OBERSEMINAR ALGEBRAIC GEOMETRY WS 21/22: DERIVED CATEGORIES OF COHERENT SHEAVES

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Overview: *Derived categories* were introduced by Verdier in his thesis (see [15] and [14]), and independently by Dold and Puppe [2]. They allow to describe the universal delta functors $R^i f : \mathcal{A} \to \mathcal{B}$, $i \geq 0$ from [3] in terms of a single functor $Rf : D^+(\mathcal{A}) \to D^+(\mathcal{B})$. The idea is very natural, and apparently goes back to Grothendieck: One replaces the abelian category $\mathcal{A}$ by the additive category $D^+(\mathcal{A})$, where the objects are cochain complexes that are bounded below, and the morphism are homotopy classes of cochain map, with quasi-isomorphisms formally inverted. In an analogous way one forms the categories $D^-(\mathcal{A})$ and $D^b(\mathcal{A})$ and $D(\mathcal{A})$.

Taking certain properties of these derived categories as axioms leads to the notation of *triangulated categories*. These are additive categories $\mathcal{D}$, where one has a *translation functor*

$$T : \mathcal{D} \longrightarrow \mathcal{D}, \quad A \longmapsto A[1]$$

and a collection of so-called *distinguished triangles* $A \to B \to C \to A[1]$ as additional structure. They take over the role of the *shift operator* $(A^n)_{n\geq 0}[1] = (A^{n+1})_{n\geq 0}$ of cochain complexes, and the *mapping cone* $C = C^\bullet(f)$ for cochain maps $f : A^\bullet \to B^\bullet$.

The goal of this Oberseminar is to study the derived category $D^b(X)$ arising form the abelian category $\mathrm{Coh}(X)$ of coherent sheaves $\mathscr{F}$ on noetherian schemes X, mainly for schemes that are are smooth and projective over a ground field k. According to a fundamental insight of Bondal and Orlov [1], one may reconstruct the ringed space X from the triangulated category $\mathcal{D} = D^b(X)$ provided that the dualizing sheaf ω_X is ample or anti-ample. This relies on viewing *Serre Duality*

$$H^i(X, \mathscr{F}) \times \mathrm{Ext}^{n-i}(\mathscr{F}, \omega_X) \longrightarrow \mathrm{Ext}^n(\mathscr{O}_X, \omega_X) = k$$

as an intrinsic symmetry of the triangulated category.

On the other hand, Mukai observed that in general such reconstructions are not possible. This is particularly striking if the invertible sheaf ω_X is trivial: For example, for any abelian variety A with dual abelian variety $\widehat{A} = \mathrm{Pic}^0_{A/k}$ the resulting triangulated categories $D^b(A)$ and $D^b(\widehat{A})$ are equivalent [9], although the schemes A and $\widehat{A}$ are in general not isomorphic, for lack of principal polarizations. The impossibility of reconstruction gives rise to the notation of *Fourier–Mukai partners*: These are smooth projective schemes with equivalent derived categories of coherent sheaves.

According to Orlov's result ([11], see also [8]), any equivalence $F : D^b(X) \to D^b(Y)$ arises from some cochain complex $\mathscr{K}^\bullet$ on the product $X \times Y$, via

$$\mathscr{F}^\bullet \longmapsto R\,\mathrm{pr}_{2,*}(L\,\mathrm{pr}_1^*(\mathscr{F}^\bullet) \otimes^L \mathscr{K}^\bullet).$$

This functor is analogous to the classical Fourier transforms $f(x) \mapsto \int f(x)e^{-2\pi xy}dx$ and the more general integral transforms $f(x) \mapsto \int f(x)K(x,y)dx$ from analysis, and therefore called *Fourier–Mukai transforms*. In turn, one regards $\mathscr{K}^\bullet$ is the *kernel* of the transform. The partnership between an abelian variety and its dual arises from the kernel $\mathscr{K}^\bullet = \mathscr{P}[0]$ stemming from the Poincaré sheaf $\mathscr{P}$ on $A \times \widehat{A}$.

Throughout the Oberseminar, we will closely follow the monograph of Huybrechts [5]. The book is the first systematic expostion of the subject, geared towards newcomers in the field and students "with basic knowledge in algebraic geometry". We will draw from what we already learned in the recent workshop and the summer school of the GRK 2240. We are less interested in elementary or formalistic aspects of derived and triangulated categories. Rather, we seek to understand the interaction between algebraic geometry and homological algebra.

Time and Place: Monday, 12:30-13:30, seminar room 25.22.03.73.

Schedule: (all dates are tentative, as ususal shifts might occur)

Talk 1: (4. Oktober), Jakob Bergqvist
Recollection on additive, abelian, derived and triangulated categories.
Summarize [5], Chapter 1–2. Explain in particular Proposition 1.49 and 1.54. Also use [15], [7] and [6].

Talk 2: (11. Oktober), Johannes Fischer
Derived categories of coherent sheaves.

Introduce $D^b(X)$, the derived category of bounded complexes of coherent sheaves. Explain how Serre Duality gives rise to a Serre functor $D^b(X) \to D^b(X)$. Discuss spanning classes and ample sequences ([5], Chapter 3).

Talk 3: (18. Oktober), Thor Wittich
Derived category and canonical bundle I.
Explain the result of Bondal and Orlov: The isomorphism class of the smooth projective scheme X can be reconstructed from the triangulated category $D^b(X)$, provided that the dualizing sheaf ω_X is ample or anti-ample ([5], Chapter 4 and [1]).

Talk 4: (25. Oktober), Thuong Tuan Dang
Fourier–Mukai transforms.
Introduce the Fourier–Mukai transforms $\Phi : D^b(X) \to D^b(Y)$ for kernels $\mathscr{P}^\bullet \in D^b(X \times Y)$, and discuss Orlov's result that any equivalence between the derived categories of smooth projective schemes is of this form. ([5], Chapter 5 and [11]).

Talk 5: (8. November), Fabian Korthauer
Derived category and canonical bundle II.
Explain that the canonical ring $R(X, \omega_X) = \bigoplus_{n \geq 0} H^0(X, \omega_X^{\otimes n})$, the Kodaira dimension $\text{kod}(X)$, and also the numerical Kodaira dimension $\nu(X) = \sup\{d \mid c_1^d \neq 0\}$ can be reconstructed from the triangulated category $D^b(X)$ ([5], Chapter 6 and [12]).

Talk 6: (15. November), Stefan Schröer
Abelian varieties.
Introduce the notion of Fourier–Mukai partnership. Explain the result of Mukai, Orlov and Polischuk: Two abelian varieties A and B have equivalent derived categories if and only if the schemes $A \times \widehat{A}$ and $B \times \widehat{B}$ are isomorphic as abelian varieties with "symplectic structures". ([5], Chapter 9 and [10], [11], [13]).

Talk 7: (22. November), Ivo Kroon
K3 surfaces.
Recall the notion of K3 surfaces, over the field $k = \mathbb{C}$. Explain Mukai's result: Two K3 surfaces X and Y have equivalent derived categories if and only if $H^\bullet(X, \mathbb{Z})$ and $H^\bullet(Y, \mathbb{Z})$, endowed with a certain Hodge structure of weight two and some quadratic form, are Hodge isometric ([5], Chapter 10).

Talk 8: (29. November), Daniel Harrer
Spherical objects.
Discuss the notation of spherical objects $\mathscr{E}^\bullet \in D^b(X)$ and explain how they induce auto-equivalences $D^b(X) \to D^b(X)$ ([5], Chapter 8).

References

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