Developable Varieties with all Singularities at Infinity
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Abstract

A projective variety is called developable if the image of its Gauss map has a smaller dimension than the variety itself. Developable varieties are always singular, and requiring that all singularities lie in a hyperplane puts a severe restriction on them. Here we refine a theorem of Wu and Zheng stating that such varieties are the union of cones if the dimension of the Gauss image is less than or equal to four. Afterwards we study their singular locus. Finally, we describe the geometry of such varieties whose Gauss image has dimension two.

1 Introduction

For an n-dimensional projective variety $X \subset \mathbb{P}^N$ we consider its rational Gauss map

$$\gamma : X \dashrightarrow \mathbb{G}(n, N), \quad x \mapsto \mathbb{T}_x X,$$

which assigns to every smooth point x of X its embedded tangent space $\mathbb{T}_x X$ as a point of the respective Grassmannian. The variety X is called developable if the dimension of the image of the Gauss map, the Gauss rank r, is less than the dimension of X. It is classically known that the general fiber of the Gauss map is a linear space of dimension $d = n - r$. Further, one knows that every developable variety is singular. In fact, on every Gauss fiber there exists a hypersurface of degree r, called the *focal hypersurface*, that lies in the singular locus of X.

Now one can ask if there exist developable varieties such that their singular locus lies on a hyperplane H_∞. We will call such a variety *affinely smooth*. Trivial examples for such varieties are cones over smooth varieties with vertices in H_∞. All affinely smooth developable varieties of Gauss rank 1 are such cones. This was originally proven by Hartman and Nirenberg in the euclidian case [10], and later by Abe in the complex [1], and by Nomizu and Pinkall as well as Opozda in the real affine case [12, 13]. After a suggestion of Bourgain, Wu worked out an example of an affinely smooth developable hypersurface of Gauss rank 2, which is not a cone. Later, Akivis and Goldberg showed that this example is locally equivalent to an earlier example of Sacksteder [4, 15, 17]. They also found new examples of affinely smooth developable varieties

[6]. In addition, Akivis and Goldberg proved that affinely smooth developable varieties of arbitrary rank, but with a certain genericity condition on the second fundamental form, are always cones[7].

Wu and Zheng examined the affinely smooth developable varieties in the context of euclidian geometry and obtained the following result, which we will describe in terms of projective geometry.

We group the d-dimensional Gauss fibers into sets, in which the fibers intersect the hyperplane H_∞ transversely in the same linear space. The closure of the union of the Gauss fibers in one of these sets is a cone, whose $(d-1)$-dimensional vertex lies in H_∞. We call such a cone a *Gauss fiber cone.* In general, the Gauss fiber cone is just the d-dimensional Gauss fiber itself. However, Wu and Zheng proved [18, Theorem 1]

Theorem (Wu, Zheng). *Let $X \subset \mathbb{P}^N$ be an affinely smooth developable variety of Gauss rank $r \leq 4$ or of Gauss fiber dimension $d = 1$. Then the Gauss fiber cones have dimension greater than d.*

Wu and Zheng also showed that under a special condition each Gauss fiber cone is the union of linear spaces of dimension greater than d or may even be a linear space itself [18, Theorem 2]. In particular, the latter will be the case if X has Gauss rank 2, which is an unpublished Theorem of Vitter [16].

Theorem (Vitter). *Let $X \subset \mathbb{P}^N$ be an affinely smooth developable variety of dimension n and Gauss rank 2. If X is not a cone then it is the union of a one-dimensional family of $(n-1)$-planes.*

For an affinely smooth developable variety of Gauss rank 3 or 4, it will be seldom the case that the above mentioned special condition holds. Here, we will extend [18, Theorem 2] to Theorem 6. As a special case it implies that the Gauss fiber cones are in many cases contained in a rather small linear space, more precisely:

Theorem. *Let $X \subset \mathbb{P}^N$ be a general affinely smooth developable variety of Gauss rank ≤ 4. Then its Gauss fiber cones are of dimension $d+1$ and each of them is contained in a linear space of dimension $d+r-1$.*

If X is of Gauss rank 2 and not a cone, then this theorem also specializes to Vitter's Theorem, since all such X are general in the sense of the theorem. In Section 1 these theorems are proved in the projective case by using standard arguments of Cartan's moving frame method.

One advantage of working in the projective space instead of the euclidian space is that one can consider the vertices of the Gauss fiber cones at infinity. The vertex of a Gauss fiber cone is the focal hypersurface of its fibers. Thus the union of their vertices is the *focal variety* of X. The focal variety together with the *focal hypercone variety*, which is the focal variety of the dual variety of X, govern the geometry of a developable variety [3, 5, 8]. We will study the focal variety of an affinely smooth variety in Section 2. We will compute its dimension, Gauss rank, and the type of its focal hypersurfaces (Theorem 9).

Finally, in Section 4 we examine the special case where X is uniruled by $(n-1)$-planes. As a special case of Corollary 11 we obtain the following strengthening of Vitter's Theorem.

Theorem. *Let $X \subset \mathbb{P}^N$ be an affinely smooth developable variety of dimension n and Gauss rank 2 which is not a cone. Then there exists a unique curve C in the hyperplane at infinity such that X is the union of a one-dimensional family of $(n-1)$-planes that contain the $(n-2)$-th osculating planes of the curve C.*

As a concrete example we get the construction of the Sacksteder-Bourgain hypersurface back, as it was discovered by Akivis and Goldberg [4]: In $\mathbb{P}^4$ take a plane conic in H_∞ and a line. Further, choose a projective one-to-one correspondence between them. Then the Sacksteder-Bourgain hypersurface is the union of those planes that are spanned by a point of the line and the tangent line to the conic at the corresponding point. Its Gauss fiber cones are plane pencils of straight lines in those planes. The locus of the centers of these pencils is the conic. This geometric description complements the analytic description of affinely smooth hypersurfaces of Gauss rank 2 in [18].

2 The Setup and the Gauss Fiber Cone Theorems

We will examine the developable varieties by using Cartan's moving frame method. Here we recall some facts, in order to fix the notations. For a complete introduction see [8] or [11]. On the projective space $\mathbb{P}^N$, we have the bundle of projective frames, consisting of bases $(e_0, \dots, e_N)$ of $\mathbb{C}^{N+1}$. The infinitesimal motion of the frame is described by

$$de_A = \omega_A^0 e_0 + \ldots + \omega_A^N e_N \quad \text{for } 0 \leq A \leq N,$$

where the ω_A^B are the Maurer-Cartan 1-forms on $\mathrm{GL}(\mathbb{C}^{N+1})$, which fulfill the Maurer-Cartan equation

$$d\omega_B^A = -\omega_C^A \wedge \omega_B^C.$$

To study the geometry of a developable variety, we work only on the submanifold of the projective frame bundle where the general frame has the following properties:

$$\begin{array}{ll} \{e_0\} & \text{is a point of } X, \\ \{e_0, \dots, e_d\} & \text{is the Gauss fiber } F \text{ of } X \text{ through } \{e_0\}, \\ \{e_0, \dots, e_n\} & \text{is the tangent space of } X \text{ in } \{e_0\}, \\ \{e_1, \dots, e_N\} & \text{is the hyperplane } H_\infty. \end{array}$$

Our adaptions to the geometry of X have the effect that

$$\begin{aligned} de_0 &= \omega^0 e_0 + \omega^\delta e_\delta + \omega^i e_i \\ de_\delta &= \qquad\quad \omega_\delta^\varepsilon e_\varepsilon + \omega_\delta^i e_i \\ de_i &= \qquad\quad \omega_i^\delta e_\delta + \omega_i^j e_j + \omega_i^\mu e_\mu \\ de_\mu &= \qquad\quad \omega_\mu^\delta e_\delta + \omega_\mu^j e_j + \omega_\mu^\nu e_\nu \end{aligned}$$

with $\omega^A := \omega_0^A$ and the index ranges $1 \leq \delta, \varepsilon \leq d$, $d+1 \leq i, j \leq n$, and $n+1 \leq \mu, \nu \leq N$.

Differentiating $\omega^\mu = 0$ resp. $\omega_\delta^\mu = 0$ and using the Cartan lemma, we find functions $q_{ij}^\mu, a_{\delta j}^i$ such that

$$\omega_i^\mu = q_{ij}^\mu \omega^j, \qquad \omega_\delta^i = a_{\delta j}^i \omega^j$$

and q^μ_{ij} as well as $q^\mu_{ik}a^k_{\delta j}$ are symmetric in i, j. Setting $Q^\mu = (q^\mu_{ij})$, $A_\delta = (a^i_{\delta j})$, $\mathcal{Q} = \operatorname{span}\{Q^\mu\}$, and $\mathcal{A} = \operatorname{span}\{A_\delta\}$, this means that QA is symmetric for $Q \in \mathcal{Q}$ and $A \in \mathcal{A}$ [8, 4.1]. From the above expression for de_δ we see that $\mathcal{A}$ describes the infinitesimal movement of $F \cap H_\infty$, thus we call it the *fiber movement system (at infinity)*.

