OBERSEMINAR ALGEBRAIC GEOMETRY SS 2021: SIMULTANEOUS RESOLUTIONS

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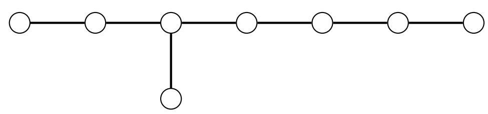
Overview: Each algebraic surface Y admits *resolutions of singularities* $f : X \to Y$, which can by obtained by repeated blowing-ups and normalizations. This is due to the work of Zariski [28] and Abhyankar [1], and a modern proof in the language of schemes was given by Lipman [19]. There is actually a unique *minimal resolution*, where the exceptional divisor $E \subset X$ does not contain (-1)-curves.

A particularly important class of normal surface singularities are the *rational double points*, where the local ring $\mathscr{O}_{Y,b}$ is Gorenstein and the stalk $R^1 f_*(\mathscr{O}_X)_b$ vanishes. These are classified in terms of the exceptional divisor $E = E_1 + \ldots + E_r$, the resulting intersection matrix $N = (E_i \cdot E_j)$ and the dual graph Γ, which takes the shape of a Dynkin diagram from the classification of simple Lie algebras [15]. For example, the rational double point of type E_8 yields the dual graph:



The rational double points are also called ADE-singularities, or du Val singularities. They are ubiquitous and often occur in a natural way, for example on canonical models for surfaces of general type, Weierstraß models for elliptic surfaces, or Kummer surfaces. The monographs of Laufer [16] and Bădescu [10], and the survey of Reid [21] contain explicit examples and further material on the resolutions of rational double points. See Lipman [18] for a comprehensive treatment of rational surface singularities.

The goal of this seminar is to understand resolutions of singularities in *families of normal surfaces*. The latter are flat morphisms $Y \to S$ where the fibers are normal surfaces, parameterized by some base scheme S. They frequently occur in higher dimensions, for example for threefolds fibered over curves, or degenerations of smooth surfaces Y_η over discrete valuation rings, or as deformations of a normal surface Y_0. The latter are induced, at least formally, from Schlessinger's *versal deformations* [22].

One may hope that the family $Y \to S$ admits a *simultaneous resolution of singularities*. This is a family $X \to S$ of smooth surfaces, together with a morphism $f : X \to Y$ that becomes fiberwise the resolution of singularities. Such an X, however, does not exist in general: Basic properties of intersection numbers for curves on surfaces preclude this, already for degenerating surfaces in $\mathbb{P}^3$.

A fundamental insight of Brieskorn ([11] and [12]) tells us that in the *complex-analytic setting*, a resolution exists if the fibers have only rational double points, and one furthermore allows a base-change along some finite surjective $S' \to S$. Here the resolution X' of the base-change $Y' = Y \times_S S'$ usually does not admit an embedding into projective space, although its fibers do.

Using *algebraic spaces*, which take over the role of the complex-analytic spaces occurring above, Michael Artin [6] extended Brieskorn's work to algebraic geometry: He showed that the resolution functor $\underline{\mathrm{Res}}_{Y/S}$, viewed as a contravariant functor on the category (Aff/S), is *representable* by an algebraic space $R = \mathrm{Res}_{Y/S}$. The existence or non-existence of simultaneous resolutions is now reflected by the geometry of the morphism $R \to S$. Indeed, Artin showed that the structure morphism is surjective on the strictly local schemes if the fibers Y_s have only rational double points. This has far-reaching consequences and applications (for examples in [23] or [24]). One should note that the diagonal $R \to R \times R$ is usually only a monomorphism, rather than an embedding. In other words, the algebraic space $R = \mathrm{Res}_{Y/S}$ may *lack this basic separation property* that is automatic for schemes. Such phenomena are also relevant in relative Picard schemes and Albanese maps, compare [17].

The goal of this seminar is to work through Artin's paper [6] and study the relevant notions on resolutions of surface singularities, flat families, functors of Artin rings, and algebraic spaces.

Time and Place: Monday, 12:30-13:30, lecture hall 5H or via Cisco Webex. We shall see what form is permissible.

Schedule:

Talk 1 (May 3), Bruno Laurent
Resolution of surface singularities.
Discuss the existence of resolutions for surface singularities. Explain normalizations,

blowing-ups, exceptional divisors, dual graphs, intersection matrices, and minimality. Give examples. ([10], Chapter 3, [19], Chapter IV, [16], [9], [14])

Talk 2: (May 10), Jakob Bergqvist
Rational singularities and rational double points.
Explain the notion of rational singularities, and in particular of rational double points. Discuss the classification of rational double points and their resolution, with explicit examples. ([10], Chapter 4, [3])

Talk 3: (May 31), Thuong Tuan Dang
Functors of Artin rings.
Discuss Schlessinger's abstract criterion for the existence of versal deformations. Explain examples: deformations of proper schemes, or affine schemes with isolated singularities. ([22])

Talk 4: (June 7), Johannes Fischer
Algebraic spaces.
Introduce the notion of algebraic spaces. Discuss how algebraic spaces arise from schemes via denormalizations along finite subschemes, or contractions of negative-definite curves on surfaces. Emphasize representability criteria. ([20], [5] and [4])

Talk 5: (June 14), Stefan Schröer
Representability of the resolution functor $\mathrm{Res}_{Y/S}$.
Introduce the resolution functor, and explain its representability. ([6])

Talk 6: (June 28), Siddarth Mathur
Brieskorn's result.
Explain that the structure morphism $\mathrm{Res}_{Y/S} \to S$ is finite and surjective, provided that the fibers Y_s have only rational double points. ([6])

Talk 7: (July 12), Daniel Harrer
Further discussions.
Discuss relations to complex-analytic geometry and positive characteristics. ([6])

Talk 8: (July 19), Fabian Korthauer
TBA

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