A normal form for curves in Grassmannians

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The purpose of this paper is to give an elementary proof of Griffiths' and Harris' normal form theorem [4, p.385].

1 Introduction

Our topic is a variety V in $\mathbb{P}_{N-1}$ which is the trace of an $(n-1)$-dimensional linear subspace moving with one complex parameter. More precisely, we consider a curve in a Grassmannian, i.e. a holomorphic mapping $\Phi : S \to \mathbb{G}(n-1, N-1)$, where S is a Riemann surface and $\mathbb{G}(n-1, N-1)$ denotes the Grassmannian, the set of $(n-1)$-planes in $\mathbb{P}_{N-1}$. If V is not linear, then one can get such a map Φ as the desingularisation of the Fano variety $\mathbb{F}_{n-1}(V)$ of V, that is the variety in $\mathbb{G}(n-1, N-1)$ consisting of the $(n-1)$-planes contained in V. For technical reasons we prefer to view Φ as a map $\Phi : S \to \mathrm{G}(n, N)$ into the Grassmannian $\mathrm{G}(n, N)$, the set of n-dimensional subspaces in $\mathbb{C}^N$.

The structure theorem we want to prove is

Theorem. *Let $\Phi : S \to \mathrm{G}(n, N)$ be a curve, then there exists a unique $r \in \mathbb{N}$, as well as unique $a_1 \geq a_2 \geq \ldots \geq a_r > 0$ and a unique linear subspace $V \subseteq \mathbb{C}^N$ with $\sum_{i=1}^{r} a_i + \dim V = n$ and (in general not unique) curves $\varphi_1, \ldots, \varphi_r : S \to \mathrm{G}(1, N)$ such that*

$$\Phi = \varphi_1^{(a_1-1)} \oplus \ldots \oplus \varphi_r^{(a_r-1)} \oplus V$$

and

$$\Phi^{(1)} = \varphi_1^{(a_1)} \oplus \ldots \oplus \varphi_r^{(a_r)} \oplus V.$$

Hereby, $\varphi_i^{(a_i-1)}$, resp. $\varphi_i^{(a_i)}$, denotes the (a_i-1)–th, resp. (a_i)–th, osculating curve of $\varphi_i : S \to \mathrm{G}(1, N) = \mathbb{P}_{N-1}$ and $\Phi^{(1)}$ is the natural generalisation of the first osculating curve to the case of a curve in $\mathrm{G}(n, N)$, where n is arbitrary.

Applying the theorem to the classical case of ruled surfaces in $\mathbb{P}_3$, we obtain that the developable surfaces (that is the case $r = 1$) are either tangent surfaces or cones. This was already proved in [1] and [3].

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2 Addition and decomposition of curves

The main tool in this examination and the reason why we work with a *one* dimensional complex manifold S is the following ([5, p.263])

Lemma 1 *If $0 \neq \widetilde{\Psi} = (\psi_1, \dots, \psi_N) : U \subset \mathbb{C} \to \mathbb{C}^N$, U connected, is a holomorphic map, then there is a unique continuation Ψ of $\mathbb{P}(\widetilde{\Psi}) : U \setminus \{z \in U \mid \widetilde{\Psi}(z) = 0\} \to \mathbb{P}_{N-1}$ to $\Psi : U \to \mathbb{P}_{N-1}$.*

Now it is easy to define the *direct sum* $\Phi \oplus \Psi$ of two curves $\Phi : S \to \mathrm{G}(n, N)$ and $\Psi : S \to \mathrm{G}(m, N)$ for which there exists a point $t \in S$ with $\Phi(t) \cap \Psi(t) = 0$, so that $(\Phi \oplus \Psi)(s) = \Phi(s) \oplus \Psi(s)$ for $s \in S$ up to isolated points.