The second fundamental form of X in $x = \{e_0\}$ is the second differential of e_0 modulo the tangent space

$$\mathrm{II}_x = d^2 e_0 = \omega^\mu_i \omega^i e_\mu = q^\mu_{ij}\omega^i\omega^j e_\mu \mod \{e_0, \ldots, e_n\}.$$

Thus the linear system $\mathcal{Q}$ describes the quadrics of the second fundamental form. Since the second fundamental form is essentially the differential of the Gauss map, the singular locus of II_x is the Gauss fiber $F = \{e_0, \ldots, e_d\}$.

Now we recall that X is singular along a hypersurface in a Gauss fiber F. Let $e = \lambda^0 e_0 + \lambda^\delta e_\delta \in F \subset X$ be a point of the fiber. We determine the tangent space of X at e. Since $F \subset X$, we have $F \subset \mathbb{T}_e X$; thus, we may compute modulo F

$$de = (\lambda^0\omega^i + \lambda^\delta\omega^i_\delta)e_i = (\lambda^0\delta^i_j + \lambda^\delta a^i_{\delta j}\omega^j)e_i \mod \{e_0, \ldots, e_d\}.$$

The point $e \in X$ is smooth iff $\mathbb{T}_e X = \mathbb{T}_x X$. This will be the case if the matrix $\lambda^0 E_r + \lambda^\delta A_\delta$ is invertible. Note that this is a local computation; hence, the point e may in fact be a singular point if it is a point of selfintersection of X. On the other hand, points $e = \lambda^0 e_0 + \lambda^\delta e_\delta \in F$ with $\det(\lambda^0 E_r + \lambda^\delta A_\delta) = 0$ will always be singular in X. They form the degree r *focal hypersurface* of the Gauss fiber F. The closure of the union of the focal hypersurfaces is the *focal variety* $X_f \subseteq \operatorname{Sing} X$ of X.

The linear system $\mathcal{A}$ has the following invariant description. Let $\mathcal{F} \subset \mathbb{G}(d, N)$ be the Gauss fiber variety, i.e. the closure of the set of all d-dimensional Gauss fibers of X. At a smooth point $F \in \mathcal{F}$ we have

$$T_F\mathcal{F} \subset T_F\mathbb{G}(d, N) = \operatorname{Hom}(F, \mathbb{C}^{N+1}/F).$$

For a smooth $x \in F \subset X$, there is a canonical isomorphism $T_F\mathcal{F} \cong \mathbb{T}_x X/F$. Further, we use the fact that the images of all maps $T_F\mathcal{F} \subset \operatorname{Hom}(F, \mathbb{C}^{N+1}/F)$ lie in $\mathbb{T}_x X$ to rewrite our above inclusion as

$$F \longrightarrow \operatorname{Hom}(T_F\mathcal{F}, \mathbb{C}^{N+1}/F) = \operatorname{End}(\mathbb{T}_x X/F).$$

The image of $F \cap H_\infty$ under this map is the liner system $\mathcal{A}$.

The linear system $\mathcal{A}$ corresponds to the conullity operators of [18]. In particular, they have the same properties, which we summarize in the following proposition.

Proposition 1 *Let $X \subseteq \mathbb{P}^N$ be an affinely smooth developable variety. Then X and the linear systems $\mathcal{A}$ and $\mathcal{Q}$ have the following properties:*

1. *$A \in \mathcal{A}$ is nilpotent.*

2. *QA is symmetric for all $Q \in \mathcal{Q}$ and $A \in \mathcal{A}$.*

3. $\operatorname{Sing} \mathcal{Q} = \{v \in \mathbb{C}^r \mid Q(v, \mathbb{C}^r)\} = 0$.

Proof. The properties 2 and 3 were mentioned above. Due to our adaptions of the frame, the point $e = \lambda^0 e_0 + \lambda^\delta e_\delta$ can only be singular for $\lambda^0 = 0$. We conclude that the focal hypersurface of the Gauss fiber is the linear space $\{e_1, \ldots, e_d\} = F \cap H_\infty$; therefore, $\det\left(\lambda^0 E_r + \lambda^\delta A_\delta\right) = (\lambda^0)^r$. This shows that 0 is the only eigenvalue of the matrix $A = \lambda^\delta A_\delta$, i.e. A is nilpotent. $\square$

While properties 2 and 3 hold for any developable variety, property 1 is equivalent to the fact that for an affinely smooth developable variety the focal hypersurface of a Gauss fiber F is the linear space $F \cap H_\infty$.

Analogous linear systems $\mathcal{A}$ and $\mathcal{Q}$ with the same properties were already encountered by Wu and Zheng in their euclidian setting [18, Section 1]. (The elements of $\mathcal{A}$ correspond to the conullity operators there.) The properties 2 and 3 are local properties, hence they appear in the euclidian as well as in the projective case. Property 1 is Abel's Nilpotency Theorem for complete manifolds in the euclidian setting [2]. The computations in [18, Prop. 1] indicate that a proof of it can be based on the smoothness of the manifold along a complete Gauss fiber. Such a proof would be similar to the above argument.

Now we fix a point $x \in X$ and study the linear systems $\mathcal{A}$ and $\mathcal{Q}$ with the help of linear algebra. We denote the *common kernel* of a linear system of endomorphisms $\mathcal{A}$ by $\ker \mathcal{A}$ and the span of the image of all endomorphisms, the *image space*, by $\operatorname{Im} \mathcal{A}$.

If $\dim \mathcal{A} = 1$ and an $A \in \mathcal{A}\backslash\{0\}$ is nilpotent, then we have $\ker \mathcal{A} = \ker A \neq 0$. In addition to that, Wu and Zheng showed in [18, Proposition 2] that for $r \leq 4$ a linear system $\mathcal{A}$ with the above properties always has a nontrivial common kernel. In their Proposition 3 they give without a proof a list of linear subspaces of $\operatorname{End}(\mathbb{C}^r)$ for $r \leq 4$ with the property that with respect to a suitable chosen basis the above linear system $\mathcal{A}$ is contained in one of those. However, not for all linear subspaces $\mathcal{A}$ of their list one can find a linear system $\mathcal{Q}$ such that $\mathcal{A}$ and $\mathcal{Q}$ have the properties of Proposition 1. Therefore, we will here refine their list in such way that such a $\mathcal{Q}$ can always be found. This improvement will be needed in the proof of Theorem 9. The proof of the following proposition is contained in the appendix.

Proposition 2 *Let $\mathcal{A}$ be a nontrivial linear system of endomorphisms of $\mathbb{C}^r$ and $\mathcal{Q}$ a linear system of symmetric bilinear forms of $\mathbb{C}^r$ with*

1. *every $A \in \mathcal{A}$ is nilpotent,*
2. *the bilinear form $Q(\,\cdot\,, A(\,\cdot\,))$ is symmetric for every $A \in \mathcal{A}$ and $Q \in \mathcal{Q}$,*
3. $\operatorname{Sing} \mathcal{Q} = \{v \in \mathbb{C}^r \mid Q(v, \mathbb{C}^r) = 0 \ \forall Q \in \mathcal{Q}\} = 0.$

Let l be the rank of a generic matrix of $\mathcal{A}$. Then there exists a basis of $\mathbb{C}^r$ such *that $\mathcal{A}$ is contained in the following linear systems of matrices*

$r \setminus l$	1	2	3
2	$\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$		
3	$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & x & * \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}$	
4	$\begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & x & * \\ 0 & 0 & * & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & x & y & * \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *)	$\begin{pmatrix} 0 & x & y & * \\ 0 & 0 & * & y \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix}$

*) *If the system $\mathcal{Q}$ contains a matrix of full rank, $y = z$, otherwise $y = 0$.*

For $l = 1$ the inequality $\dim \mathcal{Q} \geq \dim \mathcal{A}$ holds.

In particular, the linear system $\mathcal{A}$ always has a nontrivial common kernel.

For $r \geq 5$ there are examples of linear systems $\mathcal{A}$ with the above properties which have only a trivial common kernel.

All of the above linear systems can appear for affinely smooth developable varieties. This was shown in [18, 3, Lemma 2] for linear systems $\mathcal{A}$, $\mathcal{Q}$ with $\dim \mathcal{Q} = 1$, but one can simply add more equations of the same type to extend the lemma to arbitrary dimensional linear systems $\mathcal{Q}$.

Now we turn these technical results into geometric statements. The *Gauss fiber cones* of X are defined as follows: Let $F \in \mathcal{F} \subset \mathbb{G}(d, N)$ be one of the d-dimensional linear Gauss fibers of X that intersects H_∞ transversely in a $(d-1)$-plane V. The Gauss fiber cone with vertex V is the closure of the union of all Gauss fibers that intersect the hyperplane transversely in V. The dimension of the Gauss fiber cones was computed by Wu and Zheng in Euclidian context. Reproving their theorem in projective context, we get the chance to study the cone vertices at infinity which are the focal hypersurfaces of the fibers.

Theorem 3 *Let $X \subseteq \mathbb{P}^N$ be an affinely smooth developable variety of Gauss fiber dimension d. Let c be the dimension of the common kernel of the fiber movement system $\mathcal{A}$ of X at a general point and l be the rank of a general matrix of $\mathcal{A}$.*

Then the Gauss fiber cones of X have dimension $d + c$. The closure of the union of their vertices is the focal variety. It has dimension $l + d - 1 \leq \dim X - 2$.

