} & \longrightarrow & \widehat{Y} \\ & (s,t,\lambda) & \longmapsto & \sum_{i=0}^{n-4}\lambda_i\varrho_i(s,t) \end{array}$$

be an adapted parameterization of $\widehat{Y}$, i.e., the Gauss fibers of $\widehat{Y}$ are spanned by the vectors $\{\varrho_i\}_i$ and the asymptotic submanifolds are the images of Φ for fixed t. The asymptotic spaces and the tangent spaces, which depend only on s and t, are — assuming ϱ_0 is general —

$$\begin{aligned} A_{(s,t)} &:= \left\{\varrho_i, \tfrac{\partial\varrho_0}{\partial s}\right\} &&= \left\{\varrho_i, \tfrac{\partial\varrho_i}{\partial s}\right\} \\ T_{(s,t)}\widehat{Y} &:= \left\{\varrho_i, \tfrac{\partial\varrho_0}{\partial s}, \tfrac{\partial\varrho_0}{\partial t}\right\} &&= \left\{\varrho_i, \tfrac{\partial\varrho_i}{\partial s}, \tfrac{\partial\varrho_i}{\partial t}\right\}. \end{aligned}$$

Since $\frac{\partial}{\partial s}$ is an asymptotic direction, we have $\frac{\partial^2\varrho_i}{(\partial s)^2}\in T_{(s,t)}\widehat{Y}$. We claim that $\frac{\partial^2\varrho_i}{(\partial s)^2}\notin A_{(s,t)}$ for some i, w.l.o.g. $i=0$. Otherwise, arbitrary high derivatives of ϱ_i with respect to s lie in $A_{(s,t)}$ by induction; hence arbitrary high derivatives of Φ with respect to s and λ lie in $A_{(s,t)}$, and we conclude that the image of Φ with fixed t is the linear space $A_{(s,t)}$, contradicting our assumptions. Therefore,

$$\frac{\partial^2\varrho_0}{(\partial s)^2}\neq 0 \mod A_{(s,t)} \quad\text{and}\quad T_{(s,t)}\widehat{Y} = A_{(s,t)}+\left\{\frac{\partial^2\varrho_0}{(\partial s)^2}\right\} = A_{(s,t)}+\left\{\frac{\partial^2\varrho_i}{(\partial s)^2}\right\},$$

and we find functions $\zeta_i(s,t)$ with

$$\frac{\partial\varrho_i}{\partial t} = \zeta_i\frac{\partial^2\varrho_0}{(\partial s)^2}.$$

Next we choose the family $\mathcal{G}$ by choosing a parameterized surface $\varphi(s,t)$ not contained in H_∞ and setting

$$\mathcal{G}_{(s,t)} := A_{(s,t)}+\{\varphi(s,t)\} = \left\{\varrho_i, \frac{\partial\varrho_0}{\partial s}, \varphi\right\} = \left\{\varrho_i, \frac{\partial\varrho_i}{\partial s}, \varphi\right\}.$$

Now the family $s\mapsto\mathcal{G}_{(s,t)}$ is supposed to swept out a Gauss rank 1 variety, hence

$$\left(\left\{\frac{\partial\varrho_i}{\partial s}, \frac{\partial^2\varrho_0}{(\partial s)^2}, \frac{\partial\varphi}{\partial s}\right\}+\mathcal{G}\right)/\mathcal{G} = \left(\left\{\frac{\partial^2\varrho_0}{(\partial s)^2}, \frac{\partial\varphi}{\partial s}\right\}+\mathcal{G}\right)/\mathcal{G}$$

is one–dimensional [FP, 2.3.5], i.e., using $\frac{\partial^2\varrho_0}{(\partial s)^2}\notin\mathcal{G}$ there exists a function $\xi(s,t)$ with

$$\frac{\partial\varphi}{\partial s} = \xi\frac{\partial^2\varrho_0}{(\partial s)^2} \mod \mathcal{G}.$$

The variety X is locally the image of

$$\begin{array}{rccc} \Psi: & (\mathbb{C}^2,0)\times\mathbb{C}^{n-1} & \longrightarrow & \widehat{X} \\ & (s,t,\lambda,\mu,\nu) & \longmapsto & \sum_{i=0}^{n-4}\lambda_i\varrho_i(s,t)+\mu\frac{\partial\varrho_0}{\partial s}+\nu\varphi. \end{array}$$

Its tangent space is

$$\begin{aligned} T_{\Psi(s,t,\lambda,\mu,\nu)}\widehat{X} &= \left\{\varrho_i, \tfrac{\partial\varrho_0}{\partial s}, \varphi, \mu\tfrac{\partial^2\varrho_0}{(\partial s)^2}+\nu\tfrac{\partial\varphi}{\partial s}, \textstyle\sum\lambda_i\tfrac{\partial\varrho_i}{\partial t}+\mu\tfrac{\partial^2\varrho_0}{\partial s\partial t}+\nu\tfrac{\partial\varphi}{\partial t}\right\} \\ &= \left\{\varrho_i, \tfrac{\partial\varrho_0}{\partial s}, \varphi, (\mu+\nu\xi)\tfrac{\partial^2\varrho_0}{(\partial s)^2}, \left(\textstyle\sum\lambda_i\zeta_i\right)\tfrac{\partial^2\varrho_0}{(\partial s)^2}+\mu\tfrac{\partial^2\varrho_0}{\partial s\partial t}+\nu\tfrac{\partial\varphi}{\partial t}\right\}. \end{aligned}$$

Note that for a sufficient general choice of φ the $\frac{\partial \varphi}{\partial t}$ will not be contained in the span of the other occurring vectors. Therefore, the tangent space of X is precisely constant along the image of Ψ with fixed s, t and fixed ratio of μ and ν. This shows that X has Gauss rank 3. Further, an adapted parameterization of X is given by $\Psi(s,t,\lambda,\mu,\mu u)$ whose image for fixed (s,t,u) are the Gauss fibers of X. For general (s,t,u) the dimension of $T_{\Psi(s,t,\lambda,\mu,\mu u)}\widehat{X}$ will drop only at $\mu = 0$, i.e., the focal variety of X is given by the image of $\Psi(s,t,\lambda,0,0)$, which is $Y \subset H_\infty$. $\square$

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