OBERSEMINAR ALGEBRAIC GEOMETRY: ÉTALE COHOMOLOGY WINTER SEMESTER 2025/26

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Overview: The étale cohomology groups $H^i(X, \mathbb{Q}_\ell(j))$ for a scheme X of finite type over a ground field k yield powerful tools in algebraic geometry to understand both geometric and arithmetic aspects. These $\mathbb{Q}_\ell$-vector spaces can be seen as analogues of the classical singular cohomology groups for topological spaces, and many of the crucial properties carry over, in one form or another (finiteness, cup products, cycle classes, Künneth Formula, Poincaré Duality, base-change). An important consequence is that the Betti numbers b_i, which intuitively count the number of i-dimensional "holes" in a topological space, remain meaningful for schemes. The étale cohomology groups also incorporate properties of the ground field k. In fact, one should view $H^i(X \otimes k^{\text{sep}}, \mathbb{Q}_\ell(j))$ as fiber of a local system $\underline{H}^i(X, \mathbb{Q}_\ell(j))$ over the étale site of the base scheme $S = \text{Spec}(k)$.

By Grothendieck's insight, one may replace the Zariski topology of all open sets $U \subset X$, which is too coarse to yield analogues of singular cohomology, by the category of all étale morphisms $V \to X$, endowed by what is now called Grothendieck topology. Another crucial insight was that coefficients like $\mathbb{Q}$ or $\mathbb{Z}$ do not work, and have to be essentially replaced by torsion groups. Thanks to Kummer theory, the sheaf of roots of units $F = \mu_{\ell^\nu}$ for some prime $\ell > 0$ that is invertible in the ground field gives best results. Forming the inverse limit for $H^i(X, \mu_{\ell^\nu}^{\otimes j})$ and tensoring with the field of fractions for $\mathbb{Z}_\ell$ then yields the vector spaces $H^i(X, \mathbb{Q}_\ell(j))$.

Working in a relative setting with morphisms $f : X \to Y$, one can form $R^i f_*(F)$. These higher direct images satisfy two fundamental base-change properties: If f is proper, the formation of $R^i f_*(F)$ commutes with arbitrary base-change. If f is merely quasicompact and quasiseparated, the formation commutes at least with smooth base-change. In a somewhat imprecise manner, this is often called proper and smooth base-change, respectively.

The goal of this Oberseminar is to familiarize with the étale cohomology groups, to gain an understanding for the base-change theorems, and see the role of the theory in some concrete situations.

Time and Place: Monday, 12:30–13:30, seminar room 25.22.03.73.

Schedule: (all dates are tentative and shifts are likely to occur, for example due to internal talks or guests)

Talk 1 (20. October)
Ping:
The point of departure: singular cohomology.
Recall singular cohomology groups $H^i(X, \mathbb{Z})$. Discuss their computability for CW-complexes via the cellular chain complex, and Poincaré Duality for closed oriented manifolds ([9], Theorem 3.5 and Proposition 3.38, or any other useful source). Point out that the latter is a very strong condition (compare [14], Chapter II, Theorem 5.3). Also mention the Ehresmann Lemma ([7] or [18], Section 9.1.1), and the base-change property for proper continuous maps ([4], first page of Chapter IV; compare also [16], Theorem 4.4).

Talk 2 (27. October)
Schröer:
Étale morphisms, Grothendieck topologies, and sites.
Recall the notion of étale morphisms $U \to X$ of schemes (for example [13], Chapter I, §3), and explain that this generalizes the local homeomorphisms from topology. Discuss that one may replace the collection of open sets $U \subset X$ by the category of étale morphisms $V \to X$; the latter come with a Grothendieck topology and form the étale site, with ensuing notions of sheaves and cohomology (for example [4], Chapter I).