We think of the Grassmannian $\mathrm{G}(n, N)$ as a submanifold of the projective space $\mathbb{P}(\bigwedge^n \mathbb{C}^N)$ by the Plücker-embedding $V = span\{v_1, \dots, v_n\} \mapsto \mathbb{P}(v_1 \wedge \dots \wedge v_n)$. So for any point of S we can take local liftings of Φ and Ψ, i.e. curves $\widetilde{\Phi} : U \to \bigwedge^n \mathbb{C}^N \setminus \{0\}$ and $\widetilde{\Psi} : U \to \bigwedge^m \mathbb{C}^N \setminus \{0\}$ with $\mathbb{P}(\widetilde{\Phi}) = \Phi$, resp. $\mathbb{P}(\widetilde{\Psi}) = \Psi$, and define $\Phi \oplus \Psi$ on U to be the continuation of $\mathbb{P}(\widetilde{\Phi} \wedge \widetilde{\Psi})$. A short calculation shows that these local definitions are the same in their overlapping areas, so we get the desired global curve $\Phi \oplus \Psi : S \to \mathrm{G}(n + m, N)$.

It is also possible to define the sum $\Phi + \Psi$ and the intersection $\Phi \cap \Psi$ of two curves $\Phi : S \to \mathrm{G}(n, N)$ and $\Psi : S \to \mathrm{G}(m, N)$, such that up to isolated points we have $(\Phi + \Psi)(s) = \Phi(s) + \Psi(s)$ and $(\Phi \cap \Psi)(s) = \Phi(s) \cap \Psi(s)$.

Introducing the notation $\dim \Phi = n$ for $\Phi : S \to \mathrm{G}(n, N)$, we have

Remark 2 $\dim(\Phi \cap \Psi) + \dim(\Phi + \Psi) = \dim \Phi + \dim \Psi$.

Using the well-known holomorphic duality $\mathcal{D}$ between the Grassmannians $\mathrm{G}(n, N)$ and $\mathrm{G}(N - n, N)$:

$$\begin{array}{rccc} \mathcal{D}: & \mathrm{G}(n, N) & \to & \mathrm{G}(N - n, N) \\ & V & \mapsto & \{w \in \mathbb{C}^N \mid \forall v \in V : \; w^{\mathrm{T}} \cdot v = 0\}, \end{array}$$

we see that both constructions are connected similar to the sum and intersection of ordinary subspaces of $\mathbb{C}^N$

$$\Phi \cap \Psi = \mathcal{D}(\mathcal{D}\Phi + \mathcal{D}\Psi) \quad \textit{and} \quad \Phi + \Psi = \mathcal{D}(\mathcal{D}\Phi \cap \mathcal{D}\Psi).$$

Likewise, it is possible to decompose a curve into the sum of smaller ones.

Proposition 3 *Given $\Phi : S \to \mathrm{G}(n, N)$ and $\Psi : S \to \mathrm{G}(m, N)$ with $\Psi \subseteq \Phi$ (i.e. $\forall s \in S : \Psi(s) \subseteq \Phi(s)$), then there exist $\varphi_1, \dots, \varphi_{m-n} : S \to \mathrm{G}(1, N)$, such that $\Phi = \Psi \oplus \varphi_1 \oplus \dots \oplus \varphi_{m-n}$.*

Proof. First choose $t \in S$ and vectors $v_1, \dots, v_{n-m} \in \mathbb{C}^N$ such that $\Phi(t) = span\{\Psi(t), v_1, \dots, v_{n-m}\}$. Now choose $(N - n + 1)$-dimensional subspaces V_i with $V_i \cap \Phi(t) = \mathbb{C} \cdot v_i$, finally define $\varphi_i := \Phi \cap V_i := \mathcal{D}(\mathcal{D}\Phi \oplus \mathcal{D}V_i) \subseteq \Phi$. Then we have $\varphi_1 \oplus \dots \oplus \varphi_{n-m} \oplus \Psi \subseteq \Phi$ and comparing dimensions we see, that both sides must be equal. □

3 The curves $\Phi^{(1)}$ and $\Phi_{(1)}$

Next we want to study the infinitesimal behavior of a curve Φ, which we think of as a moving n–plane. We make the following auxiliary

Definition 4 *A moving point p in Φ near $t \in S$ is a holomorphic mapping $p : U \to \mathbb{C}^N \setminus \{0\}$ defined in a neighbourhood U of S so that $p(s) \in \Phi(s)$ for all $s \in U$.*

As a measure of the movement of Φ we define a new curve, $\Phi^{(1)}$.

