# Oberseminar Algebraic Geometry: *Perfectoid spaces;* Summer 2025

Stefan Schröer, Otto Overkamp

**Overview:** The theory of *perfectoid spaces* was introduced by Peter Scholze his 2011 doctoral thesis (identical with his article [10]), which led to him being awarded a Fields Medal in 2018. The goal of this semester's Oberseminar will be to gain an understanding of the foundations of this theory. One if its important technical feature is that it operates within the framework of *adic spaces,* which are somewhat akin to analytic spaces and allow one to consider limit processes. The main innovation was that perfectoid spaces provide a dictionary (called *tilting*) which allows one to translate from a language which speaks about certain geometric objects in mixed characteristic into one that talks about similar geometric objects in equal characteristic.

As a major application, we shall see how perfectoid spaces can be used to prove many new cases of the *weight-monodromy conjecture.* Interestingly, the while this application is striking, the weight-monodromy conjecture hardly features in [10]. Rather, the conjecture was already well-known in the equal characteristic case, and the mixed characteristic case follows by applying this translation.

Beginning from what turned out to be a special case of the general theory (the theorem of Fontaine-Wintenberger), we will work through Scholze's thesis, which will be our main reference. For survey articles, see [1, 9, 11, 12, 13]. See [6, 7, 8] as well as [4] for some technical background.

**Time and place:** Monday, 12:30 - 13:30; seminar room 25.22.03.73.
**Schedule** *(all dates are tentative):*

1. **The Theorem of Fontaine-Wintenberger:** (7 April; Otto) Explain the construction of a *canonical* isomorphism between the absolute Galois groups of $\mathbf{F}_p((t))$ and $\mathbf{Q}_p((p^{1/p^\infty}))$. Explain why this also yields a canonical isomorphism between the absolute Galois groups of $\mathbf{F}_p((t^{1/p^\infty}))^\wedge$ and $\mathbf{Q}_p((p^{1/p^\infty}))^\wedge$. Finally, give a categorical interpretation of this isomorphism in terms of étale algebras. Follow [2, 3, 14].

2. **Perfectoid fields:** (28 April; Bianca) Introduce the notion of *perfectoid fields* following [10, Definition 3.1] and explain the results of [10, Section 3]. Explain in detail how the *tilt* $K^\flat$ of a perfectoid field $K$ of characteristic 0 is constructed. Explain

the significance of the surjectivity condition in the definition (e. g. by showing that $\mathbf{Q}_p^\flat = \mathbf{F}_p$).

3. **Almost Mathematics:** (12 May; Stefan) Let $K$ be a perfectoid field and let $\mathfrak{m} \subseteq K^\circ$ be the maximal ideal. Explain the notions of *almost module* over $K^\circ$, *almost algebra* over $K^\circ$, and of *module* over an almost algebra over $K^\circ$. Explain how classical notions from commutative algebra, such as *finitely generated/presented, flat, projective,* and *étale* translate to the almost world. Follow [10, Section 4] and [4].

4. **TBA:** (TBA, TBA) -

5. **Perfectoid algebras:** (19 May; Jack) Explain the notion of *perfectoid algebra* over a perfectoid field $K$ [10, Definition 5.1]. In particular, introduce the notion of *Banach $K$-algebra* and as much of the section on almost mathematics [10, Section 4] as is necessary. Give examples of perfectoid $K$-algebras.

6. **Tilting perfectoid algebras:** (2 June; Ivo) Explain [10, Proposition 5.17]. In particular, explain the tilting functor for perfectoid $K$-algebras. Finally, explain the proof of the *tilting equivalence* [10, Theorem 5.2]. Use as much of [10, Section 4] as is necessary (perhaps as a black box).