Talk 3 (3. November)
Ritschel:
Étale cohomology and Galois cohomology.
Let K be a field, K^{sep} a separable closure, and $G = \text{Gal}(K^{\text{sep}}/K)$ the resulting Galois group. Recall that Galois cohomology $H^i(G, M)$ is a special case of group cohomology (for example [8], Section 4.5). Discuss that this can be viewed as étale cohomology on the scheme $S = \text{Spec}(K)$, and compute $H^i(G, K^{\text{sep}})$ (loc. cit., Proposition 5.7.8 and Theorem 4.5.3)

Talk 4 (17. November)
Rodatz:
Picard groups, Brauer groups, and Hilbert 90.

Explain the Kummer sequence $0 \to \mu_\ell \to \mathbb{G}_m \stackrel{\ell}{\to} \mathbb{G}_m \to 0$ and discuss how to compute the étale cohomology provided that $\ell > 0$ is invertible on the scheme X. and in particular on curves ([8], Theorem 7.2.9). Highlight the close connection to Picard groups and Brauer groups (confer [13], Chapter IV), and briefly dip into the Artin–Schreier sequence $0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \to \mathbb{G}_a \to 0$ and the resulting long exact sequences ([8], Theorem 7.2.3).

Talk 5 (24. November)
Reichardt:
Betti numbers, ℓ-adic cohomology, and cycle classes.

Mention how Serre used supersingular elliptic curves E to show that there is no meaningful cohomology theory with values in finitely generated abelian groups ([6], Introduction). Discuss the definition of the ℓ-adic cohomology groups

$$H^i(X, \mathbb{Q}_\ell(j)) = \varprojlim_{j \geq 0} H^i(X, \mu_{\ell^\nu}^{\otimes j}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

and the ensuing Betti numbers b_i. Motivate the Tate twists given by j via the cycle class map, following [5], Section 2.

Talk 6 (1. December)
Bode:
The Base Change Theorem for proper morphisms.

The goal of this talk is to elucidate the following foundational result: Let $f : X \to S$ be a proper morphism, and F be an abelian sheaf on X that is annihilated by some arbitrary $\ell > 0$. Then the stalks $R^i f_*(F)$ at a geometric point $s : \operatorname{Spec}(\Omega) \to X$ can be identified with the étale cohomology $H^i(X_s, F|X_s)$ of the fiber, and thus commutes with arbitrary base-changes $S' \to S$. Highlight how this general result is reduced to rather elementary properties of families of curves ([4], Chapter IV and [8], Section 7.3).

Talk 7 (8. December)
Zock:
The Base Change Theorem by smooth morphisms I.
This and the next talk require some coordination. The goal is to elucidate the following foundational result ([4], Chapter V, Theorem 3.2): Let $f : X \to S$ be a separated morphism of finite type, and F be an abelian sheaf annihilated by some $\ell > 0$ that is invertible on S. Then the formation of $R^i f_*(F)$ commutes with smooth base-changes $S' \to S$. In this talk, concentrate on local acyclicity, and why ℓ must be invertible, following [4], Chapter V, Section 2 and [8], Section 7.6.

Talk 8 (15. December)
Monreal:
The Base Change Theorem by smooth morphisms II.
Continue the previous talk to elucidate the proof of the Base-Change Theorem by Smooth morphisms, following [4], Chapter V, Section 3 and [8], Section 7.7.

Talk 9 (5. January)
Schröer:
K3 surfaces.
Recall the notion of K3 surfaces, following the lecture notes [17]. Discuss the upper bound on the rank of $\mathrm{Pic}(X \otimes K^{\mathrm{alg}})$ for a K3 surfaces X over a number field K in terms of Frobenius eigenvalues on étale cohomology at primes of good reduction (loc. cit., Theorem 2.12).

Talk 10 (12. January)
Otto Overkamp:
Point counting.
Discuss the Zeta function $Z(X, t)$ for a smooth projective scheme X over a finite field $k = \mathbb{F}_p$, and its expression via reverse characteristic polynomials $P_i(X, t)$ of the Frobenius on étale cohomology, following [13], Chapter VI, §12. Elucidate what this concretely means for K3 surfaces, following [10], Chapter 4, Section 4.1, and [17], Example 3.16.

References

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