Definition of $\Phi^{(1)}$.
An illustrative description of $\Phi^{(1)}$ is given by

$$(\Phi^{(1)})(s) = \{p'(s) \mid p \textit{ a moving point of } \Phi \textit{ near } s\} \supseteq \Phi(s),$$

where p' denotes the derivative, as usual. Unfortunately, this description is only valid up to isolated points, so we choose a different approach, which also shows that $\Phi^{(1)}$ is holomorphic.

First we define $\Phi^{(1)}$ locally. For any point of S choose a neighbourhood U and n moving points $p_1, \ldots, p_n$ on it, such that $\Phi(s) = span\{p_1(s), \ldots, p_n(s)\}$ on U. Let $V_s := span\{p_1(s), \ldots, p_n(s), p'_1(s), \ldots, p'_n(s)\}$ and $r := \max_{s \in U} \dim V_s - n$ and finally $\widetilde{s} \in U$ such that $\dim V_{\widetilde{s}} = n + r$. After renumbering we can assume $V_{\widetilde{s}} = span\{p_1(\widetilde{s}), \ldots, p_n(\widetilde{s}), p'_1(\widetilde{s}), \ldots, p'_r(\widetilde{s})\}$; then we define

$$(\Phi^{(1)})(s) = \mathbb{P}(p_1(s) \wedge \ldots \wedge p_n(s) \wedge p'_1(s) \wedge \ldots \wedge p'_r(s))$$

on U, where we again continue into the exceptional set $X := \{s \in U \mid p_1(s) \wedge \ldots \wedge p_n(s) \wedge p'_1(s) \wedge \ldots \wedge p'_r(s)) = 0\}$.

In order to show that these local pieces of $\Phi^{(1)}$ are the same at the intersections, we simply show that the new definition agrees with the old one on $U \setminus X$, which was free of any choices. Therefore we can claim that for $t \in U \setminus X$ is

$$(\Phi^{(1)})(t) = \{p'(t) \mid p \textit{ a moving point of } \Phi \textit{ near } t\}.$$

For the "$\subseteq$ " inclusion we note that $p_i(t)$ and $q_i(s) := (s - t)p_i(s)$ are moving points of Φ, and $q'_i(t) = (t - t)p'_i(t) + p_i(t) = p_i(t)$. For the opposite inclusion we have $p \in \Phi = span\{p_1, \ldots, p_n\}$, so we can find holomorphic functions α_i, such that $p = \sum_{i=1}^{n} \alpha_i p_i$. It follows that

$$p'(t) = \sum \alpha_i(t)p'_i(t) + \sum \alpha'_i(t)p_i(t),$$

i.e. $p'(t) \in span\{p_1(t), \ldots, p_n(t), p'_1(t), \ldots, p'_n(t)\} = span\{p_1(t), \ldots p_n(t), p'_1(t), \ldots, p'_r(t)\}$. The last two terms are equal, because $t \notin X$.

Lemma 5 $\Phi^{(1)} = \Phi \iff \Phi$ *constant.*

Proof. This is a reformulation of lemma 1 in [2]. □

We define $\Phi^{(0)} := \Phi$ and $\Phi^{(k+1)} := \left(\Phi^{(k)}\right)^{(1)}$ for $k \geq 0$.

Let us apply these constructions to the lowest dimensional curves $\varphi : S \to \mathrm{G}(1, N) = \mathbb{P}_{N-1}$. Locally we have $\varphi^{(k)} = span\{p, p', p'', \ldots, p^{(k)}\}$, where p is a moving point of φ. If $\dim \varphi^{(k)} = k+1$, then these curves are called osculating curves and $\varphi^{(1)}(s)$ is the tangent to $\varphi : S \to \mathbb{P}_{N-1}$ at $\varphi(s)$, $\varphi^{(2)}(s)$ is the osculating plane and so on.

