OBERSEMINAR ALGEBRAIC GEOMETRY WINTER SEMESTER 2023/24: IRREDUCIBILITY OF THE SPACE OF CURVES

STEFAN SCHRÖER

Overview: Roughly speaking, each algebraic curve X of genus $g \geq 2$ can be viewed as a point in another scheme, the moduli space M_g. Already Riemann [8] understood that such curves X depend on $3g-3$ moduli, which can also be seen as the dimension of M_g. On may view the latter as a quotient of some part of the Hilbert scheme of $\mathbb{P}^{5g-5}$ by an algebraic group. The action is usually not free, in light of possible automorphisms of X, and the quotient has singularities in general. A clearer, simpler and more accurate picture emerges if one regards the moduli problem as a stack $\mathscr{M}_g$.

A natural question is whether the space M_g is connected, or even irreducible. This was analyzed and answered by Deligne and Mumford in their ground-breaking paper [4], building on ideas of Grothendieck. Along the way, they initiated the theory of algebraic stacks.

The goal of this Oberseminar is to work through this classical paper. By this we will gain familiarity with the notion of stable curves, together with their relations to Néron models, learn about deformation theory, also in an arithmetic setting, encounter transcendental methods, and see the usefulness of algebraic stacks. Additional information can be found in the monographs [5], [7], [3], [1].

Throughout, we should use the notation that is by now established, where $\overline{M}_g$ denotes the moduli space of stable curves, and M_g is the open set of smooth curves.

Time and Place: Monday, 12:30-13:30, seminar room 25.22.03.73.

Schedule: (all dates are tentative, and shifts are likely to occur, for example due to internal talks or guest talks)

Talk 1 (18. Oktober)
Fabian Korthauer:
Stable curves.
Introduce the notion of stable curves X and their families, discuss meaningful examples, introduce the Hilbert scheme $H_g \subset \mathrm{Hilb}_{\mathbb{P}^{5g-5}/k}$ of tricanonical curves, and establish that $\mathrm{Ext}^2(\Omega^1_{X/k}, \mathscr{O}_X)$ vanishes ([4], §1, pp. 76–79).

Talk 2 (13. Oktober)
Cesar Hilario:
Versal deformations.
Give a brief recollection of Schlessinger's theory of versal deformations [9], and apply it to the the Hilbert scheme $\overline{H}_g$ and the moduli space $\overline{M}_g$ ([4], §1, pp. 79–83).

Talk 3 (30. October)
Quentin Posva:
Local description and automorphisms.
Explain the local description of the universal curve over $\overline{H}_g \otimes W(k)$, and also the results on the automorphism groups of stable curves ([4], §1, pp. 83–87).

Talk 4 (6. November)
Otto Overkamp:
Stable Reduction and Néron models.
Discuss stable reduction for algebraic curves over a discrete valuation ring, and the relation to Néron models ([4], §2). Additional information can be found in [3] and [2].

Talk 5 (13. November)
Ivo Kroon:
The first proof.
Guide us through the first proof of the connectedness and irreducibility of the space of curves ([4], §3).

Talk 6 (18. December)
Thor Wittich:
Algebraic stacks.
Review the basic theory of stacks, following [4], §4. Additional information is contained in [7] or [6].

Talk 7 (8. January)
Stefan Schröer:
Jacobi and Teichmüller structures.
Introduce Jacobi structures of level $n \geq 0$, Teichmüller structures of level G, and describe the connected components of ${}_G\mathscr{M}_g$ ([4], §5, Theorem 5.13 and Lemma 5.10).

Talk 8 (15. January)
Jan Hennig:
The second proof.
Show that the geometric fibers of ${}_n\mathscr{M}_g$ over the ring $\mathbb{Z}[e^{2\pi i/n}, 1/n]$ are connected ([4], §5, Theorem 5.15).

References

[1] E. Arbarello, M. Cornalba, P. Griffiths: Geometry of algebraic curves. II. Springer, Heidelberg, 2011.
[2] M. Artin, G. Winters: Degenerate fibres and stable reduction of curves. Topology 10 (1971), 373–383.
[3] S. Bosch, W. Lütkebohmert, M. Raynaud: Néron models. Springer, Berlin, 1990.
[4] P. Deligne, D. Mumford: The irreducibility of the space of curves of given genus. Publ. Math. Inst. Hautes Étud. Sci. 36 (1969) 75–109.
[5] J. Harris, I. Morrison: Moduli of curves. Springer, New York, 1998.
[6] G. Laumon, L. Moret-Bailly: Champs algebriques. Springer, Berlin, 2000.
[7] M. Olsson: Algebraic spaces and stacks. American Mathematical Society, Providence, RI, 2016.
[8] B. Riemann: Theorie der Abel'schen Functionen. J. Reine Angew. Math. 54, 115–155 (1857).
[9] M. Schlessinger: Functors of Artin rings. Trans. Amer. Math. Soc. 130 (1968), 208–222.