7. **Étale algebras:** (9 June; Hugo) Show that the classical notion of *étale algebra* behaves well with respect to perfectoid algebras (cf. [10, Proposition 5.23]) and prove Theorem 5.25 of [10], which shows that tilting preserves étaleness. Take care to explain how the proof uses various intermediate equivalences of categories (cf. the diagram at the bottom of p. 282 in [10]) featuring almost objects. If time permits, explain why this result is sometimes called an *almost purity theorem.*

8. **Adic spaces:** (16 June; Julian) Introduce the notions of *affinoid $K$-algebra* and of its associated *adic space* [7, Definition on p. 521]. Explain the different kinds of points of an adic space, and why the structure presheaf of an adic space need not be a sheaf. Finally, explain what the *étale cohomology* of an adic space is. Explain the comparison theorems [6, Theorem 3.8.1] and [8, Theorem 3.6(a)].

9. **Perfectoid spaces:** (23 June; Jan) Introduce the notion of *perfectoid space* [10, Definition 6.15] over a perfectoid field $K$, as well as the *tilting equivalence* [10, Proposition 6.17] between the categories of perfectoid spaces over $K$ and $K^\flat$. Set out the main properties of perfectoid spaces (in particular [10, Theorem 6.3]). Explain why the étale sites of a perfectoid space $X$ and its tilt $X^\flat$ can be identified canonically. Finally, explain the consequences of this result for the étale cohomology of $X$ and $X^\flat$.

10. **The weight-monodromy conjecture:** (30 June; Chen) Explain the *weight-monodromy conjecture* for a smooth projective variety $Z$ over $\mathbf{Q}_p$ (or over $\mathbf{F}_p((t))$; for example, following [10, Section 9]). Assuming that $Z$ is a hypersurface in some projective space

(or more generally, a smooth set-theoretic complete intersection in a smooth toric variety), show how one can associate with $Z$ a perfectoid space, and explain how the mixed-characteristic case is deduced from the equal-characteristic case. Explain the importance of the *approximation algorithm* (cf. [10, Lemma 6.5]) in the proof.

## References


[1] Bhatt, B. *What is a Perfectoid Space?* Notices AMS, vol. 61, No. 9 (2014), pp. 1082-1084.

[2] Fontaine, J.-M., Wintenberger, J.-P. *Extensions algébrique et corps des normes des extensions APF des corps locaux.* C. R. Acad. Sci. Paris Sér. A-B, 288 (1979), pp. 441-444.

[3] Fontaine, J.-M., Wintenberger, J.-P. *Le "corps des normes" de certaines extensions algébriques de corps locaux.* C. R. Acad. Sci. Paris Sér. A-B, 288 (1979), pp. 367-370.

[4] Gabber, O., Ramero, L. *Almost Ring Theory.* Lecture Notes in Math., vol. 1800, Springer, Berlin, 2003.

[5] Harris, M. *The Perfectoid Concept: The Case for an Absent Theory.* Available at `https://www.math.columbia.edu/~harris/otherarticles_files/perfectoid.pdf`.

[6] Huber, R. *Étale Cohomology of Rigid Analytic Varieties and Adic Spaces.* Aspects of Math., vol. E30, Vieweg, Braunschweig, 1996.

[7] Huber, R. *A generalization of formal schemes and rigid analytic varieties.* Math. Z., 217 (1994), pp. 513–551.

[8] Huber, R. *A finiteness result for direct image sheaves on the étale site of rigid analytic varieties.* J. Algebraic Geom., 7 (1998), 359–403.

[9] Rapoport, M. *The work of Peter Scholze.* Proc. Int. Cong. Math. 2018, Rio de Janeiro, Vol. 1, pp. 71-86

[10] Scholze. P. *Perfectoid spaces.* Publ. Math. IHÉS, Vol. 116 (2012), pp. 245-313.

[11] Scholze. P. *Perfectoid spaces and their applications.* Available at `https://www.math.uni-bonn.de/people/scholze/ICM.pdf`.

[12] Scholze. P. *Perfectoid spaces: a survey.* Available at `https://www.math.uni-bonn.de/people/scholze/CDM.pdf`.

[13] Wedhorn, T. *On the Work of Peter Scholze.* Jahresber. DMV (2019), 121, pp. 245-289.

[14] Wintenberger, J.-P. *Le corps des normes de certaines extensions infinies de corps locaux; applications*. Ann. sci. École Norm. Sup., Serie 4, Vol. 16 (1983) no. 1, pp. 59-89.