Now we come to the next construction, $\Phi_{(1)}$, which consists of the traces of moving points of Φ, for which p' is also a moving point of Φ. This might have less geometrical interpretations, but it is important, because it sometimes reverses the previous construction, e.g. $\left(\varphi^{(1)}\right)_{(1)} = \varphi$.

Definition of $\Phi_{(1)}$.
Again there is an illustrative description of $\Phi_{(1)}$

$$(\Phi_{(1)})(s) = \{p(s) \mid p \textit{ a moving point of } \Phi \textit{ near } s \textit{ with } p'(s) \in \Phi(s)\} \subseteq \Phi(s),$$

which is only valid up to isolated points. So let us take another approach.

Lemma 6 *Let $\Phi : S \to \mathrm{G}(n,N)$, $\widetilde{s} \in S$ and $r := \dim(\Phi^{(1)}) - \dim \Phi$, then there exists a neighbourhood U of $\widetilde{s}$ and moving points $p_1, \ldots, p_n$ of Φ on U, such that*

1. $\Phi = span\{p_1, \ldots, p_n\}$ *on* $U \setminus \{\widetilde{s}\}$
2. $p'_{r+1}, \ldots, p'_n \in \Phi$
3. $\Phi^{(1)} = span\{p_1, \ldots, p_n, p'_1, \ldots, p'_r\}$ *on* $U \setminus \{\widetilde{s}\}$.
   In particular $p_1, \ldots, p_n, p'_1, \ldots, p'_r$ *are linear independent on* $U \setminus \{\widetilde{s}\}$.

Proof. Looking at the definition of $\Phi^{(1)}$ we can assume that $\Phi = span\{q_1, \ldots, q_n\}$ and $\Phi^{(1)} = span\{q_1, \ldots, q_n, q'_1, \ldots, q'_r\}$ on $U \setminus \{\widetilde{s}\}$.
We define $p_i := q_i$ for $i = 1, \ldots, r$ and for $i = r+1, \ldots, n$ in the following way:
Because $q'_i \in span\{q_1, \ldots, q_n, q'_1, \ldots, q'_r\}$ on $U \setminus \{\widetilde{s}\}$ there are holomorphic functions $\alpha_i^1, \ldots, \alpha_i^n, \beta_i^1, \ldots, \beta_i^r, \gamma_i$, $\gamma_i(s) \neq 0$ for $s \in U \setminus \{\widetilde{s}\}$ (shrink U, if necessary), such that

$$\sum_{j=1}^{n} \alpha_i^j q_j + \sum_{j=1}^{r} \beta_i^j q'_j + \gamma_i q'_i = 0.$$

Define $p_i := \sum_{j=1}^{r} \beta_i^j q_j + \gamma_i q_i$, then we have

$$\begin{aligned} p'_i &= \sum_{j=1}^{r} \left(\beta_i^{j\,\prime} q_j + \beta_i^j q'_j \right) + \gamma'_i q_i + \gamma_i q'_i \\ &= \left(\sum_{j=1}^{r} \beta_i^j q'_j + \gamma_i q'_i \right) + \sum_{j=1}^{r} \beta_i^{j\,\prime} q_j + \gamma'_i q_i \\ &= -\sum_{j=1}^{n} \alpha_i^j q_j + \sum_{j=1}^{r} \beta_i^{j\,\prime} q_j + \gamma'_i q_i \in \Phi. \end{aligned}$$

Because $\gamma_i \neq 0$ on $U \setminus \{\widetilde{s}\}$, 1. and 3. also follow. □

Now we define $\Phi_{(1)}$ locally to be the continuation of $\mathbb{P}(p_{r+1} \wedge \ldots \wedge p_n)$. In order to show that these pieces of $\Phi_{(1)}$ patch together, we show that the new and the old descriptions are the same on on $U \setminus \{\widetilde{s}\}$

$$(\Phi_{(1)})(t) = \{p(t) \mid p \textit{ a moving point of } \Phi \textit{ near } t \textit{ with } p'(t) \in \Phi(t)\}.$$