This shows again how special affinely smooth developable varieties are, since the focal variety of a general developable variety has only codimension one in X. Further, from the description of the focal varieties as the union of vertices we would expect the focal variety to have dimension $r - c + d - 1$, but this is not always the same as $l + d - 1$; consider for example the case $r = 4$, $l = 2$, $\mathcal{A}^2 \neq 0$ in Proposition 2.

If the Gauss fiber cones are nontrivial, i.e. for $c > 0$, X is called a *twisted cone*. From Theorem 3 and the fact that $\ker \mathcal{A} \neq 0$ for $r \leq 4$, we immediately reobtain the following statement, which was the main result of Wu and Zheng [18].

Corollary 4 *Let $X \subseteq \mathbb{P}^N$ be an affinely smooth developable variety of Gauss fiber dimension 1 or of the Gauss rank ≤ 4. Then the variety X is a twisted cone.*

In the extreme case of $\mathcal{A} = \{0\}$, the variety X is itself a cone with a $(d-1)$-dimensional vertex at infinity. If the Gauss rank is one, we are always in this extreme case; hence, we rediscovered the following theorem of Hartman and Nirenberg in algebraic context [10].

Corollary 5 *Let X be an affinely smooth developable variety of Gauss rank 1. Then X is a cone over a smooth curve.*

Proof of the Theorem. By a change in $\{e_{d+1}, \ldots, e_n\}$ we can adapt the frame further such that the common kernel of $\omega^i_\delta = a^i_{\delta j}\omega^j$ is $\{e_{d+1}, \ldots, e_{d+c}\}$, i.e. $a^i_{\delta s} = 0$ for $d+1 \leq s \leq d+c$. We claim that

$$\{e_0, \ldots, e_{d+c}\} = \{\omega^{d+c+1}, \ldots, \omega^n\}^\perp \subset TX$$

is an integral subbundle, whose integral manifolds are the Gauss fiber cones. For the integrability we have to show by the Theorem of Frobenius that

$$d\omega^t = 0 \mod \{\omega^{d+c+1}, \ldots, \omega^n\} \quad \text{for } d+c+1 \leq t \leq n.$$

We differentiate $\omega^i_\delta = a^i_{\delta t}\omega^t$ and get

$$d\omega^i_\delta = a^i_{\delta t} d\omega^t \mod \{\omega^{d+c+1}, \ldots, \omega^n\}.$$

Since the matrices $(a^i_{\delta t})_{it}$ have no common kernel, $d\omega^t = 0$ is equivalent to $d\omega^i_\delta = 0$, both modulo $\{\omega^{d+c+1}, \ldots, \omega^n\}$. Using the Maurer-Cartan-equations we have

$$\begin{aligned} d\omega^i_\delta &= -\omega^i_\varepsilon \wedge \omega^\varepsilon_\delta - \omega^i_j \wedge \omega^j_\delta \\ &= -a^i_{\varepsilon t}\omega^t \wedge \omega^\varepsilon_\delta - \omega^i_j \wedge a^j_{\delta t}\omega^t = 0 \mod \{\omega^{d+c+1}, \ldots, \omega^n\}. \end{aligned}$$

Now on an integral manifold C of the distribution we have

$$de_\delta = \omega^\varepsilon_\delta e_\varepsilon + \omega^i_\delta e_i = \omega^\varepsilon_\delta e_\varepsilon + a^i_{\delta t}\omega^t e_i = \omega^\varepsilon_\delta e_\varepsilon.$$

This shows that the subspace $V = \{e_1, \ldots, e_d\}$ is fixed on C. Hence, C is a union of Gauss fibers $\{e_0, \ldots, e_d\}$ that contain the subspace V; thus C is a subcone of the Gauss fiber cone with vertex V. Since the tangent space to the Gauss fiber cone must lie in the common kernel of $\omega^i_\delta = a^i_{\delta j}\omega^j$ and since the cone C is maximal with this property, the cone C is equal to the Gauss fiber cone.

By Proposition 1 the union of the Gauss fiber cone vertices form a dense subset of the focal variety X_f. We compute the dimension of X_f by determining the dimension of the tangent space at a general point $e = \lambda^\delta e_\delta \in V \subset X_f$. The tangent space, which contains the linear space V, is the image of

$$de = \lambda^\delta(\omega^\varepsilon_\delta e_\varepsilon + \omega^i_\delta e_i) = \lambda^\delta a^i_{\delta j}\omega^j e_i \mod \{e_1, \ldots, e_d\}.$$

Thus it has dimension $d - 1 + \operatorname{rank} \lambda^\delta A_\delta = l + d - 1$. □

In general, the Gauss fiber cones are uniruled by d-planes, namely by the Gauss fibers, but sometimes they are uniruled by higher dimensional planes or are even linear spaces themselves. Further, they have the tendency to be contained in small linear spaces. The precise statement is the following theorem, which for special cases $\operatorname{Im} \mathcal{A} \subseteq \ker \mathcal{A}$ and $\operatorname{Im} \mathcal{A} = \ker \mathcal{A}$ was already discovered by Wu and Zheng [18, Theorem 2].

Theorem 6 *In the situation of Theorem 3 let*

$$b = \dim \operatorname{Im} \mathcal{A} \cap \ker \mathcal{A} \quad \textit{and} \quad f = \dim \operatorname{Im} \mathcal{A}.$$

Then the $(d + c)$-dimensional Gauss fiber cones are the union of $(d + b)$-dimensional subcones, each of which is contained in a $(d+f)$-dimensional linear space L.

More precisely, if S is such a subcone for a Gauss fiber cone with vertex $V \subset H_\infty$, the linear space L contains the $(d + f - 1)$-dimensional span of the tangent spaces to X_f along V and is contained in the tangent space to X at any point of S.

There are the three special cases:

- *If $\operatorname{Im} \mathcal{A} \subseteq \ker \mathcal{A}$, then the Gauss fiber cones are uniruled by $(d+f)$-planes.*
- *If $\operatorname{Im} \mathcal{A} \supseteq \ker \mathcal{A}$, then the Gauss fiber cones are contained in a $(d + f)$-plane.*
- *If $\operatorname{Im} \mathcal{A} = \ker \mathcal{A}$, then the Gauss fiber cones are linear spaces themselves.*

As said above, the special cases $\operatorname{Im} \mathcal{A} \subseteq \ker \mathcal{A}$ and $\operatorname{Im} \mathcal{A} = \ker \mathcal{A}$ were discovered by Wu and Zheng. They imply the following unpublished theorem of Vitter [16], which we now obtain as a corollary.

Corollary 7 *Let $X \subset \mathbb{P}^N$ be an affinely smooth developable variety of dimension n and Gauss rank 2. If X is not a cone then it is the union of a one-dimensional family of $(n - 1)$-planes.*

Proof of the Corollary. Since X is not a cone, the linear system $\mathcal{A}$ has a positive dimension. By Proposition 2 we have $\operatorname{Im} \mathcal{A} = \ker \mathcal{A}$, so the Gauss fiber cones themselves are linear space of dimension $d + 1 = n - 1$. □

Further, we find the following second corollary.

Corollary 8 *Let $X \subset \mathbb{P}^N$ be a general affinely smooth developable variety of Gauss rank ≤ 4 and of Gauss fiber dimension d. Then its Gauss fiber cones are of dimension $d+1$, and each of them is contained in a linear space of dimension $d + r - 1$.*

Proof of the Corollary. For a general affinely smooth developable variety a general matrix of $\mathcal{A}$ will have rank $r-1$. For $r \leq 4$ we obtain with the help of Proposition 2 that $\ker \mathcal{A} \subseteq \operatorname{Im} \mathcal{A}$ and $f = r-1$. $\square$

Proof of the Theorem. We adapt the frame further such that the image space of the maps

$$\omega^i_\delta e_i = a^i_{\delta j}\omega^j e_i : \{e_{d+1}, \dots, e_n\} \to \{e_{d+1}, \dots, e_n\}$$

is $\{e_{d+1}, \dots, e_{d+b}, e_{d+c+1}, \dots, e_{d+c+f-b}\}$. We split the above index ranges s and t into σ, s resp. τ, t, such that

$$d+1 \leq \sigma \leq d+b < s \leq d+c < \tau \leq d+c+f-b < t \leq n,$$

so

$$\begin{array}{ll} \{e_\sigma, e_s\} & \text{is the common kernel of } \omega^i_\delta e_i \\ \{e_\sigma, e_\tau\} & \text{is the image space of } \omega^i_\delta e_i \end{array}$$

This has the consequence that $\omega^\sigma_\delta = \omega^\tau_\delta = 0$ modulo $\{\omega^{d+c+1}, \dots, \omega^n\}$ and $\omega^s_\delta = \omega^t_\delta = 0$.

