OBERSEMINAR ALGEBRAIC GEOMETRY SUMMER SEMESTER 2022: MOTIVIC INTEGRATION

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Overview: The goal of this Oberseminar is to gain familiarity with certain notions from the theory of *motivic integration*, which is also the subject of an upcoming summer school at Düsseldorf in September. We want to work through selected sections of the extensive monograph of Chambert-Loir, Nicaise and Sebag [3], and will mainly concentrate on the topics *Grothendieck ring of varieties* and *Greenberg schemes*. For the motivic integrals, we will rely on some shorter lecture course of Blickle [2].

The concept of *motives* was envisioned by Grothendieck, in order to explain that all cohomology theories (Betti, Hodge, de Rham, étale, crystalline...) apparently share the same formal properties. So they all should be "variations" of one common underlying "motive", compare Mazur's discussion [5].

A theorem of Batyrev [1] asserts that compact complex manifolds X and Y that both have $c_1 = 0$ and are birational must have the same Betti numbers $b_i \geq 0$. According to Kontsevich's generalization [4], one actually has an equality of Hodge numbers $h^{pq} \geq 0$. Indeed, the method of his proof was the point of departure for motivic integration.

An important step along the way is the formation of the *Grothendieck ring of varieties* $K_0(\mathrm{Var})$. The elements are isomorphism classes of schemes that are separated and of finite type, modulo the obvious cut-and-past relations attached to closed subschemes and their open complements. It comes with certain localizations $\mathscr{M}$ and completions $\overline{\mathscr{M}}$, and it turns out these rings contain the values of the motivic integrals. They are, however, of interest in their own right.

Another key ingredient are schemes of jets and arcs, both special cases of the so-called *Greenberg schemes*, and the motivic integrals are formed with respect to measurable sets in such spaces. The proof of the Batyrev–Kontsevich Theorem relies on the so-called *transformation rule*, which describes the behaviour of motivic integrals under birational maps.

Time and Place: Monday, 12:30-13:30, seminar room 25.22.03.73.

Schedule: (all dates are tentative, as shifts due to guest talks might occur)

Talk 1: (11. April), Stefan Schröer:
The theorem of Batyrev.
Present the proof that relies on so-called p-adic integration, following [3], Chapter 1, §2.

Talk 2: (25. April), Johannes Fischer:
Additive invariants.
Introduce the Grothendieck ring of varieties $K_0(\text{Var})$, as in [3], Chapter 2, §1.

Talk 3: (2. Mai), Daniel Harrer:
Motivic measures.
Introduces motivic measures, relate them to the Grothendieck ring of varieties, and discuss various classes and relations of interest, after [3], Chapter 2, §2.

Talk 4: (9. Mai), Fabian Korthauer:
Cohomological realizations.
Discuss motivic measures that rely on Hodge theory or ℓ-adic cohomology, following [3], Chapter 2, §3.

Talk 5: (16. Mai), Jakob Bergqvist:
Localizations and completions.
Introduces the localization $\mathscr{M} = K_0(\text{Var})_L$ obtained from the Grothendieck ring of varieties by localizing the Lefschetz motive $L = [\mathbb{A}^1]$, and also its separated completion $\overline{\mathscr{M}}$ with respect to the dimensional filtration, after [3], Chapter 2, §4.

Talk 6: (23. Mai), Jan Hennig:
Bittner's presentation.
Discuss the description of $K_0(\text{Var})$ in terms of generators and relations, valid in characteristic $p = 0$, following [3], Chapter 2, §5.

Talk 7: (30. Mai), Thuong Dang:
The ring schemes $\mathscr{R}_n$.

Describe the construction that gives the structure sheaf for the Greenberg schemes, as in [3], Chapter 4, §2.

Talk 8: (13. Juni), Cesar Hilario:
Greenberg schemes.
Give the definition and basic properties of Greenberg schemes, following [3], Chapter 4, §3.

Talk 9: (20. Juni), Thor Wittich:
Motivic integration and Batyrev-Kontsevich.
Explain the definition of motivic integrals, and discuss the proof for the equality of Hodge numbers using Motivic integration, following [2], Section 1 and Section 2.5.

Talk 10: (27. Juni), Ivo Kroon:
The transformation rule.
Discuss the behaviour of motivic integrals under birational morphism, as in [2], Section 3.

References

[1] V. Batyrev: Birational Calabi–Yau n-folds have equal Betti numbers. In: K. Hulek, F. Catanese, C. Peters, M. Reid (eds.), New trends in algebraic geometry, pp. 1–11. Cambridge University Press, Cambridge, 1999.

[2] M. Blickle: A short course on geometric motivic integration. In: R. Cluckers, J. Nicaise, J. Sebag (eds.), Motivic integration and its interactions with model theory and non-Archimedean geometry. Volume I, pp. 189–243, Cambridge Univ. Press, Cambridge, 2011.

[3] A. Chambert-Loir, J. Nicaise, J. Sebag: Motivic integration. Birkhäuser/Springer, New York, 2018.

[4] M. Kontsevich: Motivic integration. Lecture at Orsay, http://www.lama.univ-savoie.fr/pagesmembres/~raibaut/Kontsevich-MotIntNotes.pdf.

[5] B. Mazur: What is ... a motive? Notices Amer. Math. Soc. 51 (2004), 1214–1216.