The "$\subseteq$" inclusion is trivial. So take a moving point p with $p'(t) \in \Phi(t)$. Since $p \in \Phi$ we have $p = \sum_{i=1}^{n} \alpha_i p_i$. Therefore,

$$p'(t) = \sum_{i=1}^{n} \alpha_i'(t) p_i(t) + \sum_{i=1}^{n} \alpha_i(t) p_i'(t).$$

We know $p'(t) \in \Phi(t)$, so, because of the choices of p_i in the lemma, we get $\alpha_i(t) = 0$ for $i = 1, \ldots, r$, i.e.

$$p(t) = \sum_{i=r+1}^{n} \alpha_i(t) p_i(t) \in (\Phi_{(1)})(t).$$

Further we define $\Phi_{(0)} := \Phi$ and $\Phi_{(k+1)} := \left(\Phi_{(k)}\right)_{(1)}$ for $k \geq 0$.

Remark 7 $\dim(\Phi^{(1)}) + \dim(\Phi_{(1)}) = 2 \dim \Phi$.

Now we can prove that these two constructions are dual.

Proposition 8 *$\Phi_{(1)} = \mathcal{D}\left((\mathcal{D}\Phi)^{(1)}\right)$ and $\Phi^{(1)} = \mathcal{D}\left((\mathcal{D}\Phi)_{(1)}\right)$*

Proof. The second assertion follows from the first by replacing Φ by $\mathcal{D}\Phi$ and applying $\mathcal{D}$. In order to prove the first we calculate on all points except for isolated points with the choice free description of the constructions. By definition

$$\mathcal{D}\left((\mathcal{D}\Phi)^{(1)}\right)(s) = \{v \in \mathbb{C}^N \mid \textit{for all moving points } p \textit{ of } \mathcal{D}\Phi \textit{ is } v^{\mathrm{T}} \cdot p'(s) = 0\}.$$

Since $(\mathcal{D}\Phi)^{(1)} \supseteq \mathcal{D}\Phi \Rightarrow \mathcal{D}\left((\mathcal{D}\Phi)^{(1)}\right) \subseteq \Phi$, we can assume that $v \in \Phi(s)$ and that $\Phi(s)$ can be written as $\Phi(s) = \{q(s) \mid q \textit{ a moving point of } \Phi \textit{ near } s\}$, so we get

$$\begin{aligned}\mathcal{D}\left((\mathcal{D}\Phi)^{(1)}\right)(s) = \{q(s) \mid\ & q \textit{ a moving point of } \Phi \textit{ near } s \textit{ such that for all} \\ & \textit{moving points } p \textit{ of } \mathcal{D}\Phi \textit{ near } s, q(s)^{\mathrm{T}} \cdot p'(s) = 0\}.\end{aligned}$$

Since $q \in \Phi$ and $p \in \mathcal{D}\Phi$ we know $q^{\mathrm{T}} \cdot p = 0$, so $(q')^{\mathrm{T}} \cdot p + q^{\mathrm{T}} \cdot p' = 0$. Applying this at the point s and $\mathcal{D}\Phi(s) = \{p(s) \mid p \textit{ a moving point of } \mathcal{D}\Phi \textit{ near } s\}$, we get

$$\begin{aligned}\mathcal{D}\left((\mathcal{D}\Phi)^{(1)}\right)(s) &= \{q(s) \mid q \textit{ a moving point of } \Phi \textit{ near } s \textit{ with for all } w \in \mathcal{D}\Phi(s) \\ &\qquad \textit{is } q'(s)^{\mathrm{T}} \cdot w = 0\} \\ &= \{q(s) \mid q \textit{ a moving point of } \Phi \textit{ near } s \textit{ with } q'(s) \in \mathcal{D}\mathcal{D}\Phi(s) \\ &\qquad = \Phi(s)\} \\ &= \Phi_{(1)}(s).\end{aligned}$$

□

4 The normal form

Now we can prove the theorem about the normal form.