We claim that

$$\{e_0, \dots, e_{d+b}\} = \{\omega^{d+b+1}, \dots, \omega^n\}^\perp \subseteq TX$$

is an integral subbundle. By the Theorem of Frobenius this is equivalent to

$$d\omega^s = d\omega^t = d\omega^\tau = 0 \quad \operatorname{mod} \{\omega^{d+b+1}, \dots, \omega^n\}.$$

This was shown for $d\omega^t$ and $d\omega^\tau$ in the proof of Theorem 3. It remains to treat

$$d\omega^s = -\omega^s_\sigma \wedge \omega^\sigma \quad \operatorname{mod} \{\omega^{d+b+1}, \dots, \omega^n\}.$$

We differentiate $\omega^s_\delta = 0$ and obtain

$$\begin{aligned} 0 = d\omega^s_\delta &= -\omega^s_\sigma \wedge \omega^\sigma_\delta - \omega^s_\tau \wedge \omega^\tau_\delta \\ &= -\omega^s_\sigma \wedge (a^\sigma_{\delta\tau}\omega^\tau + a^\sigma_{\delta t}\omega^t) - \omega^s_\tau \wedge (a^\tau_{\delta\tau'}\omega^{\tau'} + a^\tau_{\delta t}\omega^t) \\ &= -(a^\sigma_{\delta\tau'}\omega^s_\sigma + a^\tau_{\delta\tau'}\omega^s_\tau) \wedge \omega^{\tau'} - (a^\sigma_{\delta t}\omega^s_\sigma + a^\tau_{\delta t}\omega^s_\tau) \wedge \omega^t. \end{aligned}$$

By Cartan's Lemma this implies

$$a^\sigma_{\delta\tau'}\omega^s_\sigma + a^\tau_{\delta\tau'}\omega^s_\tau = a^\sigma_{\delta t}\omega^s_\sigma + a^\tau_{\delta t}\omega^s_\tau = 0 \quad \operatorname{mod} \{\omega^{d+c+1}, \dots, \omega^n\}.$$

Since $f = \dim \operatorname{Im} \mathcal{A}$, these are f linear independent linear combinations of the f forms ω^s_σ, ω^s_τ, thus

$$\omega^s_\sigma = \omega^s_\tau = 0 \quad \operatorname{mod} \{\omega^{d+c+1}, \dots, \omega^n\}.$$

Hence, $d\omega^s$ vanishes modulo $\{\omega^{d+b+1}, \dots, \omega^n\}$, and the Theorem of Frobenius applies.

Let S be an integral manifold of this distribution. We claim that on S the $(d+f)$-plane

$$L = \{e_0, e_\delta, e_\sigma, e_\tau\}$$

is fixed, and hence S is contained in L. It will be enough to show that the differentials of e_0, e_δ, e_σ, and e_τ lie in L. On S we have

$$\begin{aligned}
de_0 &= de_\delta = 0 \mod L \\
de_\sigma &= \omega_\sigma^s e_s + \omega_\sigma^t e_t + \omega_\sigma^\mu e_\mu \mod L \\
de_\tau &= \omega_\tau^s e_s + \omega_\tau^t e_t + \omega_\tau^\mu e_\mu \mod L.
\end{aligned}$$

From above we know that $\omega_\sigma^s = \omega_\tau^s = 0$ on S. Analogously, we also get $\omega_\sigma^t = \omega_\tau^t = 0$ on S. It remains to discuss $\omega_\sigma^\mu = q_{\sigma i}^\mu \omega^i$ and $\omega_\tau^\mu = q_{\tau i}^\mu \omega^i$. Because of the symmetry of QA_δ we have

$$0 = QA_\delta(\ker \mathcal{A}) = {}^t(\ker \mathcal{A}) \cdot QA_\delta \quad \Rightarrow \quad {}^t(\ker \mathcal{A}) \cdot Q \cdot \operatorname{Im} \mathcal{A} = 0,$$

hence

$$q_{\sigma\sigma'}^\mu = q_{\sigma\tau}^\mu = q_{s\sigma}^\mu = q_{s\tau}^\mu = 0.$$

Therefore, $\omega_\sigma^\mu = q_{\sigma t}^\mu \omega^t$ and $\omega_\tau^\mu = q_{\tau\tau'}^\mu \omega^{\tau'} + q_{\tau t}^\mu \omega^t$ vanish also on S.

Finally, we show that L contains the tangent space to the focal variety X_f along V. Since the tangent space at the $e = \lambda^\delta e_\delta \in V \subset X_f$ is

$$V + \operatorname{Im}(\lambda^\delta a_{\delta j}^i e_i \omega^j),$$

the span of all these will be $\{e_\delta, e_\sigma, e_\tau\} \subset L$. □

3 The Focal Variety

The geometry of any developable variety X is determined or at least strongly restricted by its focal variety and its focal hypercone variety, see [3, 5, 8] and Corollary 11 in Section 4. Since we know the fiber movement system $\mathcal{A}$ for an affinely smooth variety very well by Proposition 2, we can determine the geometry of their focal varieties. Because the focal variety lies in the hyperplane at infinity, it could not be studied by Wu and Zheng or Vitter in the euclidian context. This shows the advantage of working in the projective space, as a large part of the geometric information of X is coded in its focal variety.

Theorem 9 *Let X be an affinely smooth developable variety of Gauss rank $2 \leq r \leq 4$ which is not a cone. The Gauss fiber dimension of X is denoted by d. For the fiber movement system $\mathcal{A}$ belonging to X at a general point, we have the following invariants*

$$\begin{aligned}
l &= \textit{rank of a general matrix } A \in \mathcal{A} \\
b &= \textit{dimension of the image space } \operatorname{Im} \mathcal{A}.
\end{aligned}$$

They are restricted by $1 \leq l \leq b \leq r - 1$.

Then the focal variety X_f has dimension $d + l - 1 \leq n - 2$, and therefore it is a proper subvariety of $X \cap H_\infty$.

The Gauss rank of X_f is b. Its Gauss uniruling is a subuniruling of the uniruling of X_f given by the vertices of the Gauss fiber cones.

If X_f is developable, i.e. for $d-1 > b-l$, its focal hypersurfaces are of the following type

r	*subcase*	*focal hypersurface*
2		*hyperplane*
3		*$b-$fold hyperplane*
4	$l=1$	*$b-$fold hyperplane*
	$l=2,\ \mathcal{A}^2=0$	*quadric of rank ≤ 3*
	$l=2,\ \mathcal{A}^2\neq 0$	*$b-$fold hyperplane*
	$l=3$	*double hyperplane + hyperplane*

Proof. Let $e=\lambda^\delta e_\delta \in V \subset X_f$ be a point of the focal variety. Recall from Theorem 6 that the tangent space of X_f at e is the image of

$$de = \lambda^\delta a^i_{\delta j}\omega^j e_i \mod \{e_1,\dots,e_d\}.$$

Thus the dimension of the tangent space, and hence the dimension of X_f, is $d-1+\operatorname{rank}\lambda^\delta A_\delta = d-1+l$. Further, the tangent spaces of X_f will be constant along the smooth points of the vertex V iff nearly all matrices $\lambda^\delta A_\delta$ have the same image, i.e. if $l=b$. Otherwise, the form of the matrix systems in Proposition 2 shows that the tangent spaces will vary in a $b-l$ dimensional family along the vertex V. We see that the expected Gauss rank of X_f is $(r-c)+(b-l)$, but this will not always be the actual Gauss rank.

To determine the actual Gauss rank of X_f, we compute the second fundamental form of X_f at $z=\{e_1\}$. Our strategy is to adjust the frame such that

$$\begin{array}{ll} \{e_1,\dots,e_{d+l}\} & \text{is tangent space of } X_f \text{ at } z \text{ and} \\ \{e_1,\dots,e_{d+b}\} & \text{is span of } V \text{ and the image space of } a^i_{\delta j}\omega^j e_i. \end{array}$$

Then with the index ranges $2\leq \eta\leq d$, $d+1\leq\sigma\leq d+l$, and $d+l+1\leq\tau\leq n$, we have

$$\begin{aligned} de_1 &= \omega_1^\eta e_\eta + \omega_1^\sigma e_\sigma \mod \{e_1\} \\ \mathrm{II}_{X_f,z} = d^2 e_1 &= \omega_1^\eta\omega_\eta^\tau e_\tau + \omega_1^\sigma\omega_\sigma^\tau e_\tau + \omega_1^\sigma\omega_\sigma^\mu e_\mu \\ &= a^\tau_{\eta i}\omega^i\omega_1^\eta e_\tau + \omega_1^\sigma\omega_\sigma^\tau e_\tau + q^\mu_{\sigma i}\omega^i\omega_1^\sigma e_\mu \mod \{e_1,\dots,e_{d+l}\}. \end{aligned}$$

To express ω^i and ω_σ^τ in terms of the $\{\omega_1^1,\dots,\omega_1^{d+l}\}$ forms, we have to use Proposition 2 for the fiber movement system $\mathcal{A}$ and treat several cases separately. But it will always turn out that the Gauss fiber of X_f through z, i.e. the singular locus of $\mathrm{II}_{X_f,z}$, lies in the vertex V.

Case $l=r-1,\ r=2,3,4$.

We prove this case only for $r=4$; the cases $r=2,3$ are analogous and simpler. We adapt our frame such that the linear systems $\mathcal{A}$ and $\mathcal{Q}$ have the form of Proposition 2 and its proof. Further, we assume that A_1 is general, i.e.

$$\mathcal{A}\subseteq\begin{pmatrix} 0 & x & y & t \\ 0 & 0 & z & y \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_1=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{Q}\subseteq\begin{pmatrix} 0 & 0 & 0 & q_1 \\ 0 & 0 & q_1 & q_2 \\ 0 & q_1 & q_2 & q_3 \\ q_1 & q_2 & q_3 & q_4 \end{pmatrix}.$$

This has the effect that $\omega_1^\sigma = \omega^{\sigma+1}$ and $\omega_1^n = 0$. So, we get for the second fundamental form of the focal variety X_f

$$\begin{aligned}\mathrm{I\!I}_{X_f,z} &= \omega_1^\sigma \omega_\sigma^n e_n + q^\mu_{\sigma\sigma'+1}\omega_1^\sigma\omega_1^{\sigma'} e_\mu \\ &= \omega_1^\sigma \omega_\sigma^n e_n + \tilde{q}^\mu_{\sigma\sigma'}\omega_1^\sigma\omega_1^{\sigma'} e_\mu \mod \{e_1,\dots,e_{n-1}\}\end{aligned}$$

where the matrices $\tilde{Q}^\mu$ are such that

$$\tilde{\mathcal{Q}} \subseteq \begin{pmatrix} 0 & 0 & q_1 \\ 0 & q_1 & q_2 \\ q_1 & q_2 & q_3 \end{pmatrix}.$$

Since $0 = \operatorname{singloc}\mathcal{Q} = \operatorname{singloc}\tilde{\mathcal{Q}}$, we can conclude from the normal directions $\{e_{n+1},\dots,e_N\}$ that the singular locus of $\mathrm{I\!I}_{X_f,z}$ is contained in $V = \{e_1,\dots,e_d\}$. On the other hand, because nearly all matrices of the system $\mathcal{A}$ have the same image, the tangent spaces to X_f are constant along the smooth points of V by the argument above. Hence, the Gauss fiber of X_f through z is the whole linear space V.

To determine the structure of the focal hypersurfaces we compute the fiber movement system $\tilde{\mathcal{A}}$ for X_f. From

$$\omega_\eta^\sigma = a^\sigma_{\eta i}\omega^i = a^\sigma_{\eta\sigma'+1}\omega^{\sigma'+1} = a^\sigma_{\eta\sigma'+1}\omega_1^{\sigma'}$$

we see that the system $\tilde{\mathcal{A}}$ is of the type

$$\tilde{\mathcal{A}} \subseteq \begin{pmatrix} x & y & t \\ 0 & z & y \\ 0 & 0 & x \end{pmatrix},$$

and the focal hypersurface will be a double hyperplane plus a hyperplane or possibly a triple hyperplane if we always had $z = x$ in the initial system $\mathcal{A}$.

Case $r = 4,\ l = 2,\ \mathcal{A}^2 = 0$.

This case is analogous to the one above. By Proposition 2 we can adapt the frame such that

$$\mathcal{A} \subseteq \begin{pmatrix} 0 & 0 & x & z \\ 0 & 0 & y & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{Q} \subseteq \begin{pmatrix} 0 & 0 & q_1 & q_2 \\ 0 & 0 & q_2 & q_3 \\ q_1 & q_2 & * & * \\ q_2 & q_3 & * & * \end{pmatrix},$$

thus $\omega_1^\sigma = \omega^{\sigma+2}$ and $\omega_1^\tau = 0$. Then we have for the second fundamental form of X_f

$$\mathrm{I\!I}_{X_f,z} = \omega_1^\sigma\omega_\sigma^\tau e_\tau + \left(q_2^\mu(\omega_1^{d+1})^2 + 2q_1^\mu\omega_1^{d+1}\omega_1^{d+2} + q_2^\mu(\omega_1^{d+2})^2\right) e_\mu \mod \{e_1,\dots,e_{d+2}\}$$

and with the same arguments as in the above case, we see that the Gauss fiber of X_f through z is $V = \{e_1,\dots,e_d\}$.

To compute the fiber movement system $\tilde{\mathcal{A}}$ for X_f we look at

$$\omega_\eta^\sigma = a^\sigma_{\eta i}\omega^i = a^\sigma_{\eta\tau}\omega^\tau = a^\sigma_{\eta\sigma'+2}\omega_1^{\sigma'}$$

and see that $\tilde{\mathcal{A}}$ is of the form

$$\tilde{\mathcal{A}} \subseteq \begin{pmatrix} x & z \\ y & x \end{pmatrix};$$

thus the focal hypersurface is a quadric of rank ≤ 3.

Case $l = 1$, r arbitrary.

This case is different from the two cases above because for $b > l$ the Gauss fiber will be smaller than the vertex V. We will treat here the case where $r = 3$ and $b = 2$; the general case is only notationally more difficult. By Proposition 2 we can adapt the frame such that the matrices A_δ are of the following form

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad A_2 = \ldots = A_{d-1} = 0, \quad A_d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We conclude that $\omega_1^{d+1} = \omega_d^{d+2} = \omega^{d+3}$ and $\omega_\delta^i = 0$ for all other indices. From the proof of Proposition 2 we know $q^\mu_{d+1,d+1} = q^\mu_{d+1,d+2} = 0$. To compute the second fundamental form of X_f we must express ω^τ_{d+1} in terms of $\{\omega_1^1, \ldots, \omega_1^{d+1}\}$. Differentiating $\omega_1^\tau = 0$ gives

$$\begin{aligned}
&0 = d\omega_1^{d+2} = -\omega_d^{d+2} \wedge \omega_1^d - \omega_{d+1}^{d+2} \wedge \omega_1^{d+1} = -\omega_1^{d+1} \wedge \omega_1^d - \omega_{d+1}^{d+2} \wedge \omega_1^{d+1} \\
&0 = d\omega_1^{d+3} = -\omega_{d+1}^{d+3} \wedge \omega_1^{d+1} \\
&\Rightarrow\ \omega_{d+1}^{d+2} = \omega_1^d + f\omega_1^{d+1} \quad \text{and} \quad \omega_{d+1}^{d+3} = g\omega_1^{d+1}.
\end{aligned}$$

Thus the second fundamental form is

$$\mathrm{II}_{X_f,z} = (2\omega_1^d\omega_1^{d+1} + f(\omega_1^{d+1})^2)e_{d+2} + g(\omega_1^{d+1})^2 e_{d+3} + q^\mu_{d+1,d+3}(\omega_1^{d+1})^2 e_\mu \mod \{e_1, \ldots, e_{d+1}\},$$

and we see that its singular locus, and hence the Gauss fiber of X_f through z, is $\{e_1, \ldots, e_{d-1}\}$.

Now we compute the fiber movement system $\tilde{\mathcal{A}}$ and the focal hyperfaces of X_f. We must express ω_ζ^d and ω_ζ^{d+1} for the index range $2 \leq \zeta \leq d-1$ in terms of $\{\omega_1^d, \omega_1^{d+1}\}$. We know $\omega_\zeta^{d+1} = 0$. Differentiating $\omega_\zeta^{d+2} = 0$ we get

$$0 = d\omega_\zeta^{d+2} = -\omega_d^{d+2} \wedge \omega_\zeta^d = -\omega_1^{d+1} \wedge \omega_\zeta^d \ \Rightarrow\ \omega_\zeta^d \sim \omega_1^{d+1}$$

Therefore the fiber movement system $\tilde{\mathcal{A}}$ has the form

$$\tilde{\mathcal{A}} \subseteq \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix},$$

and the focal hypersurfaces are double hyperplanes.

Case $r = 4$, $l = 2$, $\mathcal{A}^2 \neq 0$.

By Proposition 2 the matrix system $\mathcal{A}$ has the form

$$\mathcal{A} \subseteq \begin{pmatrix} 0 & x & y & t \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{with } y = 0 \text{ or } y = z.$$

We have to divide this case into two subcases depending on whether $b = 2$ or $b = 3$. In the case of $b = 2$, we have $y = z = 0$, and the computations are analogous to the case $r = 4$, $l = 2$, $\mathcal{A}^2 = 0$. Thus we will skip this case and assume $b = 3$. Here the tangent spaces to X_f are not constant along the vertex V; so this case is similar to the case $l = 1$, $b \geq 2$. This time we adapt the frame such that with the index range $2 \leq \zeta \leq d-1$

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_\zeta = \begin{pmatrix} 0 & 0 & 0 & t_\zeta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_d = \begin{pmatrix} 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and for the second fundamental form

$$\mathcal{Q} \subseteq \begin{pmatrix} 0 & 0 & 0 & q_1 \\ 0 & q_1 & 0 & q_3 \\ 0 & 0 & q_2 & q_4 \\ q_1 & q_3 & q_4 & q_5 \end{pmatrix}.$$

In particular, we have the following equalities

$$\begin{aligned} &\omega_1^{d+1} = \omega^{d+2},\ \omega_1^{d+2} = \omega^{d+4},\ \omega_1^\tau = 0, \\ &\omega_\zeta^{d+1} = t_\zeta \omega^{d+4} = t_\zeta \omega_1^{d+2},\ \omega_\zeta^{d+2} = \omega_\zeta^\tau = 0, \\ &\omega_d^{d+3} = \omega^{d+4} = \omega_1^{d+2},\ \omega_d^{d+4} = 0. \end{aligned}$$

Therefore, as an intermediate result for the second fundamental form of X_f, we get

$$\begin{aligned} \mathrm{I\!I}_{X_f,z} = {} & \omega_1^d \omega_1^{d+2} e_{d+3} + \omega_1^\sigma \omega_\sigma^\tau e_\tau \\ & + (2 q_1^\mu \omega_1^{d+1} \omega_1^{d+2} + q_3^\mu (\omega_1^{d+2})^2) e_\mu \mod \{e_1, \ldots, e_{d+2}\}. \end{aligned}$$