Theorem. *Let $\Phi : S \to \mathrm{G}(n,N)$ be a curve and $r := \dim(\Phi^{(1)}) - n$, then there exist unique $a_1 \geq a_2 \geq \ldots \geq a_r > 0$ and a unique subspace $V \subseteq \mathbb{C}^N$ with $\sum_{i=1}^{r} a_i + \dim V = n$ and (in general not unique) curves $\varphi_1, \ldots, \varphi_r : S \to \mathrm{G}(1,N)$, such that*

$$\Phi = \varphi_1^{(a_1-1)} \oplus \ldots \oplus \varphi_r^{(a_r-1)} \oplus V$$

and

$$\Phi^{(1)} = \varphi_1^{(a_1)} \oplus \ldots \oplus \varphi_r^{(a_r)} \oplus V.$$

Proof. We proceed by induction. The case $n = 0$ is trivial. Assume $n > 0$. If $r = 0$, then $\Phi^{(1)} = \Phi = \mathit{const.} =: V$ by the lemma, so let $r > 0$. Now we can apply the induction hypothesis to $\Phi_{(1)}$ and get

$$\Phi_{(1)} = \varphi_1^{(\overline{a}_1-1)} \oplus \ldots \oplus \varphi_{\overline{r}}^{(\overline{a}_{\overline{r}}-1)} \oplus V$$

and

$$(\Phi_{(1)})^{(1)} = \varphi_1^{(\overline{a}_1)} \oplus \ldots \oplus \varphi_{\overline{r}}^{(\overline{a}_{\overline{r}})} \oplus V,$$

where $\overline{r} := \dim\left((\Phi_{(1)})^{(1)}\right) - \dim(\Phi_{(1)})$ and $\sum_{i=1}^{\overline{r}} \overline{a}_i + l = n - r$. Obviously we have $(\Phi_{(1)})^{(1)} \subseteq \Phi$, so $\overline{r} := \dim((\Phi_{(1)})^{(1)}) - \dim(\Phi_{(1)}) \leq \dim \Phi - \dim(\Phi_{(1)}) = r$. Using the proposition we find $\varphi_{\overline{r}+1}, \ldots, \varphi_r : S \to \mathrm{G}(1,N)$, such that

$$\Phi = \varphi_1^{(\overline{a}_1)} \oplus \ldots \oplus \varphi_{\overline{r}}^{(\overline{a}_{\overline{r}})} \oplus \varphi_{\overline{r}+1} \oplus \ldots \oplus \varphi_r \oplus V.$$

Define $a_i := \overline{a}_i + 1$ for $i = 1, \ldots, \overline{r}$ and $a_i := 1$ for $i = \overline{r}+1, \ldots, n$, then Φ is of the claimed form and we have $\Phi^{(1)} = \mathit{span}\{\varphi_1^{(a_1)}, \ldots, \varphi_r^{(a_r)}, V\}$. Comparing dimensions we get the intended result.

It remains to prove the uniqueness. Solving the recursion in the definition of $\Phi_{(k)}$ we get

$$\Phi_{(k)} = \varphi_1^{(a_1-1-k)} \oplus \ldots \oplus \varphi_r^{(a_r-1-k)} \oplus V,$$

where we set $\varphi_i^{(a_i-1-k)} := 0$, if $a_i - 1 - k < 0$.

So $V = \Phi_{(a_1)}$ and by inspecting the dimensions of these equations we get

$$\dim \Phi_{(k)} = l + \sum_{i=1}^{r} \max\{0, a_i - k\}.$$

Therefore the uniqueness of the a_i follows. □

Corollary. *If $\dim(\Phi^{(1)}) = \dim \Phi + 1$, then Φ is either a cone (in the projective sense, i.e. $\dim \bigcap_{s \in S} \Phi(s) \geq 1$) or the $(n-1)$-th osculating curve of a unique curve $\varphi : S \to \mathrm{G}(1,N) = \mathbb{P}_{N-1}$.*

Proof. Just the uniqueness of φ for $\Phi = \varphi^{(n-1)}$ is new, but referring to the proof above, we see that $\varphi = \varphi^{(0)} = \Phi_{(n-1)}$. □

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