To compute ω_σ^τ, we differentiate $\omega_1^\tau = 0$ and obtain

$$\begin{aligned} &0 = d\omega_1^{d+3} = -\omega_d^{d+3} \wedge \omega_1^d - \omega_\sigma^{d+3} \wedge \omega_1^\sigma = -\omega_{d+1}^{d+3} \wedge \omega_1^{d+1} - (\omega_{d+2}^{d+3} - \omega_1^d) \wedge \omega_1^{d+2} \\ &0 = d\omega_1^{d+4} = -\omega_\sigma^{d+4} \wedge \omega_1^\sigma \\ &\Rightarrow\ \omega_{d+1}^{d+3} = \omega_{d+2}^{d+3} - \omega_1^d = \omega_\sigma^{d+4} = 0 \mod \{\omega_1^{d+1}, \omega_1^{d+2}\}. \end{aligned}$$

Thus $\{e_1, \ldots, e_{d-1}\}$ lies in the singular locus of $\mathrm{I\!I}_{X_f,z}$ and expressed in the basis $\{\omega_1^d, \omega_1^{d+1}, \omega_1^{d+2}\}$ the second fundamental form is

$$Q^{d+3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & * & * \\ 1 & * & * \end{pmatrix} \quad Q^{d+4} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \quad Q^\mu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q_1^\mu \\ 0 & q_1^\mu & q_3^\mu \end{pmatrix}.$$

Hence $\{e_1, \ldots, e_{d-1}\}$ is the whole singular locus of X_f and the Gauss fiber of X_f through z.

Finally, we compute the fiber movement system $\tilde{\mathcal{A}}$ for X_f, i.e. we need to express $\omega_\zeta^d, \omega_\zeta^{d+1}, \omega_\zeta^{d+2}$ in terms of $\{\omega_1^d, \omega_1^{d+1}, \omega_1^{d+2}\}$. We already know that

$$\omega_\zeta^{d+1} = t_\zeta \omega_1^{d+2} \quad \text{and} \quad \omega_\zeta^{d+2} = 0.$$

Differentiating $\omega_\zeta^{d+3} = 0$, we get

$$0 = d\omega_\zeta^{d+3} = -\omega_d^{d+3} \wedge \omega_\zeta^d - \omega_{d+1}^{d+3} \wedge \omega_\zeta^{d+1} = -\omega_1^{d+2} \wedge \omega_\zeta^d - \omega_{d+1}^{d+3} \wedge t_\zeta \omega_1^{d+2}.$$

Since ω_{d+1}^{d+3} lies in $\{\omega_1^{d+1}, \omega_1^{d+2}\}$, ω_ξ^d must lie in $\{\omega_1^{d+1}, \omega_1^{d+2}\}$, too. Hence, the system $\tilde{\mathcal{A}}$ is of the type

$$\tilde{\mathcal{A}} \subseteq \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix},$$

and the focal hypersurface is a triple hyperplane. □

4 Varieties with a Codimension One Ruling

We have seen in Section 2 how a codimension one uniruling occurs on an affinely developable variety of Gauss rank 2 which is not a cone. Therefore, we examine now varieties with a codimension one uniruling in general. We will show how to construct such a variety with a given Gauss rank; in particular, we will discuss the Sacksteder-Bourgain hypersurface.

Let X be a variety with a codimension one uniruling, i.e. there exists an irreducible curve $B \subset \mathbb{G}(n-1, N)$ with

$$X = \bigcup_{L \in B} L \subset \mathbb{P}^N.$$

Such a variety X will be called a *twisted plane*. We take a desingularisation $\Phi : S \to B$ of B with a Riemannian surface S and recall some facts about curves in the Grassmannians from [9] and [14]. Consider the differential of Φ at a point $s \in S$

$$d_s\Phi : T_sS \longrightarrow T_{\Phi(s)}\mathbb{G}(n-1, N) \cong \mathrm{Hom}(\Phi(s), \mathbb{C}^{N+1}/\Phi(s)).$$

The rank of the linear map $d_s\Phi(v)$ at a general point $s \in S$, which is independent of the choice of the $v \in T_sS \setminus \{0\}$, is the *differential rank* r of Φ. The points $s \in S$ with $\operatorname{rank} d_s\Phi(v) = r$ are called the *regular points* of Φ. From Φ we obtain two maps

$$\begin{aligned} \Phi^{(1)} : \quad & S \longrightarrow \mathbb{G}(n-1+r, N), \quad s \longmapsto I(d_s\Phi(T_sS)) + \Phi(s) \\ \Phi_{(1)} : \quad & S \longrightarrow \mathbb{G}(n-1-r, N), \quad s \longmapsto N(d_s\Phi(T_sS)) \end{aligned}$$

which are a priori defined only on an open subset of S, but can be extended to the whole S. Obviously, this construction can be iterated by setting $\Phi^{(a+1)} := (\Phi^{(a)})^{(1)}$ resp. $\Phi_{(a+1)} := (\Phi_{(a)})_{(1)}$.

Now let us return to our variety X.

Proposition 10 *Let $\Phi : S \to \mathbb{G}(n-1, N)$ be a curve of differential rank $r \geq 1$. If $r = 1$, we assume in addition that $\Phi^{(1)}$ is nonconstant. Define*

$$X := \bigcup_{s \in S} \Phi(s) \quad \textit{and} \quad X_{(1)} := \bigcup_{s \in S} \Phi_{(1)}(s).$$

Then X is of dimension n and Gauss rank r. Its focal variety is $X_{(1)}$. If $x \in X \setminus X_{(1)}$ lies on a unique $\Phi(s)$, where $s \in S$ is a regular point of Φ, then x is a smooth point of X.

Note that the codimension of X is always greater then the differential rank by definition of the differential rank.

Proof. By [14] there exists for a general point $s \in S$ a neighborhood $U \subset S$ and functions $\varphi_1, \ldots, \varphi_n : U \to \mathbb{C}^{N+1}$ such that with $d = n - r$

$$\Phi = \{\varphi_1, \ldots, \varphi_n\}, \quad \Phi^{(1)} = \{\varphi_1, \ldots, \varphi_n, \varphi'_{d+1}, \ldots, \varphi'_n\},$$
$$\Phi_{(1)} = \{\varphi_1, \ldots, \varphi_d\}, \ \textit{i.e.}\ \varphi'_1, \ldots, \varphi'_d \in \Phi \text{ on } U.$$

The affine cone of the variety X is locally the image of

$$\Psi : U \times \mathbb{C}^n \longrightarrow \hat{X}, \quad (s, \mu) \longmapsto \sum_{i=1}^{n} \mu^i \varphi_i \ ;$$

hence, the tangent space of X at a smooth point $x = \{\Psi(s, \mu)\}$ is

$$\left\{ \sum_{i=1}^{n} \mu^i \varphi'_i, \varphi_1, \ldots, \varphi_n \right\} = \left\{ \sum_{i=d+1}^{n} \mu^i \varphi'_i, \varphi_1, \ldots, \varphi_n \right\}. \tag{$*$}$$

We find that the tangent spaces of X are constant on the smooth points of the linear space

$$L = \{\Psi(s, \mathbb{C}^d \times \mathbb{C}(\mu^{d+1}, \ldots, \mu^n))\} = \{\Psi, \varphi_1, \ldots, \varphi_d\}$$

and that L is the largest linear space with this property using that $\Phi^{(1)}$ is nonconstant for $r = 1$. Therefore, the Gauss rank of X is r.

The linear space in $(*)$, which is the candidate for the tangent space of X in x, will be of dimension less than $n+1$ iff $\mu^{d+1} = \ldots = \mu^n = 0$. Therefore, the focal hypersurface on L is given by $\{\Psi(s, \mathbb{C}^d \times 0)\} = \Phi_{(1)}(s)$ and $x \in \Phi(s) \setminus \Phi_{(1)}(s)$ is a smooth point of x if x lies on a unique $\Phi(s)$. $\square$

To examine X further, we determine the normal form of the curve Φ. We recall the necessary definitions.

For a curve $\varphi : S \to \mathbb{P}^N$, we consider the curves $\varphi^{(a)}$. If φ is locally given by $\tilde{\varphi} : U \to \mathbb{C}^{N+1}$, i.e. $\varphi = \{\tilde{\varphi}\}$, then $\varphi^{(a)}$ is given by $\{\tilde{\varphi}, \tilde{\varphi}', \ldots, \tilde{\varphi}^{(a)}\}$. We expect that $\varphi^{(a)}$ is a curve inside $\mathbb{G}(a, N)$. If that is the case, $\varphi^{(a)}$ is called the *osculating curve* of φ.

Given two curves $\varphi : S \to \mathbb{G}(k, N)$ and $\psi : S \to \mathbb{G}(l, N)$ with $\varphi(s) \cap \psi(s) = \emptyset$ for the points s of an open set $U \subseteq S$, their direct sum $\varphi \oplus \psi : S \to \mathbb{G}(k+l-1, N)$ is defined by $\varphi \oplus \psi(s) := \varphi(s) \oplus \psi(s)$ for $s \in U$ and extended to the whole S.

Now the normal form of the curve Φ expresses Φ as the sum of r osculating curves and a linear space L, i.e.

$$\Phi = \varphi_1^{(a_1)} \oplus \varphi_2^{(a_2)} \oplus \ldots \oplus \varphi_r^{(a_r)} \oplus L \quad \text{with} \ \sum_{j=1}^{r} a_j + \dim L = n - 1 - r;$$

thereby the linear space L is unique. If $L = \emptyset$ and a_r is the unique maximum of the a_i, then φ_r is also unique.

Corollary 11 *Let $X \subset \mathbb{P}^N$ be an affinely smooth twisted plane. If the Gauss rank of X is r, then the desingularisation of the ruling $\Phi : S \to \mathbb{G}(n-1, N)$ has a normal form*

$$\Phi = \varphi_1 \oplus \varphi_2^{(a_2)} \oplus \ldots \oplus \varphi_r^{(a_r)} \oplus L \quad \textit{with} \sum_{j=2}^{r} a_j + \dim L = n - 1 - r,$$

where L and the φ_i with $a_i > 0$ lie in H_∞.

In particular, if X is of Gauss rank 2 and not a cone, this means

$$\Phi = \psi \oplus \varphi^{(n-2)}.$$

Hence, X is the union of the joins of the osculating spaces of a curve in H_∞ to the corresponding points of another curve. The uniquely determined curve φ in H_∞ is called the critical curve *of X. Thus X is the union of $(n-1)$-planes that contain the $(n-2)$-th osculating planes of the critical curve.*

Note that the particular case of Gauss rank 2 strengthens Vitter's Theorem by restricting the possible one-dimensional families of $(n-1)$-planes whose union yield affinely smooth varieties of Gauss rank 2.

Proof. This is an immediate consequence of the Proposition and the fact that $\Phi_{(1)}$ of the curve Φ in the above normal form is

$$\Phi_{(1)} = \varphi_2^{(a_2-1)} \oplus \ldots \oplus \varphi_r^{(a_r-1)} \oplus L,$$

where we use the convention $\varphi^{(-1)} = \emptyset$. □

The Corollary also shows how one can attempt to construct affinely smooth twisted planes. Choose numbers $a_2, \ldots, a_r$ and a linear space $L \subset H_\infty$ with $\sum a_j + \dim L = n-1 \leq N-r$. Next choose general curves $\varphi_1, \ldots, \varphi_r$ in $\mathbb{P}^N$ under the restriction that $\varphi_i \subset H_\infty$ if $a_i > 0$. Then $\Phi = \varphi_1 \oplus \varphi_2^{(a_2)} \oplus \ldots \oplus \varphi_r^{(a_r)} \oplus L$ will be of differential rank r, and the focal variety of $X = \bigcup_{s \in S} \Phi(s)$ will lie in H_∞. To ensure that X is affinely smooth, the curve must not have points $s, t \in S$, $s \neq t$, such that $\Phi(s)$ and $\Phi(t)$ intersect outside H_∞, because $\Phi(s) \cap \Phi(t)$ will be singular points of selfintersection of X. In addition, X might have singular points due to nonregular points $s \in S$ of Φ with $\Phi(s) \not\subset H_\infty$.

The easiest way to ensure all this is to choose linear spaces $\mathbb{P}^{b_1}, \ldots, \mathbb{P}^{b_r}, L$ in $\mathbb{P}^N$ with

$$\mathbb{P}^N = \mathbb{P}^{b_1} \oplus \ldots \oplus \mathbb{P}^{b_r} \oplus L$$

and take the φ_i as rational normal curves in $\mathbb{P}^{b_i}$. Then $\Phi(s) \cap \Phi(t) = L$ for $s \neq t$, and since the osculating curves of rational normal curves have only regular points, the same holds for Φ. Further, the focal variety of the resulting X will lie in $\mathbb{P}^{b_2} \oplus \ldots \oplus \mathbb{P}^{b_r} \oplus L$.

For a concrete example let

$$\begin{aligned}
\psi : \mathbb{P}^1 &\longrightarrow \mathbb{P}^4, \qquad (s_0 : s_1) \longmapsto (s_0 : 0 : 0 : 0 : s_1) \\
\varphi : \mathbb{P}^1 &\longrightarrow \mathbb{P}^4, \qquad (s_0 : s_1) \longmapsto (0 : s_0^2 : -2s_0 s_1 : s_1^2 : 0),
\end{aligned}$$

and $\Phi = \psi \oplus \varphi^{(1)}$. We obtain an affinely smooth hypersurface X of Gauss rank 2. Its Gauss fiber cones $\Phi(s)$ are plane pencils of lines through $\varphi(s)$. The locus of the centers is the critical conic φ. This hypersurface is in fact the Sacksteder-Bourgain hypersurface mentioned in the introduction, which we now recognize as the union of the joins of points of a line to the tangent lines of the conic at corresponding points. For more details on this example see [4, 15, 17].

We close this section with a statement about the singularities of affinely smooth developable hyperfaces of Gauss rank 2.

Proposition 12 *Let $X \subset \mathbb{P}^N$ be an affinely smooth hypersurface with a codimension one ruling. Assume that X is not a cone. Then X has Gauss rank 2, and its critical curve is contained in a $(N-2)$-dimensional linear space $L \subset H_\infty$. Further, the linear space L is a component of the singular locus of X.*

Proof. By Corollary 5, the desingularisation $\Phi : S \to \mathbb{G}(N-2, n)$ of the ruling base B has the normal form $\Phi = \psi \oplus \varphi^{(N-3)}$, where the critical curve φ lies in H_∞, but ψ does not. For dimension reasons any two $(N-2)$-planes $\Phi(s)$ and $\Phi(t)$ intersect in a linear space of dimension $\geq N-4$. Their intersections will be points of selfintersection of X, and therefore singular points of X. Thus for $s \neq t$, the linear space $\Phi(s) \cap \Phi(s)$ lies in the hyperplane H_∞ and

$$\Phi(s) \cap \Phi(s) \cap H_\infty = \varphi^{(N-3)}(s) \cap \varphi^{(N-3)}(t) \quad \text{for } s \neq t \text{ with } \Phi(s), \Phi(t) \not\subset H_\infty.$$

Therefore, the $\varphi^{(N-3)}(s)$ intersect each other in codimension one. Three general of these spaces cannot contain a common $(N-4)$-plane, since otherwise all would contain this plane and X would be a cone. Hence, fixing two general $t_1, t_2 \in S$, we have

$$\varphi^{(N-3)}(s) = \left(\varphi^{(N-3)}(s) \cap \varphi^{(N-3)}(t_1)\right) + \left(\varphi^{(N-3)}(s) \cap \varphi^{(N-3)}(t_2)\right)$$

for general $s \in S$. Thus all the $\varphi^{(N-3)}(s)$ and the whole curve φ lie in the $(N-2)$-plane $L = \varphi^{(N-3)}(t_1) + \varphi^{(N-3)}(t_2) \subset H_\infty$. Counting dimensions, we get

$$L = \overline{\bigcup_{\substack{s,t \in S \\ s \neq t}} \varphi^{(N-3)}(s) \cap \varphi^{(N-3)}(t)}.$$

Further, the points of L are points of selfintersection of X, thus $L \subseteq \operatorname{Sing} X$. In fact, it must be a component of $\operatorname{Sing} X$ because $\dim L = \dim X - 1$. □

A Proof of Proposition 2

The proof is based on the Proposition [18, 2. Proposition 2] and its proof. There it was shown that $\ker \mathcal{A} \neq 0$, here we give here only the additionally needed arguments.

The case of $l = r-1$ and $r = 2, 3, 4$ was treated there completely. Besides the structure of the matrix system $\mathcal{A}$, it was shown there that the matrices of $\mathcal{Q}$ are of the form

$$Q = \begin{pmatrix} 0 & q_1 \\ q_1 & q_2 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & q_1 \\ 0 & q_1 & q_2 \\ q_1 & q_2 & q_3 \end{pmatrix} \quad \text{resp.} \quad Q = \begin{pmatrix} 0 & 0 & 0 & q_1 \\ 0 & 0 & q_1 & q_2 \\ 0 & q_1 & q_2 & q_3 \\ q_1 & q_2 & q_3 & q_4 \end{pmatrix}.$$

Case $l = 1$, r arbitrary.

For notational convenience, we assume $r = 4$, the other cases being similar. Let $A_1 \in \mathcal{A} \setminus \{0\}$. Because of $\operatorname{rank} A_1 = 1$, there exists a basis of $\mathbb{C}^r$ such that

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since all matrices of $\mathcal{A} \setminus \{0\}$ have rank 1 and are nilpotent, $\mathcal{A}$ must be contained in

$$\begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Assume there exists an $A_2 \in \mathcal{A} \setminus \mathbb{C} A_1$. With a coordinate change in the first three coordinates and subtracting a multiple of A_1, the matrix A_2 can be brought into the form

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{resp.} \quad A_2' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now we take an arbitrary $Q = (q_{ij}) \in \mathcal{Q}$ with $q_{ij} = q_{ji}$. Then the symmetry of

$$QA_1 = \begin{pmatrix} 0 & 0 & 0 & q_{11} \\ 0 & 0 & 0 & q_{12} \\ 0 & 0 & 0 & q_{13} \\ 0 & 0 & 0 & q_{14} \end{pmatrix}$$

and

$$QA_2 = \begin{pmatrix} 0 & 0 & 0 & q_{12} \\ 0 & 0 & 0 & q_{22} \\ 0 & 0 & 0 & q_{23} \\ 0 & 0 & 0 & q_{24} \end{pmatrix} \quad \text{resp.} \quad QA_2' = \begin{pmatrix} 0 & 0 & q_{11} & 0 \\ 0 & 0 & q_{12} & 0 \\ 0 & 0 & q_{13} & 0 \\ 0 & 0 & q_{14} & 0 \end{pmatrix}$$

implies

$$Q = \begin{pmatrix} 0 & 0 & 0 & q_{14} \\ 0 & 0 & 0 & q_{24} \\ 0 & 0 & q_{33} & q_{34} \\ q_{14} & q_{24} & q_{34} & q_{44} \end{pmatrix} \quad \text{resp.} \quad Q' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q_{22} & q_{23} & q_{24} \\ 0 & q_{23} & q_{33} & q_{34} \\ 0 & q_{24} & q_{34} & q_{44} \end{pmatrix}.$$

Since Q was an arbitrary matrix of $\mathcal{Q}$, the second case is impossible in view of condition 3. Thus the linear system $\mathcal{A}$ is always of the first type. In that case condition 3 implies that $\dim \mathcal{Q} \geq 2$ if $\dim \mathcal{A} \geq 2$.

Finally, if $\dim \mathcal{A} = 3$, by the same arguments as above the matrix

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

exists in $\mathcal{A}$. The symmetry of QA_3 implies

$$Q = \begin{pmatrix} 0 & 0 & 0 & q_{14} \\ 0 & 0 & 0 & q_{24} \\ 0 & 0 & 0 & q_{34} \\ q_{14} & q_{24} & q_{34} & q_{44} \end{pmatrix},$$

and we must have $\dim \mathcal{Q} \geq 3$, in order not to violate condition 3.

Case $r = 4$, $l = 2$, subcase $A^2 = 0$ for all $A \in \mathcal{A}$.

Let $A_1 \in \mathcal{A}$ be a matrix of rank 2. By the results of Wu and Zheng, there exists a basis of $\mathbb{C}^4$ such that

$$\mathcal{A} \subseteq \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From QA_1 symmetric for a $Q \in \mathcal{Q}$, we conclude that Q is of the form

$$Q = \begin{pmatrix} 0 & 0 & q_{13} & q_{14} \\ 0 & 0 & q_{14} & q_{24} \\ q_{13} & q_{14} & q_{33} & q_{34} \\ q_{14} & q_{24} & q_{34} & q_{44} \end{pmatrix} = \begin{pmatrix} 0 & Q_{12} \\ Q_{12} & Q_{22} \end{pmatrix}$$

with symmetric 2×2-matrix Q_{12}.

We claim that this implies the existence of an $Q \in \mathcal{Q}$ with full rank. Assume the contrary. Then the Q_{12} part of every $Q \in \mathcal{Q}$ is singular. The singular matrices form a quadric cone inside the space of symmetric matrices $\mathrm{Sym}(2, \mathbb{C}) \cong \mathbb{C}^3$, given by the determinant. The largest linear spaces in this cone are the ruling lines. This implies that all Q_{12} parts of the $Q \in \mathcal{Q}$ are linearly dependent; hence, they will have a common kernel. This common kernel gives rise to a common kernel of the matrices $Q \in \mathcal{Q}$ itself, contradicting condition 3.

Now let $Q \in \mathcal{Q}$ be a fixed matrix of full rank. A coordinate change by

$$\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$$

leaves A_1 and the above general form of the linear system fix. Further, it transforms Q to

$$\begin{pmatrix} 0 & {}^tTQ_{12}T \\ {}^tTQ_{12}T & {}^tTQ_{22}T \end{pmatrix};$$

hence, there is a coordinate change that transforms Q to

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q_{33} & q_{34} \\ 1 & 0 & q_{34} & q_{44} \end{pmatrix}.$$

Now we take any

$$A = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{A},$$

then the symmetry of QA implies $a = d$.

Case $r = 4$, $l = 2$, subcase $A^2 \neq 0$ for a general $A \in \mathcal{A}$.

By [18, 2. Proposition 2] we know that $\ker \mathcal{A} \neq 0$, we choose coordinates such that the first coordinate is contained in $\ker \mathcal{A}$. Pick a general $A_1 \in \mathcal{A}$. Then $\operatorname{Im} A_1 \cap \ker A_1$ is a one-dimensional space. If $\operatorname{Im} A_1 \cap \ker A_1$ is contained in $\ker \mathcal{A}$, then by a coordinate change A_1 can be brought into the form

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{otherwise into} \quad A_1' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We start with the first case. The statement $(*)$ of the proof of [18, 2. Proposition 2] says that

$$A(\ker A_1) \subseteq \operatorname{Im} A_1 \quad \text{for all } A \in \mathcal{A}.$$

Hence an arbitrary $A \in \mathcal{A}$ has the form

$$A = \begin{pmatrix} 0 & a & b & c \\ 0 & d & e & f \\ 0 & g & 0 & h \\ 0 & i & 0 & j \end{pmatrix}.$$

On the other hand, since QA_1 is symmetric for $Q \in \mathcal{Q}$, Q looks like

$$Q = \begin{pmatrix} 0 & 0 & 0 & q_1 \\ 0 & q_1 & 0 & q_3 \\ 0 & 0 & q_2 & q_4 \\ q_1 & q_3 & q_4 & q_5 \end{pmatrix}.$$

Because of condition 3 there exists an $Q \in \mathcal{Q}$ with $q_1 \neq 0$. Now the symmetry of

$$QA = \begin{pmatrix} 0 & q_1 i & 0 & q_1 j \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}$$

implies $i = j = 0$, thus

$$A = \begin{pmatrix} 0 & a & b & c \\ 0 & d & e & f \\ 0 & g & 0 & h \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial of A is $x^4 + dx^3 - egx^2$. Since A is nilpotent, this implies $d = 0$ and $eg = 0$. At this point the symmetry of QA is equivalent to

$$0 = q_1 e - q_2 g = q_1(f - a) - q_4 g = q_1 b + q_3 e - q_2 h. \qquad (*)$$

If $\mathcal{Q}$ contains a matrix of full rank, i.e. with $q_1 q_2 \neq 0$, then this together with $eg = 0$ shows $e = g = 0$, $f = a$, and $q_1 b = q_2 h$. By a scaling of the third coordinate, we can achieve that $q_1 = q_2$. Assuming that this was the case from the beginning, we get $b = h$, thus A looks like

$$A = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $\mathcal{Q}$ does not contain a matrix of full rank, then $q_2 = 0$ for all $Q \in \mathcal{Q}$, because $\mathcal{Q}$ is a linear system and there exists a matrix with $q_1 \neq 0$. Hence, a $Q \in \mathcal{Q}$ contains the vector $(-q_4\ 0\ q_1\ 0)$ in its kernel. In order to fulfill condition 3 there must be two $Q \in \mathcal{Q}$ with linear independent $(-q_4\ 0\ q_1\ 0)$. Then we conclude from $(*)$ that $e = g = b = 0$ and $a = f$, thus A looks like

$$A = \begin{pmatrix} 0 & a & 0 & c \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We turn to the second case with the matrix A_1'. Here the statement $(*)$ of the proof of [18, 2. Proposition 2] implies that all $A \in \mathcal{A}$ can be written as

$$A = \begin{pmatrix} 0 & 0 & a & b \\ 0 & c & d & e \\ 0 & f & g & h \\ 0 & 0 & i & j \end{pmatrix}.$$

Since $A + tA_1'$ is nilpotent, its characteristic polynomial is x^4 for all t. This yields the following equations

$$\begin{aligned} &f + i = 0,\ c + g + j = 0,\ ci + fj = 0, \\ &cg - fd + gj + cj - hi = 0,\ dfj - efi + chi - cgj = 0. \end{aligned} \qquad (**)$$

From the first row we get $i = -f$, $c = -g - j$, and $f(g + 2j) = 0$. Assume that $f \neq 0$ then $g = -2j$. We neglect the remaining equations and continue with

$$A = \begin{pmatrix} 0 & 0 & a & b \\ 0 & j & d & e \\ 0 & f & -2j & h \\ 0 & 0 & -f & j \end{pmatrix}.$$

We start using condition 2. From the symmetry of QA_1' we conclude

$$Q = \begin{pmatrix} q_1 & 0 & 0 & q_2 \\ 0 & 0 & 0 & q_3 \\ 0 & 0 & q_3 & q_4 \\ q_2 & q_3 & q_4 & q_5 \end{pmatrix}.$$

Now the symmetry of QA implies $q_3 f = 0$, thus $q_3 = 0$. But this contradicts condition 3; hence, matrices with $f \neq 0$ do not exit in the linear system $\mathcal{A}$.

For $f = 0$ the remaining equations of $(**)$ imply $g = j = 0$, hence

$$A = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & d & e \\ 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and we are back in the above case, since $\operatorname{Im} A_1' \cap \ker A_1' \subset \ker \mathcal{A}$. □

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